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A note on hop domination number of some special families of graphs

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Abstract

Let $G = (V, E)$ be a connected graph with neither loops nor multiple edges. A set $S \subseteq V(G)$ is said to be a hop dominating set if for every $v \in V - S$ there exists a vertex $u \in S$ such that $d_G(u, v) = 2$. The minimum cardinality of a minimal hop dominating set of G is called the hop domination number of G and denoted by $\gamma_h(G)$. In this short note, we compute hop domination number of central graph, middle graph and total graph of a star graph, path and cycle.

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Key Words and Phrases: Hop dominating set, hop domination number, central graph, middle graph and total graph.

1 Introduction

Domination in graphs is one of the interesting areas in Graph Theory which has wide applications in Engineering and Science. There are more than 300 domination parameters available in the literature. For more details on domination refer [4, 5]. Harary et al. introduced the notion of 2-step dominating sets in [2]. S.K.Ayyaswamy et al.[3, 7] initiated the study on a new domination parameter related to distance called hop domination number of a graph. Pabilona and Rara have published recently some new results on total and connected hop dominating sets under some binary operations on graphs [8, 9]. Henning and Jafari Rad have obtained recently some probabilistic bounds on hop domination number of a graph [6]. In this paper, we compute the hop domination number of some special families of graphs such as central graph, middle graph and total graph of a star graph,path and cycle.

We now present some basic definition and results.

Throughout this paper, let $G = (V, E)$ be a connected simple graph of order n and size m respectively. The central graph $C(G)$ of a graph G is a graph obtained by subdividing each edge of G exactly once and joining all the non adjacent vertices of G. The middle graph of G denoted by $M(G)$ is defined as follows: The vertex set of $M(G)$ is $V(G) \cup E(G)$. Two vertices x, y in the vertex set of $M(G)$ are adjacent in $M(G)$ if one of the following holds:

(i) x, y are in $E(G)$ and x, y are adjacent in G

(ii) $x \in V(G)$, $y \in E(G)$ and y is incident at x in G.

The total graph $T(G)$ of G is defined as follows: The vertex set of $T(G)$ is $V(G) \cup E(G)$. Two vertices x, y in the vertex set of $T(G)$ are adjacent in $T(G)$ if one of the following holds:

(i) $x, y \in V(G)$ and x is adjacent to y in G

(ii) $x, y \in E(G)$ and x is adjacent to y in G

(iii) $x \in V(G)$, $y \in E(G)$ and y is incident at x in G.

A set $S \subseteq V(G)$ is said to be a hop dominating set(HDS) if for every $v \in V - S$ there exists a vertex $u \in S$ such that $d_G(u, v) = 2$. The minimum cardinality of a minimal hop dominating set is called the hop domination number of G and denoted by $\gamma_h(G)$.

Lemma 1. For any connected graph G with at least two vertices, $2 \leq \gamma_h(G) \leq n$.

For other Graph theoretic terminologies we refer to [1].

2 Main Results

Proposition 2. For a star graph $K_{1,n}$ with n leaves,

(i) $\gamma_h(C[K_{1,n}]) = 2$

(ii) $\gamma_h(M[K_{1,n}]) = 2$

(iii) $\gamma_h(T[K_{1,n}]) = 3$

Proof. Let v be the centre of $K_{1,n}$ and $\{v_1, v_2, \cdots, v_n\}$ be its leaves.
(i) Let $\{u_1, u_2, \cdots, u_n\}$ be the set of vertices that subdivide the edges $\{vv_1, vv_2, \cdots, vv_n\}$ respectively in $C[K_{1,n}]$. One can observe that the vertex v hop dominates all leaves v_i; $1 \le i \le n$ and the vertex u_i hop dominates all other vertices u_j $(j \neq i)$ in $C[K_{1,n}]$. Therefore the set $S = \{v, u_i\}$ is a HDS of $C[K_{1,n}]$ and so $\gamma_h(C[K_{1,n}]) \le |S| = 2$. Also $2 \le \gamma_h(C[K_{1,n}])$, by Lemma 1. Thus $\gamma_h(C[K_{1,n}]) = 2$.
(ii) Let $E(K_{1,n}) = \{e_1, e_2, \cdots, e_n\}$.
In $M[K_{1,n}]$, the set $E \cup \{v\}$ induces a clique of order $n+1$ and the vertices $v_1, v_2, \cdots, v_n$ become leaves. Clearly the set $\{e_i, v_i\}$; $1 \le i \le n$, is a γ_h-set of $M[K_{1,n}]$ and hence $\gamma_h(M[K_{1,n}]) = 2$.
(iii) As $T[K_{1,n}]$ contains a triangle and no leaves, it is easy to observe that $\gamma_h(T[K_{1,n}]) \ge 3$. Also $E(K_{1,n}) \cup \{v\}$ forms a clique of order $n+1$ in $T[K_{1,n}]$. Therefore the set $S = \{v_i, e_i, v\}$; $1 \le i \le n$, is a HDS of $T[K_{1,n}]$ and so $\gamma_h(T[K_{1,n}]) \le |S| = 3$. Thus $\gamma_h(T[K_{1,n}]) = 3$. □

Corollary 3. For a complete bipartite graph $K_{m,n}$ with $m, n \ge 2$, $\gamma_h(C[K_{m,n}]) = 2$.

Proposition 4. For a complete graph K_n with $n \ge 3$ vertices, $\gamma_h(C[K_n]) = \lfloor \frac{n}{2} \rfloor + 1$.

Theorem 5. For a path P_n with $n \ge 2$ vertices, $\gamma_h(C[P_n]) = 2$.

Proof. Let $V(P_n) = \{v_1, v_2, \cdots, v_n\}$. Let u_i be the vertex that subdivides the edge v_iv_{i+1} of P_n; $1 \le i \le n$ in $C[P_n]$. In $C[P_n]$, any vertex $v_i (1 \le i \le n)$ is adjacent to $n-2$ vertices of $V(P_n)$. Therefore a vertex u_j that subdivides an edge v_jv_{j+1} for some $j(1 < j < n)$

hop dominates all vertices of $V(P_n)-\{v_j, v_{j+1}\}$ and also the vertices u_{j-1} and u_{j+1}. The vertex v_j hop dominates v_{j+1} and all other u_is; $i \notin \{j-1, j+1\}$. Thus $\{u_j, v_j\}$ is a HDS of $C[P_n]$ and so $\gamma_h(C[P_n]) \leq 2$. But always $2 \leq \gamma_h(C[P_n])$, by Lemma 1. Thus $\gamma_h(C[P_n]) = 2$. □

Corollary 6. For a cycle C_n with $n \geq 3$ vertices, $\gamma_h(C[C_n]) = 2$.

Theorem 7. $\gamma_h(M[P_n]) = \begin{cases} \lceil \frac{n}{2} \rceil, & n \equiv 0, 1, 3, 5 \ (mod\ 6) \\ \lceil \frac{n}{2} \rceil + 1, & n \equiv 2, 4 \ (mod\ 6) \end{cases}$

Proof. Let $V(P_n) = \{v_1, v_2, \cdots, v_n\}$ and $E(P_n) = \{e_1, e_2, \cdots, e_{n-1}\}$. Clearly $V(M[P_n]) = V(P_n) \cup E(P_n)$. In $M[P_n]$, $deg(e_i) = 4$ for $1 < i < n-1$ and $deg(e_1) = deg(e_{n-1}) = 3$. A vertex e_i in $M[P_n]$ hop dominates the vertices $v_{i-1}, v_{i+2}, e_{i-2}$ and e_{i+2};$i > 2$. One can observe that a γ_h-set of $M[P_n]$ contains the set $S = \{e_2, e_3, e_4, e_8, e_9, e_{10}, \cdots, e_{6r-4}, e_{6r-3}, e_{6r-2}\}$.

Case $1 : n = 6r$

The vertex e_2 hop dominates the vertices $\{v_1, v_4, e_4\}$, e_3 hop dominates the vertices $\{v_2, v_5, e_1, e_5\}$, e_4 hop dominates the vertices $\{v_3, v_6, e_6, e_2\}$ and so on. Therefore the set S itself is a γ_h-set of $M[P_n]$ containing $\lceil \frac{n}{2} \rceil$ vertices. Also one can observe that any HDS of $M[P_n]$ contains at least $3r$ vertices. Therefore $\gamma_h(M[P_n]) = \lceil \frac{n}{2} \rceil$.

Case $2 : n = 6r + 1$

In this case the set $S \cup \{v_{6r+1}\}$ is a γ_h-set of $M[P_n]$. Therefore $M[P_n] = 3r + 1 = \lceil \frac{n}{2} \rceil$.

Case $3 : n = 6r + 2$

The set $S \cup \{e_{6r+1}, v_{6r+2}\}$ is a γ_h-set of $M[P_n]$. Therefore $\gamma_h(M[P_n]) = 3r + 2 = \lceil \frac{n}{2} \rceil + 1$.

Case $4 : n = 6r + 3$

The set $S \cup \{e_{6r+2}, v_{6r+3}\}$ is a γ_h-set of $M[P_n]$. Therefore $\gamma_h(M[P_n]) = 3r + 2 = 3(\frac{n-3}{6}) + 2 = \frac{n+1}{2} = \lceil \frac{n}{2} \rceil$.

Case $5 : n = 6r + 4$

The set $S \cup \{e_{6r+2}, e_{6r+3}, v_{6r+3}\}$ is a γ_h-set of $M[P_n]$. Therefore, $\gamma_h(M[P_n]) = 3r + 3 = 3(\frac{n-4}{6}) + 3 = \frac{n}{2} + 1 = \lceil \frac{n}{2} \rceil + 1$.

Case $6 : n = 6r + 5$

In this case $S \cup \{e_{6r+2}, e_{6r+3}, e_{6r+4}\}$ is a γ_h-set of $M[P_n]$. So, $\gamma_h(M[P_n]) = 3r + 3 = 3(\frac{n-5}{6}) + 3 = \frac{n-5}{2} + 3 = \frac{n+1}{2} = \lceil \frac{n}{2} \rceil$. □

Theorem 8. $\gamma_h(T[P_n]) = \begin{cases} \lceil \frac{n}{2} \rceil, & n \equiv 0,2,3,5 \ (mod\ 6) \\ \lfloor \frac{n}{2} \rfloor, & n \equiv 1 \ (mod\ 6) \\ \lceil \frac{n}{2} \rceil + 1, & n \equiv 4 \ (mod\ 6) \end{cases}$

Proof. Let $V(P_n) = \{v_1, v_2, \cdots, v_n\}$ and $E(P_n) = \{e_1, e_2, \cdots, e_{n-1}\}$. Then by definition of total graph $V(T[P_n]) = V(P_n) \cup E(P_n)$. It is easy to observe that every γ_h-set of $T[P_n]$ contains the set $S = \{v_3, v_4, v_5, v_9, v_{10}, v_{11}, \cdots, v_{6r-3}, v_{6r-2}, v_{6r-1}\}$ of cardinality $3r$.

Case $1 : n = 6r$
The set S itself is a γ_h-set of $T[P_n]$ and so $\gamma_h(T[P_n]) = 3r = 3(\frac{n}{6}) = \lceil \frac{n}{2} \rceil$.

Case $2 : n = 6r + 1$ In this case also, S is a γ_h-set of $T[P_n]$ and hence $\gamma_h(T[P_n]) = 3r = 3(\frac{n-1}{6}) = \frac{n-1}{2} = \lfloor \frac{n}{2} \rfloor$.

Case $3 : n = 6r + 2$
The set $S \cup \{v_{6r}\}$ is a γ_h-set of $T[P_n]$. Therefore $\gamma_h(T[P_n]) = 3r + 1 = 3(\frac{n-2}{6}) + 1 = \frac{n-2}{2} + 1 = \lceil \frac{n}{2} \rceil$.

Case $4 : n = 6r + 3$
The set $S \cup \{v_{6r+1}, v_{6r+2}\}$ is a γ_h-set of $T[P_n]$. Therefore $\gamma_h(T[P_n]) = 3r + 2 = 3(\frac{n-3}{6}) + 2 = \frac{n+1}{2} = \lceil \frac{n}{2} \rceil$.

Case $5 : n = 6r + 4$
In this case, $S \cup \{v_{6r+2}, v_{6r+3}, v_{6r+4}\}$ is a γ_h-set of $T[P_n]$. Therefore we have $\gamma_h(T[P_n]) = 3r + 3 = 3(\frac{n-4}{4}) + 3 = \frac{n}{2} + 1 = \lceil \frac{n}{2} \rceil + 1$.

Case $6 : n = 6r + 5$
$S \cup \{v_{6r+2}, v_{6r+3}, v_{6r+4}\}$ is a γ_h-set of $T[P_n]$. So, $\gamma_h(T[P_n]) = 3r + 3 = 3(\frac{n-5}{6}) + 3 = \frac{n-5}{2} + 3 = \frac{n+1}{2} = \lceil \frac{n}{2} \rceil$. □

Similarly one can prove the following results

(i) $\gamma_h(M[C_n]) = \begin{cases} \lceil \frac{n}{2} \rceil, & n \equiv 0,1,3,5 \ (mod\ 6) \\ \lceil \frac{n}{2} \rceil + 1, & n \equiv 2,4 \ (mod\ 6) \end{cases}$

(ii) $\gamma_h(T[C_n]) = \begin{cases} \lceil \frac{n}{2} \rceil, & n \equiv 0,1,3,5 \ (mod\ 6) \\ \lceil \frac{n}{2} \rceil + 1, & n \equiv 2,4 \ (mod\ 6) \end{cases}$

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