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A Certain Classes of Univalent Functions of Order Gamma having Positive Coefficients Attributed to An Extension of Al-Oboudi Operator

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Abstract

In this paper, a new class $\kappa - ST^{\lambda}(\gamma, \delta, \beta)$consisting of κ −uniformly starlike, analytic, univalent functions of order γ and type δ with positive coefficientsand another new class $\kappa - UCV^{\lambda}(\gamma, \delta, \beta)$ consisting ofκ − uniformly convex, analytic, univalent functions of order γ and type $\delta$with positive coefficients attributed to an extension of Al-Oboudi operator are introduced. We acquire coefficient estimates, distortion theorems, closure theorems and convolution results for the functions belonging to the mentioned classes.

Keywords: Al-Oboudi operator, analytic, convex, convolution, starlike, univalent.

1. Introduction

Let $\mathcal{A}$ denotes the class of functions of the form

$$f(z) = z + \sum_{i=2}^{\infty} a_i z^i \quad (a_i \geq 0) \tag{1}$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} ; |z| < 1\}$.
In this paper, we make use ofan operator D_{λ}^{β} which is an extension of Al-Oboudi operator$D_{\lambda}^{n}(\lambda \geq 0; n \in \mathbb{N})$.
This linearoperator is defined by $D_{\lambda}^{\beta}: \mathcal{A} \to \mathcal{A}$,

$$D_{\lambda}^{\beta} f(z) = z + \sum_{i=2}^{\infty} [1 + (i-1)\lambda]^{\beta} a_i z^i$$

$(\lambda, \beta \in \mathbb{R} ; a_i \geq 0; \beta \geq 0; \lambda \geq 0; z \in U)$.
Many authors (see for example[1],[2],[3]) have studied various classes of the Al-Oboudi operator and its extension.
Let us now define two new subclasses $\kappa - ST^{\lambda}(\gamma, \delta, \beta)$and $\kappa - UCV^{\lambda}(\gamma, \delta, \beta)of\mathcal{A}$.

Definition1.1.
[5] Let $\kappa - ST^{\lambda}(\gamma,\delta,\beta)$denotes the class of $\kappa -$ uniformly starlike, analytic,univalent functions of order γ and type δwhich satisfies the inequality

$$R_e \left\{\frac{D_{\lambda}^{\beta+1} f(z)}{D_{\lambda}^{\beta} f(z)} - \gamma\right\} > \kappa \left|\frac{D_{\lambda}^{\beta+1} f(z)}{D_{\lambda}^{\beta} f(z)} - \delta\right| \quad (2)$$

$(\lambda, \beta \geq 0; 0 \leq \gamma < \delta \leq 1;\ \kappa(1-\delta) < (1-\gamma);\ z \in U).$
.

Definition 1.2.
Let $\kappa - UCV^{\lambda}(\gamma,\delta,\beta)$denotes the class of$\kappa -$ uniformly convex, analytic, univalent functions of orderγ and type δif

$$R_e \left\{1 + \frac{D_{\lambda}^{\beta+2} f(z)}{D_{\lambda}^{\beta+1} f(z)} - \gamma\right\} > \kappa \left|1 + \frac{D_{\lambda}^{\beta+2} f(z)}{D_{\lambda}^{\beta+1} f(z)} - \delta\right| \quad (3)$$

$(\lambda, \beta \geq 0; 0 \leq \gamma < \delta \leq 1;\ \kappa(1-\delta) < (1-\gamma);\ z \in U).$

It follows from (2) and (3) that

$$f \in \kappa - UCV^{\lambda}(\gamma,\delta,\beta) \iff z f' \in \kappa - ST^{\lambda}(\gamma,\delta,\beta)\ .$$

By specializingγ, δ, κandλ, we acquire the subsequent subclasses examined by other researchers.

(i) $1 - ST^{\lambda}(\alpha,1,\beta) = TS_p^{\lambda}(\alpha,\beta)$

and

$1 - UCV^{\lambda}(\alpha,1,\beta) = TV^{\lambda}(\alpha,\beta)$
(see [3]);

(ii) $\kappa - ST^{1}(\alpha,0,1) = \kappa - ST(\alpha,1) = SD(\kappa,\alpha)$

and

$\kappa - UCV^{1}(\alpha,0,1) = \kappa - UCV(\alpha,1) = KD(\kappa,\alpha)$
(see [4], [5], [6]);

(iii) $1-ST^1(\alpha,1,0) = TS_p^1(\alpha,0) = 1- ST(\alpha,1) = S_p(\alpha) = T^*\left(\frac{1+\alpha}{2}\right)$

and

$1-UCV^1(\alpha,1,0) = TV^1(\alpha,0) = 1- UCV(\alpha,1) = UCV(\alpha) = C^*\left(\frac{1+\alpha}{2}\right)$
(see [3],[5],[7]);

(iv) $\kappa - ST^1(0,1,0) = \kappa - ST(0,1) = \kappa - ST$

and

$\kappa - UCV^1(0,1,0) - \kappa - UCV(0,1) = \kappa - UCV$
(see [5],[8],[9]);

(v) $1-ST^1(0,1,0) = TS_p^1(0,0) = 1-ST(0,1) = 1-ST = S_p$

and

$1-UCV^1(0,1,0) = TV^1(0,0) = 1-UCV(0,1) = 1-UCV = UCV$

(see [3],[5], [7], [8],[9],[10],[11],[12],[13]).
.

The objective of this article is to study various amusing properties of thefunctions in the classes
$\kappa - ST^\lambda(\gamma,\delta,\beta)$ and$\kappa - UCV^\lambda(\gamma,\delta,\beta)$.

2. Coefficient Estimates

Theorem2.1.
A function $f(z)$of the form (1) belongs to the class$\kappa - ST^\lambda(\gamma,\delta,\beta)$if and only if

$$\sum_{i=2}^{\infty}[1+(i-1)\lambda]^\beta\,[\kappa(1-\delta+(i-1)\lambda)+(1-\gamma+(i-1)\lambda)]a_i < (1-\gamma)-\kappa(1-\delta) \qquad (4)$$

$(0 \le \gamma < \delta \le 1; \lambda,\beta \ge 0;\ a_i \ge 0; \kappa(1-\delta) < (1-\gamma))$.

Proof . Let $f \in \kappa - ST^\lambda(\gamma,\delta,\beta)$

$(0 \le \gamma < \delta \le 1; \lambda,\beta \ge 0; \kappa(1-\delta) < (1-\gamma))$.

Then

$$R_e\left\{\frac{D_\lambda^{\beta+1}f(z)}{D_\lambda^{\beta}f(z)}-\gamma\right\}-\kappa\left|\frac{D_\lambda^{\beta+1}f(z)}{D_\lambda^{\beta}f(z)}-\delta\right|>0$$

$$R_e\left\{\frac{1+\sum_{i=2}^{\infty}[1+(i-1)\lambda]^{\beta+1}a_iz^{i-1}}{1+\sum_{i=2}^{\infty}[1+(i-1)\lambda]^{\beta}a_iz^{i-1}}-\gamma\right\}-\kappa\left|\frac{1+\sum_{i=2}^{\infty}[1+(i-1)\lambda]^{\beta+1}a_iz^{i-1}}{1+\sum_{i=2}^{\infty}[1+(i-1)\lambda]^{\beta}a_iz^{i-1}}-\delta\right| > 0$$

$$R_e\left\{(1-\gamma)+\sum_{i=2}^{\infty}[1+(i-1)\lambda]^{\beta}[1+(i-1)\lambda-\gamma]\,a_iz^{i-1}\right\} -\kappa\left|(1-\delta)+\sum_{i=2}^{\infty}[1+(i-1)\lambda]^{\beta}[1+(i-1)\lambda-\delta]\,a_iz^{i-1}\right|>0$$

Letting z to take real values and $|z|\to 1$, we acquire

$$(1-\gamma)-\kappa\,(1-\delta)-\sum_{i=2}^{\infty}[1+(i-1)\lambda]^{\beta}\,[\kappa(1-\delta+(i-1)\lambda)+(1-\gamma+(i-1)\lambda)]a_i > 0$$

This implies

$$\sum_{i=2}^{\infty}[1+(i-1)\lambda]^{\beta}\,[\kappa(1-\delta+(i-1)\lambda)+(1-\gamma+(i-1)\lambda)]a_i<(1-\gamma)-\kappa(1-\delta)$$

$(0\le\gamma<\delta\le 1;\lambda,\beta\ge 0;\ a_i\ge 0;\kappa(1-\delta)<(1-\gamma))$.

Conversely,

$$R_e\left\{\frac{D_\lambda^{\beta+1}f(z)}{D_\lambda^{\beta}f(z)}-\gamma\right\}-\kappa\left|\frac{D_\lambda^{\beta+1}f(z)}{D_\lambda^{\beta}f(z)}-\delta\right|$$

$$=R_e\left\{\frac{1+\sum_{i=2}^{\infty}[1+(i-1)\lambda]^{\beta+1}a_iz^{i-1}}{1+\sum_{i=2}^{\infty}[1+(i-1)\lambda]^{\beta}a_iz^{i-1}}-\gamma\right\} -\kappa\left|\frac{1+\sum_{i=2}^{\infty}[1+(i-1)\lambda]^{\beta+1}a_iz^{i-1}}{1+\sum_{i=2}^{\infty}[1+(i-1)\lambda]^{\beta}a_iz^{i-1}}-\delta\right|$$

$$=R_e\left\{\frac{(1-\gamma)+\sum_{i=2}^{\infty}[1+(i-1)\lambda]^{\beta}[1+(i-1)\lambda-\gamma]\,a_iz^{i-1}}{1+\sum_{i=2}^{\infty}[1+(i-1)\lambda]^{\beta}a_iz^{i-1}}\right\} -\kappa\left|\frac{(1-\delta)+\sum_{i=2}^{\infty}[1+(i-1)\lambda]^{\beta}[1+(i-1)\lambda-\delta]\,a_iz^{i-1}}{1+\sum_{i=2}^{\infty}[1+(i-1)\lambda]^{\beta}a_iz^{i-1}}\right|$$

$$\geq \frac{(1-\gamma)-\kappa\,(1-\delta)-\sum_{i=2}^{\infty}[1+(i-1)\lambda]^{\beta}\,[\kappa(1-\delta+(i-1)\lambda)+(1-\gamma+(i-1)\lambda)]a_i}{\left|1+\sum_{i=2}^{\infty}[1+(i-1)\lambda]^{\beta}\,a_i z^{i-1}\right|}$$

> 0 by (4).

Thus $f \in \kappa - ST^{\lambda}(\gamma,\delta,\beta)$.

Corollary2.2.
If $f \in \kappa - ST^{\lambda}(\gamma,\delta,\beta)$, then

$$a_i \leq \frac{(1-\gamma)-\kappa(1-\delta)}{[1+(i-1)\lambda]^{\beta}\,[\kappa(1-\delta+(i-1)\lambda)+(1-\gamma+(i-1)\lambda)]} \quad (i \geq 2\,; i \in \mathbb{N}) \qquad (5)$$

The equality holds for the function

$$f(z) = z + \frac{[(1-\gamma)-\kappa(1-\delta)]}{[1+(i-1)\lambda]^{\beta}\,[\kappa(1-\delta+(i-1)\lambda)+(1-\gamma+(i-1)\lambda)]} z^i \quad (i \geq 2\,; i \in \mathbb{N}) \qquad (6)$$

By similar method adapted in Theorem2.1, we get the Theorem 2.3.

Theorem2.3.
A function$f(z)$of the form (1) belongs to the class$\kappa - UCV^{\lambda}(\gamma,\delta,\beta)$ if and only if

$$\sum_{i=2}^{\infty}[1+(i-1)\lambda]^{\beta+1}\,[\kappa(1-\delta+(i-1)\lambda)+(1-\gamma+(i-1)\lambda)]a_i < (1-\gamma)-\kappa(1-\delta) \qquad (7)$$

$(0 \leq \gamma < \delta \leq 1; \lambda,\beta \geq 0; a_i \geq 0; \kappa(1-\delta) < (1-\gamma); z \in U)$.

Corollary2.4.
If$f \in \kappa - UCV^{\lambda}(\gamma,\delta,\beta)$ then

$$a_i \leq \frac{(1-\gamma)-\kappa(1-\delta)}{[1+(i-1)\lambda]^{\beta+1}[\kappa(1-\delta+(i-1)\lambda)+(1-\gamma+(i-1)\lambda)]} \quad (i \geq 2\,; i \in \mathbb{N}) \qquad (8)$$

The equality holds for the function

$$f(z) = z + \frac{[(1-\gamma)-\kappa(1-\delta)]}{[1+(i-1)\lambda]^{\beta+1}[\kappa(1-\delta+(i-1)\lambda)+(1-\gamma+(i-1)\lambda)]} z^i \quad (i \geq 2\,; i \in \mathbb{N}) \qquad (9)$$

3. Distortion Theorems

Theorem3.1.

If the functions$f \in \kappa - ST^{\lambda}(\gamma, \delta, \beta)$, then

$$|z| - \frac{[(1-\gamma) - \kappa(1-\delta)]}{(1+\lambda)^{\beta}[\kappa(1-\delta+\lambda) + (1-\gamma+\lambda)]}|z|^2 \le |f(z)|$$
$$\le |z| + \frac{[(1-\gamma) - \kappa(1-\delta)]}{(1+\lambda)^{\beta}[\kappa(1-\delta+\lambda) + (1-\gamma+\lambda)]}|z|^2.$$

The result is sharp.

Proof Since$\psi(i) = [1+(i-1)\lambda]^{\beta}[\kappa(1-\delta+(i-1)\lambda) + (1-\gamma+(i-1)\lambda)]$ is an increasing function of i
$(i \ge 2; i \in \mathbb{N})$, considering Theorem 2.1,

$$\psi(2)\sum_{i=2}^{\infty} a_i \le \sum_{i=2}^{\infty} \psi(i)\, a_i \le (1-\gamma) - \kappa(1-\delta).$$

Therefore,

$$\sum_{i=2}^{\infty} a_i \le \frac{(1-\gamma) - \kappa(1-\delta)}{\psi(2)}.$$

Hence

$$|f(z)| \le |z| + \sum_{i=2}^{\infty} a_i\, |z|^i$$

$$|f(z)| \le |z| + \frac{[(1-\gamma) - \kappa(1-\delta)]}{(1+\lambda)^{\beta}[\kappa(1-\delta+\lambda) + (1-\gamma+\lambda)]}|z|^2.$$

Similarly we get,

$$|f(z)| \ge |z| - \sum_{i=2}^{\infty} a_i\, |z|^2$$
$$|f(z)| \ge |z| - \frac{[(1-\gamma) - \kappa(1-\delta)]}{(1+\lambda)^{\beta}[\kappa(1-\delta+\lambda) + (1-\gamma+\lambda)]}|z|^2.$$

This completes the proof of Theorem3.1.

Finally the result is sharp for the following function

$$f(z) = z + \frac{[(1-\gamma)-\kappa(1-\delta)]}{(1+\lambda)^{\beta}[\kappa(1-\delta+\lambda)+(1-\gamma+\lambda)]}z^2 \qquad (10)$$

$(0 \leq \gamma < \delta \leq 1; \lambda, \beta \geq 0; \kappa(1-\delta) < (1-\gamma))$.

Theorem3.2.
If the function $f \in \kappa - ST^{\lambda}(\gamma,\delta,\beta)$, then

$$1 - \frac{2[(1-\gamma)-\kappa(1-\delta)]}{(1+\lambda)^{\beta}[\kappa(1-\delta+\lambda)+(1-\gamma+\lambda)]}|z| \leq |f'(z)| \leq 1 + \frac{2[(1-\gamma)-\kappa(1-\delta)]}{(1+\lambda)^{\beta}[\kappa(1-\delta+\lambda)+(1-\gamma+\lambda)]}|z|$$

The result is sharp.

Proof. Since $\frac{\psi(i)}{i}$ is an increasing function of i $(i \geq 2)$, considering Theorem 2.1, we acquire

$$\frac{\psi(2)}{2}\sum_{i=2}^{\infty} i a_i \leq \sum_{i=2}^{\infty}\psi(i)\, a_i \leq (1-\gamma)-\kappa(1-\delta).$$

Therefore,

$$\sum_{i=2}^{\infty} i\, a_i \leq \frac{2[(1-\gamma)-\kappa(1-\delta)]}{\psi(2)}.$$

Hence,

$$|f'(z)| \leq \left[1+\sum_{i=2}^{\infty} i\, a_i |z|\right] \leq \left[1+\frac{2[(1-\gamma)-\kappa(1-\delta)]}{(1+\lambda)^{\beta}[\kappa(1-\delta+\lambda)+(1-\gamma+\lambda)]}|z|\right]$$

Similarly we get,

$$|f'(z)| \geq \left[1-\sum_{i=2}^{\infty} i\, a_i |z|\right] \geq \left[1-\frac{2[(1-\gamma)-\kappa(1-\delta)]}{(1+\lambda)^{\beta}[\kappa(1-\delta+\lambda)+(1-\gamma+\lambda)]}|z|\right]$$

Finally the result is sharp for the function $f(z)$ given by (10).

By similar method adapted in the Theorem 3.1 and Theorem 3.2, we get Theorem 3.3 and Theorem 3.4

Theorem3.3.
If the function $f \in \kappa - UCV^{\lambda}(\gamma,\delta,\beta)$, then

$$|z| - \frac{[(1-\gamma)-\kappa(1-\delta)]}{(1+\lambda)^{\beta+1}[\kappa(1-\delta+\lambda)+(1-\gamma+\lambda)]}|z|^2 \le |f(z)| \le |z| + \frac{[(1-\gamma)-\kappa(1-\delta)]}{(1+\lambda)^{\beta+1}[\kappa(1-\delta+\lambda)+(1-\gamma+\lambda)]}|z|^2.$$

The result becomes sharp for the following function

$$f(z) = z + \frac{[(1-\gamma)-\kappa(1-\delta)]}{(1+\lambda)^{\beta+1}[\kappa(1-\delta+\lambda)+(1-\gamma+\lambda)]}z^2 \tag{11}$$

$(0 \le \gamma < \delta \le 1; \lambda,\beta \ge 0; \kappa(1-\delta) < (1-\gamma))$.

Theorem3.4.
If the function $f \in \kappa - UCV^{\lambda}(\gamma,\delta,\beta)$, then

$$1 - \frac{2[(1-\gamma)-\kappa(1-\delta)]}{(1+\lambda)^{\beta+1}[\kappa(1-\delta+\lambda)+(1-\gamma+\lambda)]}|z| \le |f'(z)| \le 1 + \frac{2[(1-\gamma)-\kappa(1-\delta)]}{(1+\lambda)^{\beta+1}[\kappa(1-\delta+\lambda)+(1-\gamma+\lambda)]}|z|.$$

The result becomes sharp for the functiondefined in (11).

4. Closure Theorems

Theorem 4.1.
Let $f_1(z) = z$ and

$$f_i(z) = z + \frac{[(1-\gamma)-\kappa(1-\delta)]}{[1+(i-1)\lambda]^{\beta}[\kappa(1-\delta+(i-1)\lambda)+(1-\gamma+(i-1)\lambda)]}z^i \quad (i \ge 2\,; i \in \mathbb{N}).$$

Then $f \in \kappa - ST^{\lambda}(\gamma,\delta,\beta)$ if and only if it can be expressed in the form

$$f(z) = \sum_{i=1}^{\infty} \lambda_i f_i(z), \text{where } \lambda_i \ge 0 \text{ and } \sum_{i=1}^{\infty} \lambda_i = 1.$$

Proof. Let $f(z)=\sum_{i=1}^{\infty}\lambda_i f_i(z)$, where $\lambda_i \geq 0$ and $\sum_{i=1}^{\infty}\lambda_i = 1$.

We consider

$$f(z)=\sum_{i=1}^{\infty}\lambda_i f_i(z)$$

$$=\lambda_1 z+\sum_{i=1}^{\infty}\lambda_i\left[z+\frac{[(1-\gamma)-\kappa(1-\delta)]}{[1+(i-1)\lambda]^{\beta}[\kappa(1-\delta+(i-1)\lambda)+(1-\gamma+(i-1)\lambda)]}z^i\right]$$

$$=\sum_{i=1}^{\infty}\lambda_i\ z+\sum_{i=2}^{\infty}\lambda_i\frac{[(1-\gamma)-\kappa(1-\delta)]}{[1+(i-1)\lambda]^{\beta}[\kappa(1-\delta+(i-1)\lambda)+(1-\gamma+(i-1)\lambda)]}z^i$$

$$=z+\sum_{i=2}^{\infty}\lambda_i\frac{[(1-\gamma)-\kappa(1-\delta)]}{[1+(i-1)\lambda]^{\beta}[\kappa(1-\delta+(i-1)\lambda)+(1-\gamma+(i-1)\lambda)]}z^i.$$

Since

$$\sum_{i=2}^{\infty}\frac{[1+(i-1)\lambda]^{\beta}[\kappa(1-\delta+(i-1)\lambda)+(1-\gamma+(i-1)\lambda)]\,\lambda_i[(1-\gamma)-\kappa(1-\delta)]}{[1+(i-1)\lambda]^{\beta}[\kappa(1-\delta+(i-1)\lambda)+(1-\gamma+(i-1)\lambda)]}$$

$$=[(1-\gamma)-\kappa(1-\delta)]\sum_{i=2}^{\infty}\lambda_i$$

$$=[(1-\gamma)-\kappa(1-\delta)](1-\lambda_1)<(1-\gamma)-\kappa(1-\delta).$$

The condition (4) for $f(z)=\sum_{i=1}^{\infty}\lambda_i\, f_i(z)$ is satisfied.

Thus$f \in \kappa-ST^{\lambda}(\gamma,\delta,\beta)$.

Conversely, we suppose that $f \in \kappa-ST^{\lambda}(\gamma,\delta,\beta)$,

$$f(z)=z+\sum_{i=2}^{\infty}a_I z^i \quad (a_i \geq 0)$$

and we take

$$\lambda_i=\frac{[1+(i-1)\lambda]^{\beta}[\kappa(1-\delta+(i-1)\lambda)+(1-\gamma+(i-1)\lambda)]}{[(1-\gamma)-\kappa(1-\delta)]}a_i \geq 0,$$

$\big(0 \le \gamma < \delta \le 1; \lambda, \beta \ge 0; \kappa(1-\delta) < (1-\gamma)\big)$.

with $\lambda_1 = 1 - \sum_{i=2}^{\infty} \lambda_i$ so that $f(z) = \sum_{i=2}^{\infty} \lambda_I \, f_i(z)$.

Using the condition (4) we obtain,

$$\sum_{i=2}^{\infty} \lambda_i = \frac{1}{[(1-\gamma) - \kappa(1-\delta)]} \sum_{i=2}^{\infty} [1 + (i-1)\lambda]^{\beta} [\kappa(1 - \delta + (i-1)\lambda) + (1 - \gamma + (i-1)\lambda)] \, a_i$$

$$< \frac{1}{[(1-\gamma) - \kappa(1-\delta)]} [(1-\gamma) - \kappa(1-\delta)] = 1$$

so that $1 - \lambda_1 < 1$ or $\lambda_1 > 0$

This proves Theorem 4.1.

Corollary4.2.
The extreme points of $\kappa - ST^{\lambda}(\gamma, \delta, \beta)$ are $f_1(z) = z$
and

$$f_i(z) = z + \frac{[(1-\gamma) - \kappa(1-\delta)]}{[1 + (i-1)\lambda]^{\beta} [\kappa(1 - \delta + (i-1)\lambda) + (1 - \gamma + (i-1)\lambda)]} z^i \quad (i \ge 2 \, , i \in \mathbb{N}).$$

By similar method adapted in Theorem 4.1, we obtain Theorem 4.3.

Theorem4.3
Let $f_1(z) = z$ and

$$f_i(z) = z + \frac{[(1-\gamma) - \kappa(1-\delta)]}{[1 + (i-1)\lambda]^{\beta+1} [\kappa(1 - \delta + (i-1)\lambda) + (1 - \gamma + (i-1)\lambda)]} z^i \quad (i \ge 2 \, ; i \in \mathbb{N}).$$

Then $f \in \kappa - UCV^{\lambda}(\gamma, \delta, \beta)$ if and only if

$$f(z) = \sum_{i=1}^{\infty} \lambda_i f_i(z) \, , \text{where } \lambda_i \ge 0 \text{ and} \sum_{i=1}^{\infty} \lambda_i = 1.$$

Corollary4.4.
The extreme points of $f \in \kappa - UCV^{\lambda}(\gamma, \delta, \beta)$ are $f_1(z) = z$
and

$$f_i(z) = z + \frac{[(1-\gamma) - \kappa(1-\delta)]}{[1 + (i-1)\lambda]^{\beta+1} [\kappa(1 - \delta + (i-1)\lambda) + (1 - \gamma + (i-1)\lambda)]} z^i \quad (i \ge 2 \, ; i \in \mathbb{N}).$$

5. Convolution results

Definition5.1.

[14]For the two functions $f, g \in \mathcal{A}$, with

$$f(z) = z + \sum_{i=2}^{\infty} a_i z^i \ (a_i \geq 0 \, ; z \in U)$$

and

$$g(z) = z + \sum_{i=2}^{\infty} b_i z^i \ (b_i \geq 0 \, ; z \in U),$$

the convolution $f * g$is defined by

$$(f * g)(z) = z + \sum_{i=2}^{\infty} a_i b_i z^i \, .$$

Theorem5.2.

If$f, g \in \kappa - ST^{\lambda}(\gamma, \delta, \beta)$ with

$$f(z) = z + \sum_{i=2}^{\infty} a_i z^i \quad (a_i \geq 0 \, ; z \in U),$$

and

$$g(z) = z + \sum_{i=2}^{\infty} b_i z^i \quad (b_i \geq 0 \, ; z \in U),$$

then $f * g \in \kappa - ST^{\lambda}(\gamma, \delta, \beta)$.

Proof. Let $f, g \in \kappa - ST^{\lambda}(\gamma, \delta, \beta)$. Therefore

$$\sum_{i=2}^{\infty} [1 + (i-1)\lambda]^{\beta} \, [\kappa(1 - \delta + (i-1)\lambda) + (1 - \gamma + (i-1)\lambda)] a_i < (1 - \gamma) - \kappa(1 - \delta)$$

and

$$\sum_{i=2}^{\infty} [1 + (i-1)\lambda]^{\beta} \, [\kappa(1 - \delta + (i-1)\lambda) + (1 - \gamma + (i-1)\lambda)] b_i < (1 - \gamma) - \kappa(1 - \delta)$$

We know that

$$(f * g)(z) = z + \sum_{i=2}^{\infty} a_i \, b_i z^i \ (a_i b_i \geq 0; z \in U).$$

Since$g \in \mathcal{A}$, by a known result, we have

$$\sum_{i=2}^{\infty} i\, b_i \leq 1 \quad \text{and}$$

weobserve that $b_i < 1$ ($i \geq 2$).

This gives,

$$\sum_{i=2}^{\infty}[1+(i-1)\lambda]^{\beta}\,[\kappa(1-\delta+(i-1)\lambda)+(1-\gamma+(i-1)\lambda)]a_i b_i$$
$$<\sum_{i=2}^{\infty}[1+(i-1)\lambda]^{\beta}\,[\kappa(1-\delta+(i-1)\lambda)+(1-\gamma+(i-1)\lambda)]a_i$$
$$<(1-\gamma)-\kappa(1-\delta).$$

Hence
$f * g \in \kappa - ST^{\lambda}(\gamma,\delta,\beta)$.

By similar method adapted in Theorem 5.2, we obtain Theorem 5.3.

Theorem5.3.
If $f, g \in \kappa - UCV^{\lambda}(\gamma,\delta,\beta)$ with

$$f(z) = z + \sum_{i=2}^{\infty} a_i z^i \quad (a_i \geq 0; z \in U),$$

and

$$g(z) = z + \sum_{i=2}^{\infty} b_i z^i \quad (b_i \geq 0; z \in U),$$

then

$$f * g \in \kappa - UCV^{\lambda}(\gamma,\delta,\beta).$$

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