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A SUBCLASS OF MULTIVALENT FUNCTIONS WITH FINITELY MANY FIXED COEFFICIENTS

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Abstract

In this paper a new class of analytic, multivalent and normalized functions with finitely many fixed coefficients is defined. Properties like coefficient condition, extreme points and integral operators applied to functions in the class are investigated.

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1 Introduction and Preliminaries

Silverman[1] had introduced and studied univalent functions with negative coefficients.Goodman[6, 7],Srivastava et al.[11], Altintas et al.[12] and several authors had studied certain classes of convex functions and starlike functions. In recent times,Owa and Srivastava [5], Dixit and Indu Bala Misra [8],Murugusundaramoorthy and Magesh [9],Sunil Varma and Thomas Rosy [10] have studied the above classes by fixing finetely many coefficients. Researchers [2, 3, 4] had studied multivalent functions with negative coefficients. In this article,we define a subclass $TSD(n, p, \alpha)$ of $T(n, p)$ of multivalent functions by fixing finitely many coefficients and investigate the properties of functions in the subclass.

Let $T(n, p)$ denote the class of functions f of the form

$$f(z) = z^p - \sum_{k=n}^{\infty} a_{p+k} z^{p+k} \ (a_{p+k} \geq 0; p \in N = \{1, 2, 3, \dots\}; n \in N) \tag{1}$$

which are analytic in the open unit disk $U = \{z \in C : |z| < 1\}$.

We now define the following subclass.

Definition 1. A function $f \in T(n, p)$ is in the class $TSD(n, p, \alpha)$ if it satisfies the analytic criteria

$$Re\left\{\frac{f(z)}{z}\right\} \geq \alpha \left|f'(z) - \frac{f(z)}{z}\right|, \ 0 \leq \alpha < p \tag{2}$$

We now prove a necessary and sufficient condition for the functions in $T(n, p)$ to be in $TSD(n, p, \alpha)$.

Theorem 2. *A function of the form (1) is in the class $TSD(n, p, \alpha)$ if and only if*

$$\sum_{k=n}^{\infty} [1 + \alpha(p + k - 1)]|a_{p+k}| \leq 1 + \alpha(p - 1),$$
$$0 \leq \alpha < p, p, n \in N = \{1, 2, 3, \dots\} \tag{3}$$

Proof. Assume that f of the form (1) satisfies (3). Then

$$Re\left\{\frac{f(z)}{z}\right\} - \alpha \left|f'(z) - \frac{f(z)}{z}\right|$$
$$\geq 1 - \left|\frac{f(z)}{z} - 1\right| - \alpha \left|f'(z) - \frac{f(z)}{z}\right|$$

$$= 1 - \left| z^{p-1} - \sum_{k=n}^{\infty} a_{p+k} z^{p+k-1} - 1 \right|$$
$$- \alpha \left| p z^{p-1} - \sum_{k=n}^{\infty} (p+k) z^{p+k-1} a_{p+k} - z^{p-1} + \sum_{k=n}^{\infty} a_{p+k} z^{p+k-1} \right|$$
$$= 1 - \left| z^{p-1} - \sum_{k=n}^{\infty} a_{p+k} z^{p+k-1} - 1 \right|$$
$$- \alpha \left| (p-1) z^{p-1} - \sum_{k=n}^{\infty} (p+k-1) a_{p+k} z^{p+k-1} \right|$$
$$\geq 1 + |z^{p-1}| - \left| \sum_{k=n}^{\infty} a_{p+k} z^{p+k-1} \right| - 1 + \alpha(p-1)|z^{p-1}|$$
$$- \alpha \left| \sum_{k=n}^{\infty} (p+k-1) a_{p+k} z^{p+k-1} \right|$$
$$= [1 + \alpha(p-1)]|z^{p-1}| - \sum_{k=n}^{\infty} [1 + \alpha(p+k-1)] |a_{p+k}| |z^{p+k-1}|$$
$$= [1 + \alpha(p-1)] - \sum_{k=n}^{\infty} [1 + \alpha(p+k-1)] |a_{p+k}| \geq 0$$

Hence $f \in TSD(n, p, \alpha)$.

Conversely,

$$Re \left\{ \frac{f(z)}{z} \right\} - \alpha \left| f'(z) - \frac{f(z)}{z} \right| > 0$$
$$Re \left\{ z^{p-1} - \sum_{k=n}^{\infty} a_{p+k} z^{p+k-1} \right\} - \alpha \left| p z^{p-1} - \sum_{k=n}^{\infty} (p+k) a_{p+k} z^{p+k-1} \right.$$
$$\left. - z^{p-1} - \sum_{k=n}^{\infty} a_{p+k} z^{p+k-1} \right| > 0$$
$$Re \left\{ |z^{p-1}| - \sum_{k=n}^{\infty} |a_{p+k}| |z^{p+k-1}| \right\}$$
$$- \alpha \left| (p-1) z^{p-1} - \sum_{k=n}^{\infty} (p+k-1) a_{p+k} z^{p+k-1} \right| > 0$$

Letting z to take real values as $|z| \to 1$, we get

$$[1+\alpha(p-1)] - \sum_{k=n}^{\infty}[1+\alpha(p+k-1)]|a_{p+k}| \geq 0$$

which implies $\sum_{k=n}^{\infty}[1+\alpha(p+k-1)]|a_{p+k}| \leq [1+\alpha(p-1)]$. □

Remark 3. *For $p = 1$, $k = n-1$ result (3) coincides with the result obtained for the class $TSD(\alpha)$ by Sunil Varma and Thomas Rosy [10].*

Corollary 4. *For $f \in TSD(n,p,\alpha)$*

$$a_{p+k} \leq \frac{1+\alpha(p-1)}{1+\alpha(p+k-1)} \tag{4}$$

$$(0 \leq \alpha < p; k \geq n; p, n \in N = \{1, 2, 3, \dots\})$$

We now introduce the subclass $TSD(n,p,\alpha,q_{p+m})$ of $TSD(n,p,\alpha)$. This class consists of all those functions in $TSD(n,p,\alpha)$ which are of the form

$$f(z) = z^p - \sum_{i=n}^{m}\frac{[1+\alpha(p-1)]}{[1+\alpha(p+i-1)]}q_{p+i}z^{p+i} - \sum_{k=m+1}^{\infty} a_{p+k}z^{p+k} \tag{5}$$

Several interesting properties of the functions in this class are proved in the subsequent sections.

2 Coefficient Estimates

We now prove the coefficient estimate for the functions in the class $TSD(n,p,\alpha,q_{p+m})$.

Theorem 5. *A function of the form (5) is in the class $TSD(n,p,\alpha,q_{p+m})$ if and only if*

$$\sum_{k=m+1}^{\infty}[1+\alpha(p+k-1)]|a_{p+k}| \leq [1+\alpha(p-1)]\left[1-\sum_{i=n}^{m} q_{p+i}\right]$$

$$(p, n \in N = \{1, 2, 3, \dots\}) \tag{6}$$

where $0 \leq \alpha < p$, $0 \leq q_{p+i} \leq 1$ *and* $0 \leq \sum_{i=n}^{m} q_{p+i} \leq 1$.
The result is sharp.

Proof. By (5)

$$a_{p+i} = \frac{[1+\alpha(p-1)]q_{p+i}}{[1+\alpha(p+i-1)]}, \tag{7}$$

where $p, n \in N = \{1, 2, 3, \dots\}, i = n, n+1, \dots, m$,
$0 \leq q_{p+i} \leq 1, 0 \leq \sum_{i=n}^{m} q_{p+i} \leq 1$.
Which implies

$$[1+\alpha(p-1)]\sum_{i=n}^{m} q_{p+i} + \sum_{k=m+1}^{\infty}[1+\alpha(p+k-1)]|a_{p+k}| \leq [1+\alpha(p-1)]$$

$$\sum_{k=m+1}^{\infty}[1+\alpha(p+k-1)]|a_{p+k}| \leq [1+\alpha(p-1)]\left[1-\sum_{i=n}^{m} q_{p+i}\right]$$

Conversely,

$$\begin{aligned}
&Re\left\{\frac{f(z)}{z}\right\} - \alpha\left|f'(z) - \frac{f(z)}{z}\right| > 0 \\
&\geq [1+\alpha(p-1)] - \sum_{i=n}^{m}[1+\alpha(p+i-1)]|a_{p+i}| \\
&\quad - \sum_{k=m+1}^{\infty}[1+\alpha(p+k-1)]|a_{p+k}| \\
&= [1+\alpha(p-1)] - \sum_{i=n}^{m}[1+\alpha(p+i-1)]q_{p+i} \\
&\quad - \sum_{k=m+1}^{\infty}[1+\alpha(p+k-1)]|a_{p+k}|, \text{ by (7)} \\
&\geq 0, \text{ by (6)}
\end{aligned}$$

Thus $f \in TSD(n, p, \alpha, q_{p+m})$.

The sharpness of the result follows by taking

$$f(z) = z^p - \sum_{i=n}^{m} \frac{[1+\alpha(p-1)]q_{p+i}}{[1+\alpha(p+i-1)]} z^{p+i} - \frac{[1+\alpha(p-1)] - [1-\sum_{i=n}^{m} q_{p+i}]}{[1+\alpha(p+k-1)]} z^{p+k}, \quad (8)$$

where $k \geq n-1, p, n \in N = \{1, 2, 3, \dots\}$. □

The following corollary is a consequence of Theorem 5.

Corollary 6. *If f is in the class $TSD(n, p, \alpha, q_{p+m})$, then*

$$a_{p+k} \leq \frac{[1+\alpha(p-1)]\left[1 - \sum_{i=n}^{m} q_{p+i}\right]}{1+\alpha(p+k-1)}, \; k \geq m+1 \quad (9)$$

The result is sharp for the functions f given by (8).

3 Closure Theorems

Theorem 7. *The class $TSD(n, p, \alpha, q_{p+m})$ is convex.*

Proof. Let f, g be two functions in $TSD(n, p, \alpha, q_{p+m})$. Then

$$f(z) = z^p - \sum_{i=n}^{m} \frac{[1+\alpha(p-1)]q_{p+i}}{[1+\alpha(p+i-1)]} z^{p+i} - \sum_{k=m+1}^{\infty} a_{p+k} z^{p+k},$$

$$g(z) = z^p - \sum_{i=n}^{m} \frac{[1+\alpha(p-1)]q_{p+i}}{[1+\alpha(p+i-1)]} z^{p+i} - \sum_{k=m+1}^{\infty} b_{p+k} z^{p+k},$$

where $0 \leq q_{p+i} \leq 1$, $0 \leq \sum_{i=n}^{m} q_{p+i} \leq 1$.

Define

$$h(z) = \mu f(z) + (1-\mu) g(z).$$

Then

$$h(z) = z^p - \sum_{i=n}^{m} \frac{[1+\alpha(p-1)]q_{p+i}}{[1+\alpha(p+i-1)]} z^{p+i} - \sum_{k=m+1}^{\infty} [\mu a_{p+k} + (1-\mu) b_{p+k}] z^{p+k}.$$

Now,

$$\sum_{k=m+1}^{\infty}[1+\alpha(p+k-1)][\mu a_{p+k}+(1-\mu)b_{p+k}]$$
$$=\mu\sum_{k=m+1}^{\infty}[1+\alpha(p+k-1)]a_{p+k}$$
$$+(1-\mu)\sum_{k=m+1}^{\infty}[1+\alpha(p+k-1)]b_{p+k}$$
$$\leq \mu[1+\alpha(p+k-1)]\left[1-\sum_{i=n}^{m}q_{p+i}\right]$$
$$+(1-\mu)[1+\alpha(p+k-1)]\left[1-\sum_{i=n}^{m}q_{p+i}\right]$$
$$=[1+\alpha(p+k-1)]\left[1-\sum_{i=n}^{m}q_{p+i}\right]$$

which implies $h(z)\in TSD(n,p,\alpha,q_{p+m})$. □

Theorem 8. *Let*

$$f_m(z)=z^p-\sum_{i=n}^{m}\frac{[1+\alpha(p-1)]}{[1+\alpha(p+i-1)]}q_{p+i}z^{p+i} \tag{10}$$

and

$$f_{p+k}(z)=z^p-\sum_{i=n}^{m}\frac{[1+\alpha(p-1)]}{[1+\alpha(p+i-1)]}q_{p+i}z^{p+i}$$
$$-\frac{[1+\alpha(p-1)]\left[1-\sum_{i=n}^{m}q_{p+i}\right]}{1+\alpha(p+k-1)}z^{p+k},\ k\geq m+1 \tag{11}$$

Then $f\in TSD(n,p,\alpha,q_{p+m})$ if and only if f can be expressed in the form

$$f(z)=\sum_{k=m}^{\infty}\mu_{p+k}f_{p+k}(z) \tag{12}$$

where $\mu_{p+k}\geq 0$, $(k\geq m)$ and $\sum_{k=m}^{\infty}\mu_{p+k}=1$.

Proof. Suppose $f \in TSD(n,p,\alpha)$ can be expressed in the form (12). Then

$$f(z) = z^p - \sum_{i=n}^{m} \frac{[1+\alpha(p-1)]}{[1+\alpha(p+i-1)]} q_{p+i} z^{p+i} - \sum_{k=m+1}^{\infty} \frac{\mu_{p+k}[1+\alpha(p-1)]\,[1-\sum_{i=n}^{m} q_{p+i}]}{[1+\alpha(p+k-1)]} z^{p+k} \tag{13}$$

Now,

$$\begin{aligned}
&\sum_{k=m+1}^{\infty} \frac{[1+\alpha(p+k-1)]\mu_{p+k}[1+\alpha(p-1)]\,[1-\sum_{i=n}^{m} q_{p+i}]}{[1+\alpha(p+k-1)]} \\
&= \sum_{k=m+1}^{\infty} \mu_{p+k}[1+\alpha(p-1)] \left[1-\sum_{i=n}^{m} q_{p+i}\right] \\
&= [1+\alpha(p-1)] \left[1-\sum_{i=n}^{m} q_{p+i}\right] \sum_{k=m+1}^{\infty} \mu_{p+k} \\
&= [1+\alpha(p-1)] \left[1-\sum_{i=n}^{m} q_{p+i}\right] (1-\mu_{p+m}) \\
&\leq [1+\alpha(p-1)] \left[1-\sum_{i=n}^{m} q_{p+i}\right]
\end{aligned}$$

which implies $f \in TSD(n,p,\alpha,q_{p+m})$.

Conversely, for $k \geq m+1$, set

$$\mu_{p+k} = \frac{[1+\alpha(p+k-1)]a_{p+k}}{[1+\alpha(p-1)]\,[1-\sum_{i=n}^{m} q_{p+i}]}, \ k \geq m+1 \tag{14}$$

and

$$\mu_m = 1 - \sum_{k=m+1}^{\infty} \mu_{p+k} \tag{15}$$

Then f can be represented as

$$f(z) = \sum_{k=m}^{\infty} \mu_{p+k} f_{p+k}(z).$$

□

Corollary 9. *The extreme points of the class* $TSD(n,p,\alpha,q_{p+m})$ *are the functions* f_{p+k}, $(k \geq m)$ *given by (10) and (11).*

4 Integral Operator

The Alexander operator for the functions in the class $T(n,p)$ is defined as

$$I(f) = \int_0^z \frac{f(s)}{s} ds \tag{16}$$

This operator maps the class of starlike functions onto the class of convex functions. The effect of this operator on the functions in the class $TSD(n,p,\alpha,q_{p+m})$ is given in the following theorem.

Theorem 10. *Let f defined by (4) be in the class $TSD(n,p,\alpha,q_{p+m})$. Then $I(f)$ belongs to the class $TSD(n,p,\alpha,t_{p+m})$ where $t_{p+m} = \frac{q_{p+m}}{(p+m)}$*

Proof. We have

$$I(f) = z^p - \sum_{i=n}^{m} \frac{[1+\alpha(p-1)]t_{p+i}}{[1+\alpha(p+i-1)]} z^{p+i} - \sum_{k=m+1}^{\infty} \frac{a_{p+k}}{(p+k)} z^{p+k} \tag{17}$$

Now

$$\begin{aligned} &\sum_{k=m+1}^{\infty} [1+\alpha(p+k-1)] \frac{a_{p+k}}{(p+k)} \\ &\quad \le \frac{1}{(p+m+1)} \sum_{k=m+1}^{\infty} [1+\alpha(p+k-1)] a_{p+k} \\ &\quad \le \frac{1}{(p+m+1)} [1+\alpha(p-1)] \left[1 - \sum_{i=n}^{m} q_{p+i}\right] \end{aligned}$$

which implies $I(f) \in TSD(n,p,\alpha,t_{p+m})$. □

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