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Solution of Wave Equation by the Method of Separation of Variables Using the Foss Tools Maxima

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Abstract

In this paper we would like to solve one dimensional Wave Equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ under the given boundary conditions $u(0,t) = 0$, $u(l,t) = 0$, $\forall x > 0$ and $u(x,0) = f(x)$, $\frac{\partial u}{\partial t}(x,0) = g(x)$, $\forall t$ by the method of separation of variables using FOSS tools Maxima.

Key words: kill, ode2, fourie, lratsubst.

1 Introduction

Maxima: Maxima is a large computer program designed for the manipulation of algebraic expressions. You can use Maxima for manipulation of algebraic expressions involving constants, variables, and functions. It can differentiate, integrate, take limits, solve equations, factor polynomials, expand functions in power series, solve differential equations in closed form, and perform many other operations. It also has a programming language that you can use to extend Maximas capabilities.[4]

2 Wave Equation

The wave equation is an important second-order linear hyperbolic partial differential equation for the description of waves as they occur in classical physics such as sound waves, light waves and water waves. It arises in fields like acoustics, electromagnetic, and fluid dynamics.[7]

The wave equation in one space dimension can be written as follows: $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$
This equation is typically described as having only one space dimension "x", because the only other independent variable is the time "t". Nevertheless, the dependent variable "u" may represent a second space dimension, if, for example, the displacement "u" takes place in y-direction, as in the case of a string that is located in the $x - y$ plane.

The one dimensional wave equation is given by $\dfrac{\partial^2 u}{\partial t^2} = c^2 \dfrac{\partial^2 u}{\partial x^2}$ under the boundary conditions $u(0,t) = 0$, $u(l,t) = 0$, $\forall x > 0$ and $u(x,0) = f(x)$, $\dfrac{\partial u}{\partial t}(x,0) = g(x)$, $\forall t$.
Solution: The Wave equation is given by

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

The solution of equation is of the form

$$u(x,t) = X(x)\, T(t) \tag{2}$$

Substituting equation (2) in equation (1), we get

$$\frac{T''(t)}{c^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda^2 \tag{3}$$

The solution of equation (3) is the solution of the differential equation

$$T''(t) + \lambda^2 c^2 T(t) = 0 \text{ and } X''(x) + \lambda^2 X(x) = 0$$

If $\lambda > 0$, then the solution of the above equation is given by

$$T(t) = c_1 \cos(\lambda ct) + c_2 \sin(\lambda ct) \text{ and } X(x) = c_3 \cos(\lambda x) + c_4 \sin(\lambda x)$$

The solution of equation (1) is given by

$$u(x,t) = (c_1 \cos(\lambda ct) + c_2 \sin(\lambda ct))(c_3 \cos(\lambda x) + c_4 \sin(\lambda x)) \tag{4}$$

Applying the boundary condition $u(0,t) = 0$, $u(1,t) = 0$
From equation (4), we get

$$c_3 = 0,\ \lambda = \frac{n\pi}{l} \text{ where } n = 0, 1, 2, 3, ...$$

Then equation (4) reduces to

$$u(x,t) = \left(c_1 \cos\left(\frac{n\pi ct}{l}\right) + c_2 \sin\left(\frac{n\pi ct}{l}\right)\right) c_4 \sin\left(\frac{n\pi x}{l}\right)$$

$$u(x,t) = \left(A_n \cos\left(\frac{n\pi ct}{l}\right) + B_n \sin\left(\frac{n\pi ct}{l}\right)\right) \sin\left(\frac{n\pi x}{l}\right) \tag{5}$$

For each value of n equation (5) is the solution.
By Superposition principle the sum of all these solution is also a solution

$$\therefore u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{n\pi ct}{l}\right) + B_n \sin\left(\frac{n\pi ct}{l}\right)\right) \sin\left(\frac{n\pi x}{l}\right) \tag{6}$$

Applying the initial conditions to equation (6), then

$$A_n = \frac{2}{l}\int_0^1 f(x)\sin\left(\frac{n\pi x}{l}\right)dx \tag{7}$$

and

$$B_n = \frac{2}{n\pi c}\int_0^1 g(x)\sin\left(\frac{n\pi x}{l}\right)dx \tag{8}$$

Substituting equation (7) & equation (8) in equation (6) we get the required solution.

3 Algorithm

Step 1: Start

Step 2: Input $u(x,t) \leftarrow X(x) * T(t), x, t > 0, c > 0, \lambda > 0, n > 0, l > 0$

Step 3: Substitute Step 2 in $\frac{X''(x)}{X(x)} \leftarrow \frac{T''(t)}{c * T(t)}$

Step 4: Input LHS of Step 3 to $-\lambda^2$.

Step 5: Solve Step 4.

Step 6: Input RHS of Step 3 to $-\lambda^2$.

Step 7: Solve Step 6.

Step 8: Replace $\%k_1$ to $\%k_3$ and $\%k_2$ to $\%k_4$ in Step 7.

Step 9: Input $X(0) \leftarrow 0$ in Step 5. [Applying Boundary Condition]

Step 10: Input $X(l) \leftarrow 0$ in Step 5. [Applying Boundary Condition]

Step 11: Substitute Step 10 in Step 8 and Step 5.

Step 12: Substitute Step 11 in Step 2.

Step 13: Simplify Step 12.

Step 14: Input $A[n] \leftarrow \%k_1 * \%k_3$ and $B[n] \leftarrow \%k_1 * \%k_4$

Step 15: Solve $A[n]$ $\left[as\ \frac{2}{(l)} * integrate\left(f(x) * \sin\left(\frac{n\pi x}{l}\right), x, 0, l\right)\right]$

Step 16: Solve $B[n]$ $\left[as \quad \frac{2}{(n\pi c)} * integrate\Big(g(x) * \sin\left(\frac{n\pi x}{l}\right), x, 0, l\Big)\right]$

Step 17: Put Step 15 and Step 16 in Step 13.

Step 18: Solve for summation $n \leftarrow 1$ to ∞ of Step 17.

Step 19: Output Step 18.

4 Problem

A tightly stretched string with fixed end points $x = 0$ and $x = 1$ is initially at rest in its equilibrium position with density is 1mass unit/volume and tension is 1 unit. if it is set vibrating giving each point a velocity $x(1-x)$, find its displacement function.

The vibration of the string are governed by one dimensional wave equation $\frac{\partial^2 \mathbf{u}}{\partial \mathbf{t}^2} = \frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2}$ under the boundary conditions

i) $\mathbf{u}(\mathbf{0}, \mathbf{t}) = \mathbf{0}$, $\mathbf{u}(\mathbf{l}, \mathbf{t}) = \mathbf{0}$, $\forall \mathbf{t} \geq \mathbf{0}$

ii) $\mathbf{u}(\mathbf{x}, \mathbf{0}) = \mathbf{0}$, $\frac{\partial \mathbf{u}}{\partial \mathbf{t}}(\mathbf{x}, \mathbf{0}) = \mathbf{lx} - \mathbf{x^2}$ for $\mathbf{0} < \mathbf{x} < \mathbf{1}$

5 Maxima Program:

```
kill(all) $
load("fourie") $
load("lrats") $
g(x,t) := diff(u(x,t),t,2) = c * diff(u(x,t),x,2);
assume (n > 0, c > 0, x > 0, t > 0, λ > 0, l > 0) $
c : 1 $
l : l $
u(x,t) := X(x) * T(t) $
F(x,t) := g(x,t)/(c * u(x,t)) $
x1 : ode2(rhs(F(x,t)) = -lambda^2, X(x), x) $
define (X(x), rhs(x1)) $
t2 : ode2(lhs(F(x,t)) = -lambda^2, T(t), t) $
t1 : subst([%k1 = %k3, %k2 = %k4], %) $
t2 : define(T(t), rhs(t1)) $
disp("applying the condition X(0)=0") $
if at ( X(x), x = 0) =%k2 then %k2 : 0 else %k1 : 0 $
disp("X(x)=",X(x)) $
```

```
if at(X(x),x=l)#0 then lambda:n*%pi/l else lambda:0 $
disp("X(x)=",X(x)) $
u1:ratsimp(u(x,t)) $
f(x):=0 $
2/l*integrate(f(x)*sin(n*%pi*x/l),x,0,l) $
B[n]:foursimp(%);
```
$g(x) := l * x - x^2$;
```
2/(n*%pi)*integrate(g(x)*sin(n*%pi*x/l),x,0,l) $
A[n]:foursimp(%);
u2:lratsubst([%k1*%k3=A[n], %k1*%k4=B[n]],u1) $
u3:sum(u2,n,1,inf) $
disp("solution is u(x,t)=",u3) $
```

Output:

(%o3) g(x,t):=diff(u(x,t),t,2)=c diff(u(x,t),x,2)

applying the condition X(0)=0

X(x)=

$\%k1 \sin(x\, lambda)$

X(x)=

$\%k1 \sin\left(\frac{\pi n x}{l}\right)$

(%o22) 0

(%o23) $g(x):=l\, x-x^2$

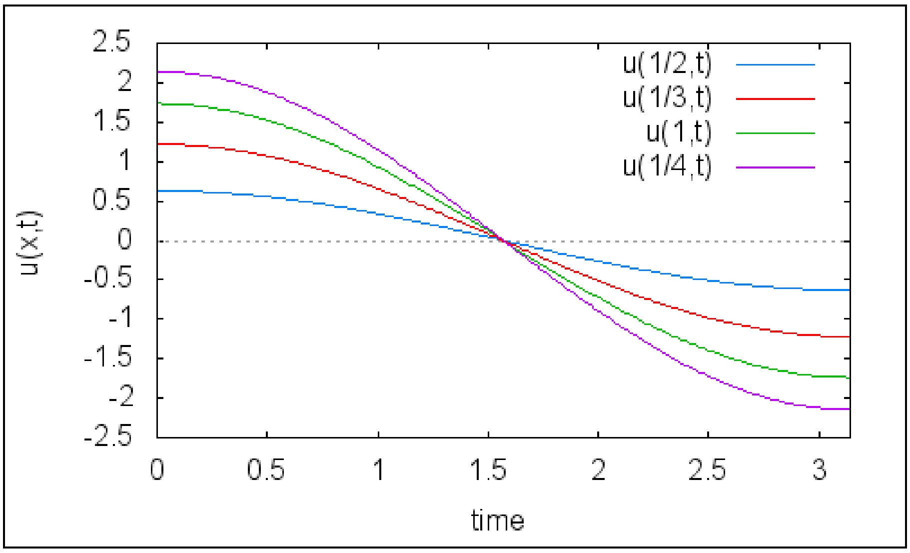
(%o25) $-\frac{4\, l^3\left((-1)^n-1\right)}{\pi^4 n^4}$

solution is u(x,t)=

$$-\frac{\sum_{n=1}^{\infty} \frac{(4\, l^3 (-1)^n - 4\, l^3) \sin\left(\frac{\pi n t}{l}\right) \sin\left(\frac{\pi n x}{l}\right)}{n^4}}{\pi^4}$$

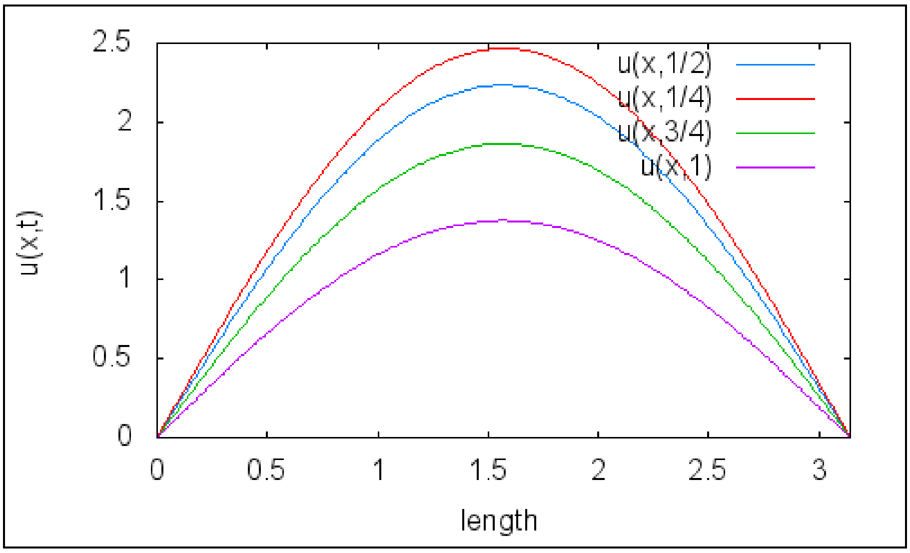
While the solution $u(x,t)$ is very complicated, in fact each term is simple. For each fixed t, $8\sin(\frac{x}{\pi})\cos(t)$ is just a constant multiple of $\sin(\frac{x}{\pi})$, as x runs from 0 to $\frac{l}{\pi}$.
Here are the graphs, at fixed t.

(%t33)



For each fixed x, the $8\sin(\frac{x}{\pi})\cos(t)$ is just a constant times $\cos(t)$. As t increases the argument of $\cos(t)$ increases by one half cycle, since $c = \sqrt{\frac{T}{\rho}}$, to increase the frequency of oscillation of a string we need to increase the tension or decrease the density or shorten the length of the string.

(%t38)



6 Conclusion

The solution obtained manually are exactly same as the solutions obtaione by Maxima program.

7 Authors Contributions

First two authors worked together for the preparation of the manuscript and both take full responsibility for the content of the paper. However second author typed the paper and as read and approved the final manuscript. The third author gave the Algorithm to the Maxima program.

8 Conflict of Interests

The authors hereby declares that there are no issues regarding the publication of this paper.

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