INTEGRATED MODEL WITH FLEXIBLE PRODUCTION RATE FOR A DETERIORATING ITEM AND PARTIAL BACKLOGGING UNDER THE EFFECT OF INFLATION IN FUZZY FRAMEWORK

Vandana¹ Dharmendra² and S.R.Singh³

¹Department of Mathematics, K.L.Mehta Dayanand College, Faridabad-121002(Haryana) India
²Department of Mathematics, Vardhman (P.G) College, Bijnor-246701(U.P) India
³Department of Mathematics, C.C.S. University Meerut-246701 Corresponding author
¹vandna.acme@gmail.com,²dharmendrayadav3580@gmail.com,³shivrampundir@gmail.com

Abstract: In the recent years, the academicians and the inventory practitioners pay considerable attention for the coordination of the manufacturer, wholesaler and retailer. Traditionally, in supply chain the primary focus of the timing of the delivery quantities. But now-a-days they share the information regarding the demand of customer and inventory. By doing this operating cost of the inventory system reduces. In the traditional supply chain inventory model, it is usually assumed that the retailer has storage space with unlimited capacity. Warehouse with infinite storage capacity in main market is not possible in real life situation. So this assumption made by the inventory practitioners is not practicable. Hence, there is a requirement to deal the supply chain inventory model in different scenario. There are many factors which influence the retailer to buy more than their own warehouse capacity. So, in this situation it is beneficiary for the retailer to take rented warehouse house at some distance from the market place. Since, the rented warehouse (RW) incurs additional cost of inventory handling and has a better preservation facility as compared with own warehouse so we may assume that there is higher holding cost and lower deterioration cost for the rented warehouse. To reduce the inventory costs, retailer stores inventory in OW before RW but clear the inventory in RW before OW.

Goyal (1976) was the first researcher who introduces the idea of joint total cost for the supplier and the buyer. Pake and Cohen (1993) used stochastic sub-models to calculate the values of the included random variables for analyzing the integrated system. Gyna and Bhabha (1999) explored a single manufacturing system for procurement of raw materials with a multi-ordering policy that minimized the total inventory costs of both the raw materials and the finished goods. Sarkar et al. (2000) explored a supply chain model for determining an optimal ordering policy under inflation and allowable shortages. Chien and Lin (2004) investigated the optimal order interval and discount price such that the joint total cost was minimized during a finite planning horizon. Ahmed et al. (2008) have coordinated a two level supply chain in which they considered production interruptions for restoring the quality of the production process. Jha and Shanker (2009) considered a two-echelon supply chain inventory problem consisting of a single-vendor and a single-buyer. In the system under study, a vendor produced a product in a batch production environment and supplied it to a buyer. Also, buyer’s lead time was controllable which can be shortened at an added cost and all shortages were backordered. Sajadieh et al. (2010) developed an integrated vendor–buyer model for a two-stage supply chain. The vendor manufactures the product and delivers it to the buyer. The items delivered were presented to the end customers in a display area. Here, the demand was assumed to be positively dependent on the amount of items displayed. The objective was to maximize total supply chain profit. Yadav et al. (2013) developed three-stage supply chain (SC) coordination focusing on the fuzziness as well as randomness aspect of demand and production rate. Handa et al. (2014) developed a mixed integrated manufacturing-inventory model where a manufacturer produces item in a batch production environment and supplies it to a set of buyers.

Most of the classical supply chain inventory models assumed the potential worth of the inventory remains constant during the storage period. But in real situation, deterioration occurs when inventory present physically in warehouse. Thus, deterioration of inventory items present is stock is a realistic phenomenon which produces direct impact on the inventory policy. Deterioration is the change, damage, decay, spoilage, evaporation, obsolescence, pilferage, and loss of utility or loss of marginal value of a commodity that results in decreasing usefulness from...
the original one. Most products such as medicine, blood, fish, alcohol, gasoline, vegetables and radioactive chemicals have finite shelf life, and start to deteriorate once they are replenished. In addition, for certain types of commodities, deterioration is usually observed during their normal storage period. Incorporating this phenomenon, Chung and Huang (2007) developed two-warehouse inventory model for deteriorating items under trade credit financing. Min et al. (2010) developed an inventory model for deteriorating items under stock-dependent and two-level trade credit to study the retailer’s optimal ordering policy. Soni (2013) extended the work of Min et al. (2010) by incorporating a constraint on the maximum inventory level. Yadav et al. (2015) developed the retailer’s inventory model in fuzzy environment for deteriorating items to determine the optimal cycle time and payment time for a retailer.

In most of the inventory models unrealistically assume that during stock-out either all demand is backlogged or all is lost. In reality often some customers are willing to wait until replenishment, especially if the wait will be short, while others are more impatient and go elsewhere. The backlogging rate depends on the time to replenishment—the longer customers must wait, the greater the fraction of lost sales. Wee (1999) formulated an inventory model with quantity discount, pricing and partial backordering when the product in stock deteriorates with times.

Skouri and Papachristos (2003) proposed an algorithm to find the optimal stopping and restarting manufacturing times for EOQ model with deteriorating items and time-dependent partial backlogging. Wee et al. (2005) developed a two-warehouse inventory model in inflationary environment with partial backordering and Weibull distribution deterioration. Lo et al. (2007) investigated a two-echelon system with partial backlogging, two-parameter deterioration rate and constant rate of demand. Pentico et al. (2009) proposed the new and/or alternative approaches to solve the EPQ model with partial backordering.

Kumar et al. (2013) developed an inventory model with two-warehouse system under the effect of learning and partial backordering in inflationary environment. Goyal et al. (2015) investigated an EOQ model in which the demand of items is fuzzy in nature and depends on the frequency of advertisement. Learning effect on number of defective items present in each lot is considered and the possibility of lost sale and backorder are also analyzed.

From the above literature it is observed that most of the inventory practitioners assumed that the different cost associated to inventory system is precise in nature. However, these parameters deviate from one cycle to another cycle due to vagueness and impreciseness. For example, the ordering cost is influenced by various factors such as transportation cost, fuel prices etc. and these costs continuously changes due to inflation and many other factors. So, in the presence of various uncertainties it is difficult to calculate different cost such as ordering cost, holding cost, purchasing cost, shortages cost and lost sale cost precisely. Moreover, in many cases sufficient historical data are also not available to the inventory practitioners especially for newly launched product or launching of product with new specification or due to the change in supply chain environment. All of these raise the challenge in front of the inventory practitioner to determine the inventory policy. To cope up with the presence of imprecision it is better to characterize cost parameters using fuzzy numbers. Many inventory problems incorporating fuzziness have been investigated (see Vijayan and Ouyang, 2008; Handfield et al., 2009; Sadjadi et al., 2010; Liu, 2012; Mahata and Goswami, 2013 and references therein).

From the above literature survey it is observed that most of the inventory practitioner developed two-warehouse system for retailer point of view. While from literature it is observed that decision taken as integrated point of view is more beneficiary in place of taking by individual entities. Thus previous works fall short of regarding supply chain perspective. Therefore in the present study a supply chain system consisting of one manufacturer, one wholesaler and one retailer (having two different warehouses one known as own-warehouse (OW) and other is rented warehouse (RW)) has been considered. It has been assumed that there are different deterioration rate at OW and RW due to different handling facility. Here, shortages are allowed and partially backlogged at retailer end only. The holding cost at RW is higher as compared to OW. Whole of the study is carried out in inflationary environment. Further, different cost associated with inventory may be flexible with some vagueness as to their values. For this type of parameters statistical estimations proved to be inefficient due to lack of historical data. In this situation, a suitable way to handle the imprecise is to consider as fuzzy parameters. Due to this the objective function becomes impreciseness in nature. To obtain the optimal values of the decision variables an equivalent deterministic expression of objective function can be obtained by defuzzifying using modified graded mean integration method. A numerical example and sensitivity analysis are presented to illustrate the models.

2. ASSUMPTIONS AND NOTATIONS

2.1 ASSUMPTIONS:-

In developing the mathematical models of the inventory system for this paper, the following common assumptions are used:
1. Supply chain system consists of one manufacturer, one wholesaler and one retailer has been considered.
2. System delivers a perishable item to the end customer.
3. There is no lead time at the end of retailer and wholesaler.
4. Shortages are allowed at the retailer ends while it is not allowed at wholesaler ends. Shortages are partially backlogged.
5. Retailer has two warehouses i.e., own warehouse (OW) of fixed capacity (W) and rented warehouse (RW) of infinite capacity.
6. The holding costs in RW are higher than those in OW.
7. Unit manufacturing cost is the function of production rate.
8. Deterioration occurs as soon as items are received into inventory.
9. There is no replacement or repair of deteriorating items during the period under consideration.
10. Different costs associated to system consider constant for crisp model while for fuzzy model they are fuzzy in nature.

2.2 Notations:-

With help of following notations, mathematical model for integrated system has been formulated in this chapter.

- **W**: Capacity of OW
- **I_w(t)**: The inventory level of manufacturer at any time 't'
- **I_w(t)**: The inventory level at the wholesaler
- **I_r(t)**: The inventory level at the OW at time ‘t’ of retailer
- **I_r(t)**: The inventory level at the RW at time ‘t’ of retailer
- **S(t)**: The shortage level at the retailer ends
- **P**: Production rate
- **M(P)**: Unit manufacturing cost ($/unit/unit time) which is imprecise in nature is given as

\[
\bar{M} = (M_1, M_2, M_3, M_4)
\]

\[
\tilde{G} = (G_1, G_2, G_3, G_4)
\]

\[
\tilde{B} = (B_1, B_2, B_3, B_4)
\]

- **\bar{\tilde{h}}_r**: Shortage cost per lost sales per unit time ($/unit/unit time) which is imprecise in nature is given as

\[
\tilde{h}_r = (h_{r1}, h_{r2}, h_{r3}, h_{r4})
\]

- **\bar{\tilde{h}}_w**: Shortage cost per unit time ($/unit/unit time) which is imprecise in nature is given as

\[
\tilde{h}_w = (h_{w1}, h_{w2}, h_{w3}, h_{w4})
\]

- **\bar{\tilde{p}}_R**: The purchasing cost for the retailer per unit which is imprecise in nature is given as

\[
\tilde{p}_R = (p_{R1}, p_{R2}, p_{R3}, p_{R4})
\]

- **\bar{\tilde{p}}_W**: The purchasing cost for the wholesaler per unit which is imprecise in nature is given as

\[
\tilde{p}_W = (p_{W1}, p_{W2}, p_{W3}, p_{W4})
\]

3. Formulation of Crisp Integrated Inventory Model

In this supply chain system, we assume that wholesaler and retailer worked as single entity and plays the role of leader in this supply chain system as they have to satisfy the demand of customer. Here, wholesaler place the order of \(Q_w\) units to the manufacturer and these units shipped to the retailer in k shipment of equal size of \(Q_k\). On bases of optimal
policy adopted by wholesaler and retailer, manufacturer finds the optimal production rate so that its total inventory cost is minimum.

Many inventory models have been developed for the static environment where the demand rate is assumed to be constant. However, it is observed that in real situations, constant demand can be applicable only for the maturity phase of the product. Many products, such as fashion accessories, clothes, mobile phones, need to prove their worth before they are generally accepted. Hence, it is reasonable to approximate the demand for a product to be represented by a time dependent function during its growth stage.

Therefore, the differential equation for raw material inventory level \( I_m(t) \) representing above system during \( t_1 \) is as follows:

\[
\frac{dI_m(t)}{dt} = -P, \quad 0 \leq t \leq t_1
\]  

(1)

With boundary condition \( I_m(0) = Q_w \) and \( I_m(t_1) = 0 \)

On solving the above differential equation with the help of boundary condition, we get

\[
I_m(t) = Q_w - Pt, \quad 0 \leq t \leq t_1
\]

(2)

Using the condition \( I_m(t_1) = 0 \), we get

\[
t_1 = \frac{Q_w}{P}
\]

(3)

Holding cost for raw material

\[
= \frac{h_r^4}{2P} \left( 1 - \frac{rQ_w}{3P} \right)
\]

Manufacturing cost of \( Q_w \) units

\[
= M(P) \int_0^{t_1} Q_w e^{-rt} dt = \frac{M(P)Q_w^2}{P} \left( 1 - \frac{rQ_w}{2P} \right)
\]

Setup cost for the manufacturer per setup = \( A_m \)

Therefore, the manufacturer’s total inventory cost is the sum of holding cost of raw material and finished product, manufacturing cost and setup cost.

Total inventory cost

\[
= \frac{(h_r+h_f)Q_w^2}{2P} \left( 1 - \frac{rQ_w}{3P} \right) + \frac{M(P)Q_w^2}{P} \left( 1 - \frac{rQ_w}{2P} \right) + A_m
\]

Hence, the inventory cost of manufacturer per unit time is

\[
TIC_M = \frac{(h_r+h_f)Q_w}{2} \left( 1 - \frac{rQ_w}{3P} \right) + M(P)Q_w \left( 1 - \frac{rQ_w}{2P} \right) + \frac{A_mP}{Q_w}
\]

(4)

3.2 Formulation of Inventory Model for Wholesaler

Now for the wholesaler, the ordering quantity is equal to the inventory needed for the ‘k’ periods at the retailer ends and the amount inventory which deteriorates during the wholesaler’s inventory cycle. During the \( k^{th} \) interval of wholesaler there is no
inventory and after receiving $Q_w$ of the product at the end of $T_2$. $Q_w$ quantity is sent to the retailer. Therefore there is no deterioration during the $k^{th}$ interval of the wholesaler. Now, the order quantity of the wholesaler is

$Q_w = k Q_e + \text{ Deteriorated units during the wholesaler's inventory cycle}$

(5)

Inventory Level of Wholesaler

$$
\text{Level of wholesaler at } (k-1)T_1
$$

$$
\text{Level of wholesaler at } (k-1)T_2
$$

$$
\text{Level of wholesaler at } kT_2
$$

$$
\text{Fig.2 Graphical representation of Inventory System of Wholesaler}
$$

It is obvious that one inventory period in the wholesaler consists of 'k' retailer's inventory period. Inventory level of the wholesaler is illustrated with the help of Fig.2. From the time $(k-2)T_2$ to $(k-1)T_2$, the inventory level drops by $Q_W$ and the number of units which is deteriorated during the interval $[(k-2)T_2, (k-1)T_2]$. During this interval the inventory level of the wholesaler is represented by the following differential equation: 

$$
\frac{dW}{dt} = -\gamma Q_w(t), \quad (k-2)T_2 \leq t \leq (k-1)T_2
$$

(6)

Inventory level at the wholesaler at $(k-1)T_1$ is Qk. Thus the inventory level for the period $[(k-2)T_2, (k-1)T_1]$ is 

$$
Q_w(t) = Q_k e^{\gamma [(k-1)T_1 - t]}, \quad (k-2)T_2 \leq t \leq (k-1)T_1
$$

(7)

Using this expression we can get the inventory level at $(k-2)T_2$,

$$
Q_w[(k-2)T_2] = Q_k e^{\gamma T_2}
$$

(8)

During the $[(k-3)T_2, (k-2)T_2]$ interval the inventory level of the wholesaler is represented by the following differential equation:

$$
\frac{dW}{dt} = -\gamma Q_w(t), \quad (k-3)T_2 \leq t \leq (k-2)T_2
$$

Subject to the condition that $Q_w[(k-2)T_2] = Q_k e^{\gamma T_2}$

On solving above equation by using the condition we get

$$
Q_w(t) = Q_k(1 + e^{\gamma T_2})e^{\gamma [(k-2)T_2 - t]}, \quad (k-3)T_2 \leq t \leq (k-2)T_2
$$

(9)

Thus the inventory of wholesaler during the $i^{th}$ interval can be written as follows:

$$
Q_w(t) = Q_k\left(\sum_{j=i+1}^{k} e^{(k-j)\gamma T_2}\right)e^{\gamma [(k-2)T_2 - t]}
$$

(10)

Now, we calculate the different cost associated with the wholesaler.

The purchasing cost of the wholesaler is 

$$
PCW = A_W + p_w Q_W
$$

Inventory carrying cost for the $i^{th}$ interval is

$$
Q_w(\sum_{j=i+1}^{k} e^{(k-j)\gamma T_2})(\gamma + r)\left(1 - e^{-\gamma T_2}\right)
$$

Hence, carrying cost during one cycle time of the wholesaler

$$
C_{CW} = h_w Q_w \left[\sum_{j=1}^{k-1} e^{\gamma j\gamma T_2} - \sum_{j=1}^{k-1} e^{(k-j)\gamma T_2}\right] e^{-\gamma T_2}
$$

(11)

Thus total inventory cost of the wholesaler is the sum of the purchasing cost, carrying cost and deterioration cost.

Total inventory cost of wholesaler

$$
\text{Total inventory cost of wholesaler} = A_W + p_w Q_W + h_w Q_w \left[\sum_{j=1}^{k-1} e^{\gamma j\gamma T_2} - \sum_{j=1}^{k-1} e^{(k-j)\gamma T_2}\right] e^{-\gamma T_2}
$$

(12)

Hence, total inventory cost of wholesaler per unit time is

$$
TICW = \frac{1}{T_2} \left[ A_W + p_w Q_W + h_w Q_w \left[\sum_{j=1}^{k-1} e^{(k-j)\gamma T_2} - \sum_{j=1}^{k-1} e^{(k-j)\gamma T_2}\right] e^{-\gamma T_2}\right]
$$

(13)
During the period of $[0,t_o]$, inventory level of OW is depleted due to deterioration only.

During this period inventory system of retailer can be represented by the following differential equation:

$$\frac{dL_o(t)}{dt} = -aI_o(t), \quad 0\leq t \leq t_o$$  \hspace{1cm} (15)

Subject to the condition that $L_o(0) = W$

On solving equation (15) by using above condition, we get

$$L_o(t) = We^{-at}, \quad 0 \leq t \leq t_o$$  \hspace{1cm} (16)

After $t_o$, inventory level of RW is zero so, after that demand of the customer is satisfied from the OW. Inventory level of OW during $[t_o,t_1]$ is depleted due to demand and deterioration. This situation can be described by differential equation as follows:

$$\frac{dL_o(t)}{dt} = -(a + bt) - \alpha L_o(t), \quad t_o \leq t \leq t_1$$  \hspace{1cm} (17)

Subject to the condition that $L_o(t_o) = 0$

On solving equation (17) by using the above condition

$$L_o(t) = \frac{1}{a+b} \left[ (aa - b)(e^{b(t_o - t)} - 1) + ba(t_o e^{b(t_o - t)} - t) \right], \quad t_o \leq t \leq t_1$$  \hspace{1cm} (18)

During $[t_o,t_1]$ system is out of stock situation. In this period demand is partially backlogged. During this inventory level can be represented by the following differential equation

$$\frac{dL_o(t)}{dt} = -\frac{(a + bt)}{1 + \delta (t_1 - t)}, \quad t_o \leq t \leq t_1$$  \hspace{1cm} (19)

Subject to the condition that $L_o(t_o) = 0$

On solving the differential equation (19) by using the above condition, we get

$$L_o(t) = \frac{-b}{\delta} (t - t_o) - \frac{(a+b \delta)}{\delta} \log \left( 1 + \frac{\delta}{1 + \delta (t_1 - t_o)} \right), \quad t_o \leq t \leq t_1$$  \hspace{1cm} (20)

For continuity, inventory level given by equation (16) and (18) must be same at $t = t_o$, i.e.,

$$We^{-at} = \frac{1}{a^2} \left[ (aa - b)(e^{b(t_o - t)} - 1) + ba(t_o e^{b(t_o - t)} - t) \right]$$

Which gives

$$t_o = t_o + \frac{1}{a} \log \left( \frac{a^2 e^{-at} + bat_o + aa - b}{aa + bat_o} \right)$$  \hspace{1cm} (21)

Above expression shows that $t_o$ is the function of $t_o$. Total ordering quantity of the retailer is sum of the initial inventory level of RW, OW and the total backlogged quantity. Thus

$$Q_r = I_r(0) + I_o(0) + S(T_1)$$

$$Q_r = \frac{1}{b} \left[ (aa - b)(e^{bT_1} - 1) + ba(t_o e^{bT_1}) \right] + W + \frac{b}{\delta} (T_1 - t_o) + \frac{a+b \delta}{\delta} \log \left( \frac{1 + \delta (t_1 - t_o)}{1 + \delta (t_1 - t_o)} \right)$$  \hspace{1cm} (22)

Above expression shows that order quantity is the function of $t_o$ and $T_1$.

Purchasing cost of retailer

$$PC_r = A_k + P O_k$$

Holding cost = Holding cost at the OW + Holding cost at the RW

$$HC_R = h_r \int_{0}^{T_1} I_r(t) e^{-\gamma t} dt + h_o \int_{t_o}^{T_1} I_o(t) e^{-\gamma t} dt + h_o \int_{t_o}^{T_1} I_o(t) e^{-\gamma t} dt$$

Fig.3 Graphical representation of Inventory System of Retailer
\[= h_f \int_0^{T_f} \left( \frac{1}{\sigma} \left[ (a\beta - b)(e^{\beta(T_f-t)} - 1) + b\beta \right] (t_e, e^{\beta(T_f-t)} - 1) \right) e^{-\gamma t} dt + h_a \int_0^{T_a} \left( \frac{1}{a} \left[ (aa - b)(e^{a(t_a-t)} - 1) + a\alpha \right] (t_e, e^{a(t_a-t)} - 1) \right) e^{-\gamma t} dt \]

\[= \frac{h_f}{\sigma} \left( a\beta - b \right) \left\{ \left( \frac{1}{\sigma} - \frac{t_e^\beta}{\sigma^2} \right) - \left( 1 - r_t \right) \left( \frac{1}{\sigma} + t_e \beta \right) + t_e \beta \right\} + b\beta \left( \frac{t_e^\beta}{\sigma^2} - \frac{t_e^\beta}{\sigma^2} \right) - \left( 1 - r_t \right) \frac{t_e^\beta}{\sigma^2} + t_e \beta - \left( 1 - r_t \right) \frac{t_e^\beta}{\sigma^2} + t_e \beta \]

\[= \frac{h_a}{\sigma} \left( aa - b \right) \left\{ \left( 1 - r_t \right) \left( \frac{1}{a} + r_t \right) + \left( 1 - r_t \right) \frac{t_e^\beta}{a^2} - r \left( \frac{t_e^\beta}{a^2} - \frac{t_e^\beta}{a^2} \right) \right\} \]

Deterioration cost = Deterioration cost at the OW & Deterioration cost at the RW

\[DC_R = \frac{h_a}{\sigma} \left( aa - b \right) \left\{ \left( 1 - r_t \right) \left( \frac{1}{a} + r_t \right) + \left( 1 - r_t \right) \frac{t_e^\beta}{a^2} - r \left( \frac{t_e^\beta}{a^2} - \frac{t_e^\beta}{a^2} \right) \right\} \]

\[Demand \ during \ the \ shortage \ period \ is \ partially \ backlogged. \]

The total backordering cost is given by

\[S_{CR} = s \int_0^{T_s} -\lambda_s(t) e^{-\gamma t} dt \]

\[DC_R = \frac{h_a}{\sigma} \left( aa - b \right) \left\{ \left( 1 - r_t \right) \left( \frac{1}{a} + r_t \right) + \left( 1 - r_t \right) \frac{t_e^\beta}{a^2} - r \left( \frac{t_e^\beta}{a^2} - \frac{t_e^\beta}{a^2} \right) \right\} \]

The total lost sale cost is given by

\[LSC_R = \pi \int_0^{T_s} \left( \frac{1}{\sigma} - \frac{t_e^\beta}{\sigma^2} \right) \left( \frac{1}{\sigma} - \frac{t_e^\beta}{\sigma^2} \right) e^{-\gamma t} dt \]

\[= \frac{h_a}{\sigma} \left( aa - b \right) \left\{ \left( 1 - r_t \right) \left( \frac{1}{a} + r_t \right) + \left( 1 - r_t \right) \frac{t_e^\beta}{a^2} - r \left( \frac{t_e^\beta}{a^2} - \frac{t_e^\beta}{a^2} \right) \right\} \]

Total inventory cost of retailer is the sum of purchasing cost, holding cost, deterioration cost, backordering cost

Total inventory cost of retailer = \[\rho_{pa}Q_a + \frac{h_a}{\sigma} \left( aa - b \right) \left\{ \left( 1 - r_t \right) \left( \frac{1}{a} + r_t \right) + \left( 1 - r_t \right) \frac{t_e^\beta}{a^2} - r \left( \frac{t_e^\beta}{a^2} - \frac{t_e^\beta}{a^2} \right) \right\} \]
\[
\left\{ \frac{1-t_0^2}{2t} \log(1 + \delta(T_1 - t_0)) + \frac{14\delta(T_1-t_0)}{2t} \left( T_1^2 - t_0^2 \right) \right\} + \frac{1}{\delta} \left\{ 1 + (1 + \delta T_1) \left( \frac{\delta r^2 T_1 + r^2 - 2r\delta}{\delta t} \right) \log(1 + \delta T_1 - t_0) \right\} + \frac{1}{\delta} \left[ a - \frac{(ar-b)^2}{2} \right] \left( T_1 - t_0 \right) + \frac{b-ar-\frac{by}{a}}{z} \left( T_1^2 - t_0^2 \right) + \frac{1}{\delta} \left[ \frac{a}{1 + \delta T_1} \left( \frac{ar-b}\frac{by}{a}\left( 1+\delta T_1 \right) \right) \log(1 + \delta T_1 - t_0) \right]
\]
\]

Hence, total inventory cost of retailer per unit time is
\[TIC_R = \frac{1}{t_1} \left( A_R + p_a Q_R + \frac{a}{t_1^2} \left( a\beta - b \right) \left[ \frac{1}{t_1^2} + t_0 + \frac{t_0^2}{2t_1} \right] + \frac{1}{t_1} \left( a\beta - t_0 \right) \left[ \frac{t_0^2}{2} \right] + \frac{1}{t_1} \left( a\beta - \frac{t_0^2}{2} \right] + \frac{1}{t_1} \left( \frac{a}{1 + \delta T_1} \frac{ar-b}{\frac{by}{a}} \left( 1 + \delta T_1 \right) \right) \log(1 + \delta T_1 - t_0) \right) \]

\[\text{3.4 Mathematical Formulation of Integrated System} \]

\[\text{(Leader-Follower Approach)}\]

In this situation, wholesalers and retailer jointly determined the optimal policy and on the basis of that manufacturer decide the optimal strategies. Thus, Objective function of integrated system in crisp environment is

Minimum TIC \(T_1, T_2, k\)

Subject to \(T_1, T_2 \geq 0, k\) is an integer \(23\)

Where \(TIC = TIC_W + TIC_R\)

Value of \(TIC_W\) is given in equation \(12\) whereas the values of \(TIC_R\) is given in equation \(23\).

Objective function of the manufacturer in crisp environment is

Minimum \(TIC_M(P)\)

Subject to \(P > 0\)

Where the value of \(TIC_M\) is given in equation \(4\).

4. Formulation of Fuzzy Inventory Model for Integrated System

In this section, fuzzification and defuzzification of the objective has been performed.

Fuzzification by Trapezoidal Fuzzy Numbers:

In the present scenario, it is very difficult to determine precise value of different costs associated with inventory system. So, it is more reasonable to describe it as fuzzy in nature. Therefore, all the cost parameters are considered as fuzzy variables. Suppose the different cost represented as following by using the trapezoidal fuzzy number.

\[\tilde{A}_R = \left( A_{R1}, A_{R2}, A_{R3}, A_{R4} \right), \tilde{A}_W = \left( A_{W1}, A_{W2}, A_{W3}, A_{W4} \right), \tilde{P}_R = \left( P_{R1}, P_{R2}, P_{R3}, P_{R4} \right), \tilde{P}_W = \left( P_{W1}, P_{W2}, P_{W3}, P_{W4} \right), \tilde{h}_0 = \left( h_{01}, h_{02}, h_{03}, h_{04} \right), \tilde{h}_R = \left( h_{R1}, h_{R2}, h_{R3}, h_{R4} \right), \tilde{h}_W = \left( h_{W1}, h_{W2}, h_{W3}, h_{W4} \right), \tilde{s} = \left( s_1, s_2, s_3, s_4 \right), \tilde{r} = \left( r_1, r_2, r_3, r_4 \right)\]
\[ \tilde{A}_m = (A_{m1}, A_{m2}, A_{m3}, A_{m4}) \]
\[ \tilde{h}_y = (h_{y1}, h_{y2}, h_{y3}, h_{y4}) \]
\[ \tilde{M} = (M_1, M_2, M_3, M_4) \]
\[ \tilde{G} = (G_1, G_2, G_3, G_4) \]
\[ \tilde{B} = (B_1, B_2, B_3, B_4) \]

Thus the objective function wholesaler and retailer which we have to minimized can be constructed under fuzzy framework as follows:

\[ \begin{align*}
\overline{\text{TIC}} &= \\
&= \frac{1}{7} \sum \left[ C_m \otimes F_y \otimes Q_y \otimes h_y \otimes Q_y \otimes h_y \otimes C_y \otimes C_y \right] - \\
&= e^{-(r+k)y} \left( \begin{bmatrix} t_1 \frac{a}{a_2} + \frac{a}{a_2} \\
1 - r + t_2 \frac{a}{a_2} + \frac{a}{a_2} \\
1 - r + t_3 \frac{a}{a_2} + \frac{a}{a_2} \\
1 - r + t_4 \frac{a}{a_2} + \frac{a}{a_2} \\
\end{bmatrix} \right) \otimes P \otimes W \otimes 1 - \\
&= e^{-(r+k)y} \left( \begin{bmatrix} t_1 \frac{a}{a_2} + \frac{a}{a_2} \\
1 - r + t_2 \frac{a}{a_2} + \frac{a}{a_2} \\
1 - r + t_3 \frac{a}{a_2} + \frac{a}{a_2} \\
1 - r + t_4 \frac{a}{a_2} + \frac{a}{a_2} \\
\end{bmatrix} \right) \otimes P \otimes W \otimes 1 - \\
&= e^{-(r+k)y} \left( \begin{bmatrix} t_1 \frac{a}{a_2} + \frac{a}{a_2} \\
1 - r + t_2 \frac{a}{a_2} + \frac{a}{a_2} \\
1 - r + t_3 \frac{a}{a_2} + \frac{a}{a_2} \\
1 - r + t_4 \frac{a}{a_2} + \frac{a}{a_2} \\
\end{bmatrix} \right) \otimes P \otimes W \otimes 1 - \\
&= e^{-(r+k)y} \left( \begin{bmatrix} t_1 \frac{a}{a_2} + \frac{a}{a_2} \\
1 - r + t_2 \frac{a}{a_2} + \frac{a}{a_2} \\
1 - r + t_3 \frac{a}{a_2} + \frac{a}{a_2} \\
1 - r + t_4 \frac{a}{a_2} + \frac{a}{a_2} \\
\end{bmatrix} \right) \otimes P \otimes W \otimes 1 - \\
&= e^{-(r+k)y} \left( \begin{bmatrix} t_1 \frac{a}{a_2} + \frac{a}{a_2} \\
1 - r + t_2 \frac{a}{a_2} + \frac{a}{a_2} \\
1 - r + t_3 \frac{a}{a_2} + \frac{a}{a_2} \\
1 - r + t_4 \frac{a}{a_2} + \frac{a}{a_2} \\
\end{bmatrix} \right) \otimes P \otimes W \otimes 1 - \\
&= e^{-(r+k)y} \left( \begin{bmatrix} t_1 \frac{a}{a_2} + \frac{a}{a_2} \\
1 - r + t_2 \frac{a}{a_2} + \frac{a}{a_2} \\
1 - r + t_3 \frac{a}{a_2} + \frac{a}{a_2} \\
1 - r + t_4 \frac{a}{a_2} + \frac{a}{a_2} \\
\end{bmatrix} \right) \otimes P \otimes W \otimes 1 - \\
&= e^{-(r+k)y} \left( \begin{bmatrix} t_1 \frac{a}{a_2} + \frac{a}{a_2} \\
1 - r + t_2 \frac{a}{a_2} + \frac{a}{a_2} \\
1 - r + t_3 \frac{a}{a_2} + \frac{a}{a_2} \\
1 - r + t_4 \frac{a}{a_2} + \frac{a}{a_2} \\
\end{bmatrix} \right) \otimes P \otimes W \otimes 1 - \\
&= e^{-(r+k)y} \left( \begin{bmatrix} t_1 \frac{a}{a_2} + \frac{a}{a_2} \\
1 - r + t_2 \frac{a}{a_2} + \frac{a}{a_2} \\
1 - r + t_3 \frac{a}{a_2} + \frac{a}{a_2} \\
1 - r + t_4 \frac{a}{a_2} + \frac{a}{a_2} \\
\end{bmatrix} \right) \otimes P \otimes W \otimes 1 - \\
&= e^{-(r+k)y} \left( \begin{bmatrix} t_1 \frac{a}{a_2} + \frac{a}{a_2} \\
1 - r + t_2 \frac{a}{a_2} + \frac{a}{a_2} \\
1 - r + t_3 \frac{a}{a_2} + \frac{a}{a_2} \\
1 - r + t_4 \frac{a}{a_2} + \frac{a}{a_2} \\
\end{bmatrix} \right) \otimes P \otimes W \otimes 1
\[ -e^{-(r+\gamma r)T_1 \frac{1-e^{-(k-1)r}T_2}{1-e^{-rT_2}}} + \frac{1}{T_1} \] ARB + \\
\frac{h_{ij}}{\beta_i} + \left\{ (a\beta - b) \left[ r \left( \frac{1}{\beta_i} - \frac{t_2^2}{a^2} - \frac{t_3^2}{\beta_i^2} \right) - (1 - r_t) \left( \frac{1}{\beta_i} + t_r \right) \right] + \frac{e^{\beta r_t}}{\beta_i} + b \beta \left[ r \left( \frac{t_2^2}{a^2} - \frac{t_3^2}{\beta_i^2} \right) - (1 - r_t) \left( \frac{t_2}{a} + \frac{t_3^2}{\beta_i^2} \right) \right] \right\} + \frac{B}{\beta_i} \left[ 1 - (1 - r_t)e^{-\alpha t_r} \right] - \\
\frac{3}{a} \left( \frac{e^{\alpha(T_2-T)} - 1}{a^2} \right) + \frac{t_2}{2} + t_3^2 \left( \frac{a^2}{a^2} - t_3^2 \right) + \frac{b_r}{\beta_i} \left[ (a - b) \left( \frac{1}{\beta_i} - \left( 1 - r_t \right) \left( \frac{1}{a^2} + \frac{t_2^2}{2} \right) + \right. \right. \\
\left. \left. \left( 1 - r_t \right) \left( \frac{e^{\beta r_t}}{\beta_i} + \frac{t_2^2}{2} \right) + r \left( \frac{t_2^2}{2} - \frac{t_2^2}{a^2} \right) \right\} \right\} + \\
\frac{1}{\beta_i} \left\{ (1 - r_T) \left( \frac{1}{\beta_i} + t_3 \right) \right. \\
\left. \left( \frac{e^{\beta r_t}}{\beta_i} - \frac{t_3^2}{2} \right) + b \left( 1 - r_t \right) \left( \frac{t_2^2}{2} - \frac{t_3^2}{a^2} \right) \right\} - \\
\frac{1}{\beta_i} \left\{ (1 + \delta T_1) \left( \frac{1}{\beta_i} - \frac{t_3^2}{2} \right) + b \left( 1 - r_t \right) \left( \frac{t_2^2}{2} - \frac{t_3^2}{a^2} \right) \right\} \right\}

And \( P(\hat{T}TIC) = \frac{\hat{T}TIC_M + 2\hat{T}TIC_{M2} + 2(1-\hat{C})\hat{T}TIC_{M3} + (1-\hat{C})\hat{T}TIC_{M4}}{3} \)

Where \( \hat{T}TIC_M = \frac{(h_{ij} + \gamma h_i)R_{dw}}{\beta} \left( 1 - \frac{r_{Uw}}{3\beta} \right) + \frac{Q_w}{\beta} \left( 1 - \frac{r_{Uw}}{2\beta} \right) \left( M_i + \frac{Q_i}{\beta} + B_i P \right) + \frac{\delta m}{\beta} \)

Now, problem is

Minimum \( P(\hat{T}TIC(t_r, T_1, k)) \)

Subject to \( t_r, T_1 \geq 0, k \) is an integer and and \( 0 \leq \delta \leq 1 \)

Minimum \( P(\hat{T}TIC_M(\delta)) \)

Subject to \( P > 0 \) and \( 0 \leq \delta \leq 1 \)

5. Solution Procedure

To derive the optimal solutions for wholesaler and retailer, the following classical optimization technique can be used.

Step-1: Take the partial derivatives of \( \hat{T}TIC \) with respect to \( t_r \) and \( T_1 \) and equating the results to zero. The necessary conditions for optimality are

\[
\frac{\partial P(\hat{T}TIC)}{\partial t_r} = 0 \quad \text{and} \quad \frac{\partial P(\hat{T}TIC)}{\partial T_1} = 0
\]

Step-2: The above simultaneous equations can be solved for \( t^*_r \) and \( T^*_1 \). \n
Step-3: With \( t^*_r \) and \( T^*_1 \) found in step 2, derive \( P(\hat{T}TIC(t^*_r, T^*_1)) \).

Step-4: Sufficient condition for the optimality of integrated cost of wholesaler and retailer is

\[
\frac{\partial^2 P(\hat{T}TIC)}{\partial t_r^2} > 0 \quad \text{and} \quad \frac{\partial^2 P(\hat{T}TIC)}{\partial T_1^2} > 0
\]

And \( \frac{\partial^2 P(\hat{T}TIC)}{\partial t_r^2} \cdot \frac{\partial^2 P(\hat{T}TIC)}{\partial T_1^2} - \left( \frac{\partial^2 P(\hat{T}TIC)}{\partial t_r \partial T_1} \right)^2 > 0 \)

Step-5: To derive the optimal value of \( k \) we use the following condition:

\[
P(\hat{T}TIC(t^*_r, T^*_1, k - 1)) \leq P(\hat{T}TIC(t^*_r, T^*_1, k)) \leq P(\hat{T}TIC(t^*_r, T^*_1, k + 1))
\]

Now, to derive the optimal solutions for manufacturer, the following classical optimization technique can be used.

Step-1: First we put the optimal value of \( Q_w \) in the expression of \( \hat{T}TIC_M(\delta) \).

Step-2: Find the first derivative of \( \hat{T}TIC_M(\delta) \) with respect to \( P \).

Step-3: Necessary condition for optimality is

\[
\frac{d\hat{T}TIC_M(\delta)}{dP} = 0
\]

Step-4: On solving above equation, we get the value of \( P^* \).

Step-5: Sufficient condition for the minimization of cost function is

\[
\frac{d^2\hat{T}TIC_M(\delta)}{dP^2} > 0
\]
6. Numerical Analysis:

ABC Company consists of one retailer and one wholesaler dealing with farming industry. Company purchases the product form farmer who worked as manufacturer. Here, retailer has two warehouses one is known as own warehouse and other is rented warehouse. The capacity of own warehouse is 200 units, deterioration rate in OW, \( \alpha = 8\% \), deterioration rate for wholesaler ends, \( \gamma = 3\% \). The fuzzy replenishment cost of the retailer \( (A_R) \) and wholesaler \( (A_W) \) is about 1500 and 2500 respectively and fuzzy setup cost for manufacturer \( (A_M) \) is about 200. The fuzzy purchasing cost of the retailer \( (P_R) \) and wholesaler \( (P_W) \) is about 8 and 3.5 respectively. Fuzzy raw material cost \( (M) \) is about 20, fuzzy labor cost \( (A) \) is about 2400 and fuzzy energy cost \( (B) \) is about 0.01. The fuzzy holding cost of the own warehouse \( (h_R) \), rented ware house \( (h_w) \), wholesaler \( (h_W) \), for raw products \( (h_r) \) and finished items \( (h_f) \) are about 0.4, 0.5, 0.3, 2.5 and 3.0 respectively. The fuzzy shortage cost \( (s) \) per unit item is greater or less than 4 and fuzzy lost sale cost \( (\tilde{h}) \) is greater or less than 20. Now, the objective is to determine \( t_r, t_w \) and \( k \) so that the total integrated inventory cost of retailer and wholesaler is minimum and value of \( P \) so that the manufacturer inventory cost is minimum.

Here we use a general rule to transfer the linguistic data, “greater or less than X" and “about X”, into trapezoidal fuzzy numbers as“greater or less than X” = \((0.9X, 0.95X, 0.95X, 1.1X)\).

“about X” = \((0.95X, X, X, 1.1X)\).

By the above rule, the fuzzy parameters in this example can be transferred as follows:

\[
\begin{align*}
A_R &= (1425, 1500, 1500, 1650), \\
A_W &= (2375, 2500, 2500, 2750), \\
P_R &= (7.6, 8, 8, 8), \\
P_W &= (3.3, 3.5, 3.5, 3.85), \\
h_R &= (0.38, 0.4, 0.4, 0.44), \\
h_w &= (0.48, 0.5, 0.5, 0.55), \\
h_W &= (29, 0.3, 0.3, 0.33), \\
s &= (3.6, 3.8, 3.8, 4.4), \\
t &= (18, 19, 19, 22) \\
A_M &= (190, 200, 200, 220), \\
h_r &= (2.375, 2.5, 2.5, 2.75), \\
h_f &= (2.85, 3.3, 3.3, 3.3), \\
M &= (19, 20, 20, 22), \\
\tilde{g} &= (2280, 2400, 2400, 2640), \\
B &= (0.0095, 0.0101, 0.0101, 0.0111)
\end{align*}
\]

In addition to above, following parameters are also considered:

\[
a = 400, \quad b = 0.05, \quad \text{inflation rate } (r) = 0.06
\]

Now, we replace the above fuzzy parameters and other different parameters in the expression of objective function and using the above solution process we get the optimal solution of the system as follows.

6.1 Sensitive Analysis w.r.t Deterioration Rate:

Effect of deterioration on wholesaler and retailer is tabulated in Table-1. Fig.4 and Fig.5 shows the effect of deterioration rate on the inventory cost of wholesaler and retailer. On increasing the value of \( \alpha \), wholesaler cost decreases whereas in other cases all are positively correlated. It is observed that wholesaler inventory cost is highly sensitive with respect to \( \gamma \) whereas least sensitive with respect to \( \alpha \) and inventory cost of retailer is highly sensitive with respect \( \beta \) and not effected by \( \gamma \).

<table>
<thead>
<tr>
<th>% change in deterioration rate</th>
<th>% change in inventory cost w.r.t ( \alpha )</th>
<th>% change in inventory cost w.r.t ( \beta )</th>
<th>% change in inventory cost w.r.t ( \gamma )</th>
</tr>
</thead>
<tbody>
<tr>
<td>For wholesaler</td>
<td>For retailer</td>
<td>For wholesaler</td>
<td>For retailer</td>
</tr>
<tr>
<td>-50</td>
<td>-1.13</td>
<td>-0.46</td>
<td>-1.96</td>
</tr>
<tr>
<td>-25</td>
<td>0.75</td>
<td>-2.73</td>
<td>-0.99</td>
</tr>
<tr>
<td>25</td>
<td>-0.69</td>
<td>2.81</td>
<td>1.02</td>
</tr>
<tr>
<td>50</td>
<td>-1.33</td>
<td>5.7</td>
<td>2.07</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>44.99</td>
</tr>
</tbody>
</table>

So, we can conclude that while making the inventory policy for wholesaler and retailer special focus should given to control the value of \( \gamma \). By doing so total inventory cost of the integrated system reduces.

6.2 Sensitive Analysis w.r.t Inflation Rate:

Effect of inflation on wholesaler, retailer and on supply chain is tabulated in Table-2. Fig.6 shows the effect of inflation on the inventory cost of wholesaler, retailer and supply chain. Inventory cost and inflation are negatively correlated to each other means as the inflation rate increases inventory cost decreases.
6.3 Sensitive Analysis w.r.t Purchasing Cost:-
Effect of purchasing cost of wholesaler and retailer is tabulated in Table-3 and graphically it is presented in Fig. 7 and Fig.8.

Table-3: Effect of Purchasing Cost on Different Player of Supply Chain

<table>
<thead>
<tr>
<th>% Change in parameter</th>
<th>% change in inventory cost w.r.t $p_w$</th>
<th>% change in inventory cost w.r.t $p_r$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>For wholesaler</td>
<td>For retailer</td>
</tr>
<tr>
<td></td>
<td>For wholesaler</td>
<td>For retailer</td>
</tr>
<tr>
<td>-50</td>
<td>0</td>
<td>-13.99</td>
</tr>
<tr>
<td>-25</td>
<td>0</td>
<td>-7.01</td>
</tr>
<tr>
<td>25</td>
<td>0</td>
<td>6.88</td>
</tr>
<tr>
<td>50</td>
<td>0</td>
<td>14.00</td>
</tr>
</tbody>
</table>

Table-2: Effect of Inflation rate on inventory cost of wholesaler, retailer and supply chain

<table>
<thead>
<tr>
<th>% change in Inflation</th>
<th>% change in wholesaler’s inventory cost</th>
<th>% change in retailer’s inventory cost</th>
<th>% change in integrated inventory cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>-50</td>
<td>6.012</td>
<td>9.898</td>
<td>8.98</td>
</tr>
<tr>
<td>-25</td>
<td>3.011</td>
<td>5.011</td>
<td>4.02</td>
</tr>
<tr>
<td>25</td>
<td>-2.989</td>
<td>-4.999</td>
<td>-4.01</td>
</tr>
<tr>
<td>50</td>
<td>-5.012</td>
<td>-10.001</td>
<td>-7.99</td>
</tr>
</tbody>
</table>

7. Conclusion:-
In this Paper, an integrated inventory model consisting wholesaler and retailer for deteriorating items is presented where retailer has two-warehouse system to store the stock. Whole of the study is carried out under the effect of inflation and partial backordering is considered at retailer’s end. It is assumed that holding cost at RW is higher as compared to OW. To obtain the optimal solution of the supply chain Leader-Follower approach has been used. This type of scenario is observed in farming industries where farmer worked as follower and the wholesaler and retailer worked as single entity. Demand rate is taken as linear function of time which is especially suitable for seasonal products.

The major contribution of the proposed model is the consideration of supply chain, inflation, partial backordering, volume flexibility and time dependent demand for production-inventory model for a three echelon supply chain model. Consideration of impreciseness in cost parameters increases the applicability and practicality of the proposed model. As lot of models are presented in the literature but we try to enrich the literature through this work As far as our knowledge is concern this type of supply chain model is not published by any researcher.

Through sensitive analysis, we provide the platform to the decision-maker what optimal policy they adopt with respect to the change in different inventory parameters. Form sensitive analysis, it observed that inventory cost is highly sensitive with respect to $\gamma$, inflation, purchasing cost of retailer and wholesaler. While making the optimal policy, decision-maker should pay special attention on these parameters. There is ample scope for further extension of the present research. Applicability of the proposed model can be enhancing by conserving imprecise demand rate and rate of inflation. We can also consider trade credit and stock dependent demand for better coordination. Optimal strategies can be obtained by taking all players as single entity.
References:-


