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ON A CERTAIN CLASSES OF HARMONIC UNIVALENT FUNCTIONS ASSOCIATED WITH SRIVASTAVA-ATTIYA OPERATOR

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Abstract

In this paper, we define two new classes of harmonic univalent functions in the open unit disk by using Srivastava-Attiya operator. We obtain sufficient coefficient conditions for the class of harmonic univalent functions with positive coefficients which are found also to be necessary when the coefficients are negative. Properties like distortion bounds, covering theorem, extreme points, convolution and convex combination for the functions in the class of harmonic univalent functions with negative coefficients are investigated.

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Key Words and Phrases: Multivalent functions, harmonic functions, Srivastava-Attiya operator, extreme points, distortion bounds.

1 Introduction

A complex valued continuous function $w = f(z) = u(z) + iv(z)$ defined in a simply connected convex domain $D \subset C$ is said to be harmonic in D if both u and v are real-valued harmonic functions in D. In any simply connected domain D, we can write $f = h + \overline{g}$ where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|$, $z \in D$. See Clunie and Sheil-Small [4].

Let S_H be the class of functions $f = h + \overline{g}$ that are harmonic, univalent and sense-preserving in the open unit disk $U = \{z : |z| < 1\}$ with the normalization

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = \sum_{k=1}^{\infty} b_k z^k, \quad |b_1| < 1 \tag{1}$$

Several authors (see for example [4],[5],[6],[7],[8],[9]) had studied different subclasses of the class S_H of harmonic functions.

Srivastava-Attiya operator [1] (see also Raducanu and Srivastava [2], Prajapat and Goyal [3], Tulasiram et al. [10]) introduced and investigated the linear operator $J_{\mu,b} : A \to A$ where A is the class of all analytic functions defined in the open unit disk U. For the functions $f(z)$ of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, the Srivastava-Attiya operator, $J_{\mu,b}$ is defined as follows.

$$J_{\mu,b} f(z) = z + \sum_{k=2}^{\infty} C_k(\mu, b) a_k z^k$$

where

$$C_k(\mu, b) = \left| \left(\frac{1+b}{k+b} \right)^{\mu} \right| \quad \text{and } b \in \mathbb{C} \backslash \{Z_0^-\}; \mu \in \mathbb{C}; z \in U. \tag{2}$$

In this paper, two new classes $RH(\mu, b, \beta, \gamma)$ and $RF(\mu, b, \beta, \gamma)$ $(\mu \in \mathbb{C}, b \in \mathbb{C} \backslash \{Z_0^-\}, \beta \geq 0, 0 \leq \gamma < 1)$ of harmonic univalent functions in $U = \{z : |z| < 1\}$ are introduced and studied.

2 The Classes $RH(\mu, b, \beta, \gamma)$ and $RF(\mu, b, \beta, \gamma)$

Definition 1. Let $f = h + \overline{g}$ be a harmonic function where

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = \sum_{k=1}^{\infty} b_k z^k, \quad |b_1| < 1, z \in U \tag{3}$$

The $f \in RH(\mu, b, \beta, \gamma)$, if it satisfies

$$Re\left\{\beta J_{\mu,b}\left[\frac{h(z)}{z} + \left[\overline{\frac{g(z)}{z}}\right]\right] + J_{\mu,b}[h'(z) + \overline{g'(z)}] - \beta\right\} > \gamma \tag{4}$$

where $J_{\mu,b}f(z)$ is defined by

$$J_{\mu,b}f(z) = \sum_{k=1}^{\infty}\left|\left(\frac{1+b}{k+b}\right)^{\mu}\right|[a_k z^k + \overline{b_k z^k}]$$

$$b \in \mathbb{C}\backslash\{Z_0^-\}; \mu \in \mathbb{C}; z \in U$$

Definition 2. Let $f = h + \overline{g}$ be a harmonic function where

$$h(z) = z - \sum_{k=2}^{\infty} |a_k| z^k, \quad g(z) = -\sum_{k=1}^{\infty} |b_k| z^k, \quad |b_1| < 1, z \in U \tag{5}$$

The $f \in RF(\mu, b, \beta, \gamma)$, if it satisfies

$$Re\left\{\beta J_{\mu,b}\left[\frac{h(z)}{z} + \left[\overline{\frac{g(z)}{z}}\right]\right] + J_{\mu,b}[h'(z) + \overline{g'(z)}] - \beta\right\} > \gamma \tag{6}$$

where $J_{\mu,b}f(z)$ is defined by

$$J_{\mu,b}f(z) = \sum_{k=1}^{\infty}\left|\left(\frac{1+b}{k+b}\right)^{\mu}\right|[a_k z^k + \overline{b_k z^k}],$$

$$b \in \mathbb{C}\backslash\{Z_0^-\}; \mu \in \mathbb{C}; z \in U$$

3 Main Results

Theorem 3. *Let $f = h + \overline{g}$ where h and g are given by (1). Furthermore, let*

$$\sum_{k=1}^{\infty}(k+\beta)\left|\left(\frac{1+b}{k+b}\right)^{\mu}\right|[|a_k|+|b_k|] \leq 2+\beta-\gamma, \tag{7}$$

$$|b_1| < \frac{1-\gamma}{1+\beta} < 1.$$

where $a_1 = 1, \beta \geq 0, b \in \mathbb{C}\backslash\{Z_0^-\}, \mu \in \mathbb{C}; z \in U$
Then f is harmonic univalent and sense-preserving in U and $f \in RF(\mu, b, \beta, \gamma)$

Proof. Based on the technique used in ([9],Theorem 2.1), it is preserved that f is harmonic, sense-preserving and univalent. For $|z_1| \leq |z_2| < 1$, by using (7) it can be seen that

$$\begin{aligned}
|f(z_1) - f(z_2)| &\geq |h(z_1) - h(z_2)| - |g(z_1) - g(z_2)| \\
&= \left|(z_1 - z_2) + \sum_{k=2}^{\infty} a_k(z_1^k - z_2^k)\right| - \left|\sum_{k=1}^{\infty} b_k(z_1^k - z_2^k)\right| \\
&\geq |z_1 - z_2| \Bigg[1 - \gamma - (1+\beta)|b_1| \\
&\qquad -|z_2|\sum_{k=2}^{\infty}(k+\beta)\left|\left(\frac{1+b}{k+b}\right)^{\mu}\right|(|a_k|+|b_k|)\Bigg] \\
&\geq |z_1 - z_2|[1-\gamma-(1+\beta)|b_1|](1-|z_2|) > 0.
\end{aligned}$$

Consequently, f is univalent in U. It is observed that f is sense-preserving in U, since by using (7), $|z| < 1$,

$$\begin{aligned}
|h'(z)| &\geq 1 - \sum_{k=2}^{\infty} k|a_k| \\
&\geq 1 - \gamma - \sum_{k=2}^{\infty}(k+\beta)\left|\left(\frac{1+b}{k+b}\right)^{\mu}\right||a_k| \\
&\geq \sum_{k=1}^{\infty}(k+\beta)\left|\left(\frac{1+b}{k+b}\right)^{\mu}\right||b_k| \\
&\geq |g'(z)|.
\end{aligned}$$

Now, it suffices to show that $f \in RH(\mu, b, \beta, \gamma)$ using the fact

$$Re\ w > \gamma \text{ if and only if } |1 + w - \gamma| > |1 - w + \gamma|$$

Substituting h and g from (3) and using (7), it follows that

$$\begin{aligned}
&\left|1 + \beta J_{\mu,b}\left[\frac{h(z)}{z} + \left(\overline{\frac{g(z)}{z}}\right)\right] + J_{\mu,b}[h'(z) + \overline{g'(z)}] - \beta - \gamma\right| \\
&\quad - \left|1 - \beta J_{\mu,b}\left[\frac{h(z)}{z} + \left(\overline{\frac{g(z)}{z}}\right)\right] - J_{\mu,b}[h'(z) + \overline{g'(z)}] + \beta + \gamma\right| \\
&= \left|1 + \beta\left[\sum_{k=1}^{\infty}\left|\left(\frac{1+b}{k+b}\right)^{\mu}\right|[a_k z^{k-1} + \overline{b_k z^{k-1}}]\right.\right. \\
&\quad \left.\left. + \left[\sum_{k=1}^{\infty}\left|\left(\frac{1+b}{k+b}\right)^{\mu}\right|[k a_k z^{k-1} + \overline{k b_k z^{k-1}}] - \beta - \gamma\right]\right]\right| \\
&\quad - \left|1 - \beta\left[\sum_{k=1}^{\infty}\left|\left(\frac{1+b}{k+b}\right)^{\mu}\right|[a_k z^{k-1} + \overline{b_k z^{k-1}}]\right.\right. \\
&\quad \left.\left. - \left[\sum_{k=1}^{\infty}\left|\left(\frac{1+b}{k+b}\right)^{\mu}\right|[k a_k z^{k-1} + \overline{k b_k z^{k-1}}] + \beta + \gamma\right]\right]\right| \\
&= \left|1 + \beta(a_1 + \overline{b_1}) + (a_1 + \overline{b_1})\right. \\
&\qquad \left. + (\beta + k)\sum_{k=2}^{\infty}\left|\left(\frac{1+b}{k+b}\right)^{\mu}\right|(a_k z^{k-1} + \overline{b_k z^{k-1}}) - \beta - \gamma\right| \\
&\quad - \left|1 - \beta(a_1 + \overline{b_1}) - (a_1 + \overline{b_1})\right. \\
&\qquad \left. - (\beta + k)\sum_{k=2}^{\infty}\left|\left(\frac{1+b}{k+b}\right)^{\mu}\right|(a_k z^{k-1} + \overline{b_k z^{k-1}}) + \beta + \gamma\right|
\end{aligned}$$

$$
\begin{aligned}
&= \Bigg|1+\beta+\beta\overline{b_1}+1+\overline{b_1}-\beta-\gamma \\
&\quad +\sum_{k=2}^{\infty}(k+\beta)\left|\left(\frac{1+b}{k+b}\right)^{\mu}\right|(a_k z^{k-1}+\overline{b_k z^{k-1}})\Bigg| \\
&\quad -\Bigg|1-\beta-\beta\overline{b_1}-1-\overline{b_1}+\beta+\gamma \\
&\quad -\sum_{k=2}^{\infty}(k+\beta)\left|\left(\frac{1+b}{k+b}\right)^{\mu}\right|(a_k z^{k-1}+\overline{b_k z^{k-1}})\Bigg| \\
&= \Bigg|2-\gamma+(1+\beta)\overline{b_1}+\sum_{k=2}^{\infty}(k+\beta)\left|\left(\frac{1+b}{k+b}\right)^{\mu}\right|(a_k z^{k-1}+\overline{b_k z^{k-1}})\Bigg| \\
&\quad -\Bigg|\gamma-(1+\beta)\overline{b_1}-\sum_{k=2}^{\infty}(k+\beta)\left|\left(\frac{1+b}{k+b}\right)^{\mu}\right|(a_k z^{k-1}+\overline{b_k z^{k-1}})\Bigg| \\
&\geq 2-2\gamma-2(1+\beta)|b_1|-2\sum_{k=2}^{\infty}(k+\beta)\left|\left(\frac{1+b}{k+b}\right)^{\mu}\right|(|a_k|+|b_k|)|z|^{k-1} \\
&= 2-2\gamma+2(1+\beta)|a_1|-2\sum_{k=1}^{\infty}(k+\beta)\left|\left(\frac{1+b}{k+b}\right)^{\mu}\right|(|a_k|+|b_k|)|z|^{k-1} \\
&\qquad\qquad\qquad\qquad \because\ a_1=1 \\
&= 2-2\gamma+2+2\beta-2\sum_{k=1}^{\infty}(k+\beta)\left|\left(\frac{1+b}{k+b}\right)^{\mu}\right|(|a_k|+|b_k|)|z|^{k-1} \\
&\qquad\qquad\qquad\qquad \because\ |z|<1 \\
&> 2[2+\beta-\gamma-\sum_{k=1}^{\infty}(k+\beta)\left|\left(\frac{1+b}{k+b}\right)^{\mu}\right|(|a_k|+|b_k|) \\
&\geq 0 \qquad \text{by (7)}
\end{aligned}
$$

The harmonic mapping

$$
f(z)=z+\sum_{k=2}^{\infty}\frac{(1-\gamma)x_k}{(k+\beta)\left|\left(\frac{1+b}{k+b}\right)^{\mu}\right|}z^k+\sum_{k=1}^{\infty}\frac{(1-\gamma)\overline{y_k}}{(k+\beta)\left|\left(\frac{1+b}{k+b}\right)^{\mu}\right|}\overline{z^k} \tag{8}
$$

where $\sum_{k=2}^{\infty}|x_k|+\sum_{k=1}^{\infty}|y_k|=1$, $\beta\geq 0$ and $0\leq\gamma<1$, shows that the coefficient bound given by (7) is sharp. □

In the next theorem, we will prove a necessary and sufficient condition for functions f to belong to $RH(\mu,b,\beta,\gamma)$.

Theorem 4. *Let $f = h + \overline{g}$ where h and g are given by (5). Then $f \in RF(\mu, b, \beta, \gamma)$ if and only if*

$$\sum_{k=1}^{\infty}(k+\beta)\left|\left(\frac{1+b}{k+b}\right)^{\mu}\right|[|a_k| + |b_k|] \leq 2 + \beta - \gamma \tag{9}$$

where $a_1 = 1$, $\beta \geq 0$, $0 \leq \gamma < 1$ and $z \in U$.

Proof. Assume that $f \in RF(\mu, b, \beta, \gamma)$. Then from (4) with h and g given by (5), it can be found that

$$Re\left[\beta J_{\mu,b}\left[\frac{h(z)}{z} + \left(\frac{\overline{g(z)}}{z}\right)\right] + J_{\mu,b}[h'(z) + \overline{g'(z)}] - \beta\right| > \gamma$$

$$Re\left[\beta\left[\frac{\left[z - \sum_{k=2}^{\infty}\left|\left(\frac{1+b}{k+b}\right)^{\mu}\right||a_k|z^k\right]}{z} - \frac{-\sum_{k=1}^{\infty}\left|\left(\frac{1+b}{k+b}\right)^{\mu}\right||b_k|\overline{z^k}}{z}\right]\right.$$

$$\left. + \left[1 - \sum_{k=2}^{\infty}\left|\left(\frac{1+b}{k+b}\right)^{\mu}\right|k|a_k|z^{k-1} - \sum_{k=1}^{\infty}\left|\left(\frac{1+b}{k+b}\right)^{\mu}\right|k|b_k|z^{k-1}\right] - \beta\right] > \gamma$$

$$\Rightarrow Re\left[\left[\beta - \beta\sum_{k=2}^{\infty}\left|\left(\frac{1+b}{k+b}\right)^{\mu}\right||a_k|z^{k-1} - \beta\sum_{k=1}^{\infty}\left|\left(\frac{1+b}{k+b}\right)^{\mu}\right||b_k|\overline{z^{k-1}}\right]\right.$$

$$\left. 1 - \sum_{k=2}^{\infty}\left|\left(\frac{1+b}{k+b}\right)^{\mu}\right|k|a_k|z^{k-1} - \sum_{k=1}^{\infty}\left|\left(\frac{1+b}{k+b}\right)^{\mu}\right|k|b_k|\overline{z^{k-1}}\right] > \gamma$$

$$\Rightarrow Re\left[1 - \sum_{k=2}^{\infty}(\beta+k)\left|\left(\frac{1+b}{k+b}\right)^{\mu}\right||a_k|z^{k-1}\right.$$

$$\left. - \sum_{k=1}^{\infty}(\beta+k)\left|\left(\frac{1+b}{k+b}\right)^{\mu}\right||b_k|\overline{z^{k-1}}\right] > \gamma$$

$$\Rightarrow Re\left[1 + 1 + \beta - \left[\sum_{k=2}^{\infty}(\beta+k)\left|\left(\frac{1+b}{k+b}\right)^{\mu}\right|(|a_k|z^{k-1} + |b_k|\overline{z^{k-1}})\right]\right] > \gamma$$

$(0 \leq \gamma < 1, \beta \geq 0, b \in \mathbb{C}\backslash\{Z_0^-\}, \mu \in \mathbb{C})$

Now choose z to be real and let $z \to 1^-$, then

$$2+\beta-\sum_{k=1}^{\infty}(k+\beta)\left|\left(\frac{1+b}{k+b}\right)^{\mu}\right|[|a_k|+|b_k|] \geq \gamma$$
$$\Rightarrow -\sum_{k=1}^{\infty}(k+\beta)\left|\left(\frac{1+b}{k+b}\right)^{\mu}\right|[|a_k|+|b_k|] \geq \gamma-2-\beta$$
$$\Rightarrow \sum_{k=1}^{\infty}(k+\beta)\left|\left(\frac{1+b}{k+b}\right)^{\mu}\right|[|a_k|+|b_k|] \leq 2+\beta-\gamma$$

which is the assertion (9) of Theorem 4.

Conversely, assume that (9) holds true. Then from (4) with h and g given by (5) and on using (9) for $|z|<1$, it can be found that

$$Re\left\{\beta J_{\mu,b}\left[\frac{h(z)}{z}+\left(\overline{\frac{g(z)}{z}}\right)\right]+J_{\mu,b}[h'(z)+\overline{g'(z)}]-\beta\right\}$$
$$=\left\{2+\beta-\sum_{k=1}^{\infty}(k+\beta)\left|\left(\frac{1+b}{k+b}\right)^{\mu}\right|[|a_k|+|b_k|]z^{k-1}\right\}$$
$$\geq 2+\beta-\sum_{k=1}^{\infty}(k+\beta)\left|\left(\frac{1+b}{k+b}\right)^{\mu}\right|[|a_k|+|b_k|]z^{k-1} > \gamma$$

This shows that $f \in RF(\mu,b,\beta,\gamma)$. □

Theorem 5. *Let $f \in RF(\mu,b,\beta,\gamma)$. Then for $|z|=r<1$,*

$$|f(z)| \leq (1+|b_1|)r+\frac{|(2+b)^{\mu}|}{(2+\beta)|(1+b)^{\mu}|}(1-(1+\beta)|b_1|-\gamma)r^2$$

and

$$|f(z)| \geq (1-|b_1|)r-\frac{|(2+b)^{\mu}|}{(2+\beta)|(1+b)^{\mu}|}(1-(1+\beta)|b_1|-\gamma)r^2$$

These bounds are sharp.

Proof. Let $f = h + \overline{g}$ where h and g are defined as in (5). Then

$$\begin{aligned} |f(z)| &= \left| z - \sum_{k=2}^{\infty} |a_k| z^k - \sum_{k=1}^{\infty} |b_k| \overline{z}^k \right| \\ &\geq (1 - |b_1|)r - \frac{|(2+b)^\mu|}{(2+\beta)|(1+b)^\mu|} \\ &\qquad \times \sum_{k=2}^{\infty} (k+\beta) \left| \left(\frac{1+b}{k+b} \right)^\mu \right| (|a_k| + |b_k|) r^2 \\ &\geq (1 - |b_1|)r - \frac{|(2+b)^\mu|}{(2+\beta)|(1+b)^\mu|} (1 - \gamma - (1+\beta)|b_1|) r^2 \end{aligned}$$

The proof of upper bounds of $|f(z)|$ is similar.
Hence, these bounds are sharp and equalities occur if

$$f(z) = z + |b_1|\overline{z} + \frac{|(2+b)^\mu|}{(2+\beta)|(1+b)^\mu|} (1 - (1+\beta)|b_1| - \gamma) \overline{z}^2$$

and

$$f(z) = (1 - |b_1|)z - \frac{|(2+b)^\mu|}{(2+\beta)|(1+b)^\mu|} (1 - (1+\beta)|b_1| - \gamma) z^2, \quad |b_1| < 1.$$

□

Corollary 6. *Let $f \in RF(\mu, b, \beta, \gamma)$. Then*

$$\left\{ w : |w| < \frac{(1 - |b_1|)(2+\beta)|(1+b)^\mu| - |(2+b)^\mu|(1 - (1+\beta)|b_1| - \gamma)}{(2+\beta)|(1+b)^\mu|} \right\} \subset f(U)$$

Next we determine, a representation theorem for functions $f \in RF(\mu, b, \beta, \gamma)$.

Theorem 7. *Let $f = h + \overline{g}$, where h and g are defined in (5). Then $f \in RF(\mu, b, \beta, \gamma)$ if and only if*

$$f(z) = \sum_{k=1}^{\infty} (l_k h_k(z) + m_k g_k(z)), \ z \in U$$

where

$$h_1(z) = z, \quad h_k(z) = z - \frac{(1-\gamma)}{(k+\beta) \left| \left(\frac{1+b}{k+b} \right)^\mu \right|} z^k, \quad k = 2, 3, \ldots$$

$$g_k(z) = z - \frac{(1-\gamma)}{(k+\beta)\left|\left(\frac{1+b}{k+b}\right)^{\mu}\right|}\overline{z}^k, \quad k = 1, 2, 3, \ldots$$

and

$$\sum_{k=1}^{\infty}(l_k + m_k) = 1, \quad l_k \geq 0, m_k \geq 0.$$

In particular, the extreme points of $RF(\mu, b, \beta, \gamma)$ are $\{h_k\}$ and $\{g_k\}$.

Proof. Let

$$\begin{aligned} f(z) &= \sum_{k=1}^{\infty}(l_k h_k(z) + m_k g_k(z)) \\ &= \sum_{k=1}^{\infty}(l_k + m_k)z - \sum_{k=2}^{\infty}\frac{(1-\gamma)}{(k+\beta)\left|\left(\frac{1+b}{k+b}\right)^{\mu}\right|}l_k z^k \\ &\quad - \sum_{k=1}^{\infty}\frac{(1-\gamma)}{(k+\beta)\left|\left(\frac{1+b}{k+b}\right)^{\mu}\right|}m_k \overline{z}^k \\ &= z - \sum_{k=2}^{\infty}a_k z^k - \sum_{k=1}^{\infty}b_k \overline{z}^k \end{aligned}$$

where $a_1 = 1$ and

$$a_k = \frac{(1-\gamma)}{(k+\beta)\left|\left(\frac{1+b}{k+b}\right)^{\mu}\right|}l_k \quad (k \geq 2)$$

$$b_k = \frac{(1-\gamma)}{(k+\beta)\left|\left(\frac{1+b}{k+b}\right)^{\mu}\right|}m_k \quad (k \geq 1)$$

Since

$$\begin{aligned} &\sum_{k=2}^{\infty}\frac{(k+\beta)\left|\left(\frac{1+b}{k+b}\right)^{\mu}\right|}{(1-\gamma)}|a_k| + \sum_{k=1}^{\infty}\frac{(k+\beta)\left|\left(\frac{1+b}{k+b}\right)^{\mu}\right|}{(1-\gamma)}|b_k| \\ &= \sum_{k=2}^{\infty}\frac{(k+\beta)\left|\left(\frac{1+b}{k+b}\right)^{\mu}\right|}{(1-\gamma)}\left(\frac{(1-\gamma)}{(k+\beta)\left|\left(\frac{1+b}{k+b}\right)^{\mu}\right|}\right)l_k \\ &\quad + \sum_{k=1}^{\infty}\frac{(k+\beta)\left|\left(\frac{1+b}{k+b}\right)^{\mu}\right|}{(1-\gamma)}\left(\frac{(1-\gamma)}{(k+\beta)\left|\left(\frac{1+b}{k+b}\right)^{\mu}\right|}\right)m_k \end{aligned}$$

$$= \sum_{k=1}^{\infty}(l_k + m_k) - l_1 = 1 - l_1 \leq 1$$

$$\Rightarrow \sum_{k=1}^{\infty}(k+\beta)\left|\left(\frac{1+b}{k+b}\right)^{\mu}\right|(|a_k| + |b_k|) \leq 1 - \gamma + 1 + \beta$$

$$= 2 + \beta - \gamma$$

and so $f \in RF(\mu, b, \beta, \gamma)$.

Conversely, suppose that $f \in RF(\mu, b, \beta, \gamma)$.
Let

$$l_k = \frac{(k+\beta)\left|\left(\frac{1+b}{k+b}\right)^{\mu}\right|}{(1-\gamma)}|a_k|, \quad (k = 2, 3, \dots)$$

$$m_k = \frac{(k+\beta)\left|\left(\frac{1+b}{k+b}\right)^{\mu}\right|}{(1-\gamma)}|b_k|, \quad (k = 1, 2, \dots)$$

and $l_1 = 1 - \sum_{k=2}^{\infty} l_k - \sum_{k=1}^{\infty} m_k$.

Then note that by Theorem 4, $0 \leq l_k \leq 1$ $(k = 2, 3, \dots)$, $0 \leq m_k \leq 1$ $(k = 1, 2, 3, \dots)$ and $l_1 \geq 0$. Consequently $f(z) = \sum_{k=1}^{\infty}(l_k h_k(z) + m_k g_k(z))$, as required. □

The invariance property of the class $RF(\mu, b, \beta, \gamma)$ under convolution and convex combination of its elements are now studied.

Definition 8. Let

$$q(z) = z - \sum_{k=2}^{\infty}|a_k|z^k - \sum_{k=1}^{\infty}|b_k|\overline{z}^k \quad \text{and}$$

$$Q(z) = z - \sum_{k=2}^{\infty}|A_k|z^k - \sum_{k=1}^{\infty}|B_k|\overline{z}^k, \quad z \in U.$$

Then the convolution of q and Q is given by

$$(q * Q)(z) = q(z) * Q(z).$$

This can be written as

$$(q * Q)(z) = z - \sum_{k=2}^{\infty}|a_k||A_k|z^k - \sum_{k=1}^{\infty}|b_k||B_k|\overline{z}^k \qquad (10)$$

Theorem 9. *Let $q \in RF(\mu, b, \beta, \gamma)$ and $Q \in RF(\mu, b, \beta, \delta)$ for $0 \leq \delta \leq \gamma < 1$. Then*

$$(q * Q) \in RF(\mu, b, \beta, \gamma) \subseteq RF(\mu, b, \beta, \delta).$$

Proof. The convolution of $q * Q$ is defined by (10). It is enough to show that the coefficients of $q * Q$ satisfy the condition given in Theorem 4.

For $Q \in RF(\mu, b, \beta, \delta)$, $|A_k| \leq 1$ and $|B_k| \leq 1$. Now for the convolution $q * Q$, it follows that

$$\begin{aligned} &1 + \beta + \sum_{k=2}^{\infty}(k+\beta)\left|\left(\frac{1+b}{k+b}\right)^{\mu}\right||a_k||A_k| + \sum_{k=1}^{\infty}(k+\beta)\left|\left(\frac{1+b}{k+b}\right)^{\mu}\right||b_k||B_k| \\ &\quad \leq 1 + \beta + \sum_{k=2}^{\infty}(k+\beta)\left|\left(\frac{1+b}{k+b}\right)^{\mu}\right||a_k| + \sum_{k=1}^{\infty}(k+\beta)\left|\left(\frac{1+b}{k+b}\right)^{\mu}\right||b_k| \\ &\quad \leq 2 + \beta - \gamma \\ &\quad \leq 2 + \beta - \delta \end{aligned}$$

yielding the desired result. □

Theorem 10. *The family $RF(\mu, b, \beta, \gamma)$ is closed under convex combination.*

Proof. Let $f_i \in RF(\mu, b, \beta, \gamma)$, where

$$f_i(z) = z - \sum_{k=2}^{\infty}|a_{k,i}|z^k - \sum_{k=1}^{\infty}|b_{k,i}|\overline{z}^k, \text{ for } i = 1, 2, \ldots, n.$$

Then for $\sum_{i=1}^{n} m_i = 1$, $0 \leq m_i \leq 1$, the convex combination of f_i may be written as

$$\sum_{i=1}^{n} m_i f_i(z) = z - \sum_{k=2}^{\infty}\left(\sum_{i=1}^{n} m_i|a_{k,i}|\right)z^k - \sum_{k=1}^{\infty}\left(\sum_{i=1}^{n} m_i|b_{k,i}|\right)\overline{z}^k.$$

Using (9), one can see that

$$\begin{aligned} &1+\beta+\sum_{k=2}^{\infty}(k+\beta)\left|\left(\frac{1+b}{k+b}\right)^{\mu}\right|\left(\sum_{i=1}^{n}m_i|a_{k,i}|\right)\\ &\quad+\sum_{k=1}^{\infty}(k+\beta)\left|\left(\frac{1+b}{k+b}\right)^{\mu}\right|\left(\sum_{i=1}^{n}m_i|b_{k,i}|\right)\\ &\quad=\sum_{i=1}^{n}m_i\left(\sum_{k=1}^{\infty}(k+\beta)\left|\left(\frac{1+b}{k+b}\right)^{\mu}\right||a_{k,i}|+\sum_{k=1}^{\infty}(k+\beta)\left|\left(\frac{1+b}{k+b}\right)^{\mu}\right||b_{k,i}|\right)\\ &\quad\leq 2+\beta-\gamma \end{aligned}$$

Therefore $\sum_{i=1}^{n} m_i f_i \in RF(\mu, b, \beta, \gamma)$.
The proof of the theorem is completed. □

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