LINEAR TOPOLOGICAL SPACES OF CONTINUOUS VECTOR-VALUED FUNCTIONS

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Linear Topological Spaces of Continuous Vector-Valued Functions

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Introduction

This monograph is devoted to the study of linear topological spaces of continuous bounded vector-valued functions endowed with the uniform, strict, compact-open and weighted topologies. In the past four decades, several major results on vector-valued function spaces have been extended to the non-locally setting. These include generalized versions of some classical results such as the Stone-Weierstrass theorem, the Arzela-Ascoli theorem, and the Riesz representation theorem. These also include maximal ideal spaces of function algebras, separability and transseparability, weak approximation, composition operators, general strict topology on topological modules, the mean value theorem and almost periodicity for vector-valued functions, and non-Archimedean function spaces. Our main objective is to present recent developments in these areas. Several examples and counter-examples are included in the text. Background material on Topology and Functional Analysis is included in the Appendix. Also, an up-to-date bibliography is included to assist research in further studies.

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Preface

The purpose of this monograph is to present an up-to-date study of linear topological spaces of continuous bounded vector-valued functions endowed with the uniform, strict, compact-open and weighted topologies. In the past four decades, several new results have been obtained in this direction. The main topics include generalized versions of some classical results such as the Stone-Weierstrass theorem, the Arzela-Ascoli theorem, and the Riesz representation theorem. These also include the study of maximal ideal spaces of function algebras, separability and trans-separability, weak approximation, composition operators, general strict topology on topological modules, the mean value theorem and almost periodicity for vector-valued functions, and non-Archimedean function spaces. Our main objective is to present recent developments in the non-locally convex setting, that is, in topological vector spaces, not necessarily locally convex. So it covers both the cases: (a) locally convex spaces, (b) topological vector spaces which are not locally convex. We mention that "non-convexity" has also been the main theme in the monographs by L. Waelbroeck (1971), N. Adasch, B. Ernst and D. Keim (1978), N.J. Kalton, N.T. Peck and J.W. Roberts (1984), S. Rolewicz (1985) and A. Bayoumi (2003); see also the papers by W. Robertson (1958), V. Klee (1960a, 1960b) and J. Kakol (1985, 1987, 1990, 1992).

Our exposition relies mostly on the original papers (published during 1972-2012) with some additional clarifications. We have tried to make the text easily readable and as self-contained as possible. The only prerequisites for reading the book are topology, functional analysis and measure theory of the undergraduate level (e.g. the material given in H.L. Royden’s book: Real Analysis). For this purpose, we have included background material on Topology and Functional Analysis (such as topological spaces, topological vector spaces, non-Archimedean functional analysis, topological algebras and measure theory) in Appendix A. Also, an up-to-date bibliography is included to assist research in further studies. The monograph is intended basically a monograph for researchers working in vector-valued function spaces. However, those working in this and
related fields may conveniently choose some topics for a one or two semester course work. In fact, the author has been using portions of the text in his lectures to M.Sc./M.Phil. students as a one semester course before initiating their research in a relevant field of function spaces.

To introduce the material in its historical perspective, consider a topological space $X$ and a topological vector space $E$ over $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$). Let $C(X, E)$ (resp. $C_b(X, E)$) be the vector space of all continuous (resp. continuous and bounded) $E$-valued functions on $X$; if $E = \mathbb{K}$, these spaces are simply denoted by $C(X)$ and $C_b(X)$. If $X$ is compact, then $C(X, E) = C_b(X, E)$ and $C(X) = C_b(X)$; in this case the uniform topology (i.e. the sup norm topology) $u$ is the appropriate one to study on $C(X)$. After the appearance of M.H. Stone’s paper of 1937, the space $(C(X), u)$ in the case of compact $X$ has been an intensively studied mathematical object. Its interest arises in part from its rich structure: under the uniform topology $u$, $C(X)$ is a Banach space; under pointwise multiplication, it is an algebra; under the natural ordering, $C(X, \mathbb{R})$ is a lattice (see, e.g., Kakutani (1941), Kaplansky (1947b), Hewitt (1948), Myers (1950), M. and S. Krein (1940), etc.). If $X$ is not compact, the uniform topology $u$ on $C(X)$ is not well-defined since $C(X)$ may contain some unbounded functions. However, in this case, the compact-open topology $k$ is the most useful topology to be considered on $C(X)$ and on $C(X, E)$ with $E$ even a topological space. It has been studied by several authors over the past sixty years and significant contributions have been made in this field, among others, by Fox (1945), Arens (1946), Myers (1946), Hewitt (1948), Warner (1958), Wheeler (1976), McCoy (1980); see also the monographs by Semadeni (1971), Schmets (1983), McCoy and Ntantu (1988), Arkhangel’skii (1992) and Tkachuk (2011). If $X$ is again non-compact and we consider the spaces $C_b(X)$ and $C_b(X, E)$, then both $u$ and $k$ topologies are well-defined and we have $k \leq u$.

In 1958, R.C. Buck introduced the notion of strict topology $\beta$ on $C_b(X, E)$ in the case of $X$ locally compact and $E$ a locally convex space. The problems discussed in the Buck’s paper (1958) are:

1. Relationship between the $\beta, k$ and $u$ topologies on $C_b(X, E)$ (e.g. $k \leq \beta \leq u$ with $k = u$ iff $X$ is compact; $k = \beta$ on $u$-bounded sets);
2. Completeness of $C_b(X, E, \beta)$;
3. Stone-Weierstrass theorems for $(C_b(X, \beta)$ and $C_b(X, E_u, \beta)$;
4. Characterization of maximal $\beta$-closed ideals in $C_b(X)$;
5. Identification of the $\beta$-dual of $C_b(X)$ with the space $M(Bo(X))$ of regular Borel measures on $X$, via the integral representations;
6. The open problem whether or not $(C_b(X), \beta)$ a Mackey space.
After the appearance of the Buck’s paper, a large number of papers have appeared in the literature concerned with extending Buck’s results to more general cases or studying further properties of $\beta$, and also with obtaining some variants of $\beta$. In particular Todd (1965) and Wells (1965), independently, established the Stone-Weierstrass theorem for $(C_b(X), \beta)$. Todd (1965) also characterized the maximal $\beta$-closed $C_b(X)$-submodules of $C_b(X, E)$ while Wells (1965) identified the $\beta$-dual of $C_b(X, E)$ with a certain space $M(X, E^*)$ of $E^*$-valued measures on $X$. Conway (1967) and LeCam (1957), independently, proved that, if $X$ is locally compact and paracompact, $(C_b(X, E), \beta)$ is a Mackey space but it is not so in general. Collins and Dorroh (1968) proved that $(C_b(X), \beta)$ has the approximation property. Dorroh (1969) also showed that $\beta$ is the finest locally convex topology on $C_b(X)$ which agrees with $k$ on $u$-bounded sets.

The next major development in the study of $\beta$ topology on $C_b(X)$ has been its extension to the case of a completely regular space $X$, given, independently, by Van Rooij (1967), Giles (1971), Fremlin, Garling and Haydon (1972), Gulick (1972), Hoffman-Jørgensen (1972), and Sentilles (1972). In fact, Fremlin, Garling and Haydon (1972) and, independently, Sentilles (1972) introduced on $C_b(X)$ three types of strict topologies: the substrict topology $\beta_0$, the strict topology $\beta$ and the superstrict topology $\beta_1$ with $\beta$ being equivalent to the strict topology ‘$\beta$’ of Buck (1958) in the case of $X$ a locally compact space. They also identified the topological duals of $(C_b(X), \beta_0)$, $(C_b(X), \beta)$ and $(C_b(X), \beta_1)$ with the spaces $M_1(X)$, $M_2(X)$ and $M_3(X)$ of tight, $\tau$-additive and $\sigma$-additive measures, respectively, on $X$ (see LeCam (1957), Varadarajan (1965) and Wheeler (1983) for detail of these spaces of measures). Subsequently, several authors have further explored the properties of $\beta_0$, $\beta$ and $\beta_1$ for both $C_b(X)$ and $C_b(X, E)$, where $E$ is a normed space or a locally convex space; see, e.g., Mosiman and Wheeler (1972), Wheeler (1973), Summers (1972), Haydon (1976), Cooper (1971), Choo (1979), Fontenot (1974), Katsaras (1975), Khurana (1978a, 1978b), Morishita and Khan (1997), Zafarani (1986, 1988), and also the survey papers by Hirschfeld (1978), Collins (1976) and Wheeler (1983), and monographs by Cooper (1978), Prolla (1977), Schmets (1983), Singh and Manhas (1993).

In 1967, Nachbin studied in detail the more general notion of weighted topology $\omega_V$ on certain subspaces $C\nu_b(X)$ and $C\nu_u(X)$ of $C(X)$ and $\beta, k, u$ were shown to be the special cases of $\omega_V$ for suitable choices of $V$ consisting of non-negative upper-semicontinuous functions on $X$. Further work on $\omega_V$ has been done by Summers (1969, 1971), Prolla
Bierstedt (1973, 1975), and others. Singh and Summers (1988) and Singh and Manhas (1991, 1992) made an extensive study of composition and multiplication operators, respectively on $CV_b(X, E)$. Another useful topology on $C_b(X)$ is the $\sigma$-compact-open topology $\sigma$ which was introduced by Gulick (1972) and further studied by Gulick and Schmets (1972). The strict and related topologies have also been studied on not necessarily ‘function spaces’. For instance, Wiwegar (1961) and Cooper (1971, 1978) defined it in the form of ‘mixed topology’ on a normed space; Busby (1968) considered it on the double centralizer (or multiplier) algebra $M_d(A)$ of Banach algebra $A$; Sentilles and Taylor (1969) defined it on a Banach $A$-module; Ruess (1977) studied it on an arbitrary locally convex space (see the survey paper by Collins (1976)).

In all the above mentioned investigations about $C_b(X, E)$, $E$ has been assumed to be the scalar field or a locally convex space. The case of $E$ a general topological vector space has been first considered by Shuchat in (1972a, 1972b) where he established several useful approximation results for $(C_b(X, E), u)$ and characterized the dual of $(C_b(X, E), u)$, with $X$ a compact space and without assuming the local convexity of $E$ (see also Bierstedt (1973) and Wealbroeck (1971, 1973)). In 1979, Khan defined the $\beta$ and other related topologies on $C_b(X, E)$, where $X$ is a Hausdorff space and $E$ any Hausdorff topological vector space, and showed that $\beta$ has almost all the properties of the “strict topology” studied by the above authors. Further work in this direction has been done in later years by Khan (1980 through 2011), Khan-Rowlands (1981, 1991), Katsaras (1981, 1983), Kalton (1983), Nawrocki (1985, 1989), Abel (1987, 2004), Prolla (1993b), Khan-Thaheem (1997, 2002), Manhas-Singh (1998), Khan-Mohammad-Thaheem (1999, 2005), Khan-Oubbi (2005), Katsaras-Khan-Khan (2011), Katsaras (2011) and others. The purpose of this monograph is to present some results of these authors regarding the strict, weighted, and related topologies on $C_b(X, E)$ and $CV_b(X, E)$ in the non-locally convex setting. We also study the strict and related topologies on any topological modules. Several examples and counter-examples are included in the text.

The monograph consists of twelve chapters which are organized as follows.

In Chapter 1, we introduce and study the strict topology $\beta$ (and the related ones such as the uniform topology $u$, compact-open topology $k$, $\sigma$-compact-open topology $\sigma$) and also the more general notion of weighted topology $\omega_V$ on the function spaces $C_b(X, E)$ (resp. $CV_b(X, E)$) with $E$.
not necessarily a locally convex topological vector space. Here, we are mainly concerned with their linear topological properties.

Chapter 2 deals with completeness and quasi-completeness of function spaces under the above mentioned topologies.

In Chapter 3, we establish the Arzela-Ascoli type theorems which characterize the $\beta$-compact and $\omega_V$-precompact subsets of $C_b(X, E)$ and $CV_b(X, E)$, respectively.

Chapter 4 contains various generalizations of the famous Stone-Weierstrass theorem and related results. It is perhaps the most important one from the point of view of its applications.

Chapter 5 deals with characterization of maximal and closed ideals in $(C_b(X, E), \beta)$ and $(C(X, E), k)$, $E$ a topological algebra, and also their maximal ideal spaces $\Delta((C_b(X, E), \beta))$ and $\Delta((C(X, E), k))$.

In Chapter 6, we present various versions of the M and S. Krein type theorem for separability and trans-separability in the $\beta, k, \sigma, \text{and} u$ topologies.

Chapter 7 deals with the problem of weak approximation in $(C_b(X, E), \beta)$ and $(CV_b(X, E), \omega_V)$, using some basic terminology of Measure Theory.

In Chapter 8, we first study the integral of functions in $C_b(X, E)$ with respect to certain $E^*$-valued measures, and then obtain the Riesz representation type theorems for the dual of function spaces under various topologies.

In Chapter 9, we study necessary and sufficient conditions for certain maps $M_\theta, M_\psi, C_\phi$ and $W_{\pi, \varphi}$ to be the multiplication or composition operators on the weighted spaces $CV_b(X, E)$ and $CV_b(X, E)$. We also include characterizations of compact weighted composition operators and compact multiplication operators.

We have treated the non-commutative analogue of the strict and related topologies in Chapter 10. In fact, we introduce and study the general strict and related topologies on a topological left $A$-module $Y$. We also study properties of the topological module $Hom_A(A, Y)$ of continuous homomorphisms.

Chapter 11 consists of some results on the mean value theorem and almost periodicity for vector-valued functions.

In Chapter 12, we have considered the study of compact-open and strict topologies on non-Archimedian function spaces.

Finally, in the Appendix A, we have included basic definitions and results on Topological Spaces, Topological Vector Spaces, Non-Archimede
Functional Analysis, Topological Algebras, Topological Modules, Measure Theory and Uniform Spaces. Most of their proofs (with the exception of some selected results) are omitted since they are available in any standard book on these topics.

At the end of each chapter, the section ”Notes and Comments” contains references to the literature and gives some further information on the results in the text. Some references are also included in the text to clarify the source of information. The bibliography lists some papers that are not specifically referred but can be useful for supplementary material. We have treated the cases of strict topology and weighted topology in separate sections in most of the chapters. Any one interested only in more familiar results on uniform, strict and compact-open topologies can easily skip the material on the more abstract notion of weighted topology.

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CHAPTER 1

The Strict and Weighted Topologies

In this chapter, we study the uniform, strict, compact-open, \( \sigma \)-compact-open, weighted and other related topologies on \( C_b(X, E) \) or \( CV_b(X, E) \) and their linear topological properties in the non-locally convex setting.

Throughout this monograph, we shall assume, unless stated otherwise, that \( X \) is a completely regular Hausdorff space and \( E \) a non-trivial Hausdorff topological vector space (in short, a TVS) over the field \( \mathcal{K} = (\mathbb{R} \text{ or } \mathbb{C}) \), and we let \( W \) denote a base of open balanced neighborhoods of 0 in \( E \). Most of the undefined terminology can be found in the Appendix A (A.1-A.8).
1. The Strict and Weighted Topologies on $C_b(X, E)$

The notion of the **strict topology** $\beta$ on $C_b(X, E)$ was first introduced by Buck [Buc58] in 1958 in the case of $X$ a locally compact space and $E$ a locally convex TVS. Since then a large number of papers have appeared in the literature concerned with extending the results contained in Buck’s paper; see the Preface and the list of references. In this section, a general method for defining linear topologies on $C_b(X, E)$, called the $S$-topologies, is considered first. Using the notion of an $S$-topology, the uniform topology $\upsilon$, strict topology $\beta$, compact-open topology $k$, $\sigma$-compact-open topology $\sigma$, countable-open-topology $\sigma_o$, and pointwise topology $p$ are introduced. We include here some results from [Kh79, KR91, Kat81], where $E$ is a Hausdorff TVS but not necessarily locally convex.

**Definition.** A function $f : X \to E$ is said to be **bounded** if $f(X)$ is a bounded subset of $E$, i.e. for each $W \in W$, there exists an $r > 0$ such that $f(X) \subseteq \lambda W$ for all $\lambda \in K$ with $|\lambda| \geq r$. $f : X \to E$ is said to **vanish at infinity** if, for each $W \in W$, the set

$$F_W = \{x \in X : f(x) \notin W\}$$

is relatively compact in $X$; in particular, a function $\varphi : X \to K$ is said to **vanish at infinity** if, for each $\varepsilon > 0$, the set

$$F_\varepsilon = \{x \in X : |\varphi(x)| \geq \varepsilon\}$$

is relatively compact in $X$. The **support** of $f : X \to E$ is defined as the closure of the set $\{x \in X : f(x) \neq 0\}$ in $X$ and we write

$$\text{supp}(f) = \text{cl} - \{x \in X : f(x) \neq 0\}.$$
Since we shall frequently use the notion of a function vanishing at infinity, the following three lemmas are useful in this regard.

**Lemma 1.1.1** Let \( \varphi : X \to \mathbb{K} \) be a function. Then:

(a) \( \varphi \) vanishes at infinity iff, for each \( \varepsilon > 0 \), there exists a compact set \( K_\varepsilon \subseteq X \) such that

\[
| \varphi(x) | < \varepsilon \quad \text{for all } x \in X \setminus K_\varepsilon.
\]

(b) If \( \varphi \) is non-negative upper semi-continuous (in particular, continuous), then \( \varphi \) vanishes at infinity iff for every \( \varepsilon > 0 \), the set \( F_\varepsilon = \{ t \in X : |\varphi(t)| \geq \varepsilon \} \) is compact iff for every \( \varepsilon > 0 \), there exists a compact set \( K_\varepsilon \subseteq X \) such that

\[
| \varphi(x) | < \varepsilon \quad \text{for all } x \in X \setminus K_\varepsilon.
\]

**Proof.** (a) Suppose \( \varphi \) vanishes at infinity, and let \( \varepsilon > 0 \). Then the set \( F_\varepsilon = \{ x \in X : |f(x)| \geq \varepsilon \} \) is relatively compact in \( X \). Now the set \( K_\varepsilon = \overline{F_\varepsilon} \) is compact. If \( x \in X \setminus K_\varepsilon \), then \( x \in X \setminus F_\varepsilon \); hence \( |\varphi(x)| < \varepsilon \).

Conversely, let \( \varepsilon > 0 \), and let \( K_\varepsilon \) be the compact set such that \( |\varphi(x)| < \varepsilon \) for all \( x \in X \setminus K_\varepsilon \). If \( x \notin K_\varepsilon \), \( |\varphi(x)| < \varepsilon \) and so clearly \( x \notin F_\varepsilon \). Hence \( F_\varepsilon \subseteq K_\varepsilon \). Now \( K_\varepsilon \), being a compact subset of the Hausdorff space \( X \), is closed. Therefore \( \overline{F_\varepsilon} \subseteq \overline{K_\varepsilon} = K_\varepsilon \). Then \( \overline{F_\varepsilon} \), being a closed subset of compact set \( K_\varepsilon \), is compact.

(b) If \( \varphi \) is a non-negative upper semi-continuous, then the set \( \{ x \in X : \varphi(x) < \varepsilon \} = \{ x \in X : |\varphi(x)| < \varepsilon \} \) is open, or that \( F_\varepsilon \) is closed. Then the result follows from (a). \( \square \)

Analogously, we also obtain:

**Lemma 1.1.2** Let \( f : X \to E \) be a function. Then:

(a) \( f \) vanishes at infinity iff, for each \( W \in \mathcal{W} \), there exists a compact set \( K_W \subseteq X \) such that

\[
f(x) \in W \quad \text{for all } x \in X \setminus K_W.
\]

(b) If \( f \) is continuous, then \( f \) vanishes at infinity iff, for each \( W \in \mathcal{W} \), the set \( F_W = \{ x \in X : f(x) \notin W \} \) is compact iff for each \( W \in \mathcal{W} \), there exists a compact set \( K_W \subseteq X \) such that

\[
f(x) \in W \quad \text{for all } x \in X \setminus K_W. \square
\]

**Lemma 1.1.3.** Let \( f : X \to E \) be a function and suppose \( \varphi f \in B(X, E) \) for each \( \varphi \in B_0(X) \). Then \( f \in B(X, E) \).

**Proof.** Suppose \( f \) is not bounded. Then there exist a sequence \( \{ x_n \} \subseteq X \), and a \( W \in \mathcal{W} \) such that \( f(x_n) \notin n^2 W \). Define \( \varphi : X \to \mathbb{R} \) by

\[
\varphi(x) = 1/n \text{ if } x = x_n, \quad \text{and } \varphi(x) = 0 \text{ if } x \neq x_n \quad (n = 1, 2, \ldots).
\]
Then $\varphi \in B_0(X)$. [Clearly $\varphi$ is bounded. To show that $\varphi$ vanishes at infinity, let $\varepsilon > 0$. Choose $N \geq 1$ such that $\frac{1}{N} < \varepsilon$. Then $K = \{x_1, \ldots, x_N\}$ is compact and if $x \notin K = \{x_1, \ldots, x_N\}$, then
\[
\varphi(x) \leq \frac{1}{N + 1} < \frac{1}{N} < \varepsilon.
\]
Alternatively,
\[
\{x \in X : \varphi(x) \geq \varepsilon\} = \{x \in X : \frac{1}{N} \leq \varphi(x) \leq 1\} = \{x_1, \ldots, x_N\},
\]
which, being a finite set, is compact in $X$. Now, clearly $\varphi(x_n) = \frac{1}{n} \varphi(x_n) \notin nW$ for all $n = 1, 2, \ldots$.
This implies that $\varphi f$ is not bounded, which contradicts our hypothesis. Thus $f \in B(X, E)$. □

Remarks. (1) The above result need not hold if $X$ is not Hausdorff. For example, let $X = \mathbb{N}$ with the topology
\[
\tau = \{\mathbb{N}\} \cup \{A_n : n \in \mathbb{N}\},
\]
where $A_n = \{i : i \in N, i < n\}$ for every $n \in \mathbb{N}$. This is $T_0$ but not $T_1$ (hence also not Hausdorff). In this case, the only closed compact subset of $X$ is $\emptyset$, so the only relatively compact set is $\emptyset$. Consequently, the only member of $B_0(\mathbb{N})$ is the constant function with value 0. Now consider the TVS $E = \mathbb{R}$ and the identity function $f : \mathbb{N} \to \mathbb{R}$, $f(n) = n$, $n \in \mathbb{N}$. Then $\varphi f$ is bounded for every $\varphi \in B_0(\mathbb{N})$, because there is only the zero function $\varphi$ to consider; but $f$ is not bounded.

(2) Clearly, a finite subset $A$ of a topological space $X$ is always compact; but need not be closed. So it need not be relatively compact (e.g. if $X$ is neither $T_1$ nor regular).

Notation. For any $\varphi \in B(X)$ and $W \in \mathcal{W}$, we define
\[
N(\varphi, W) = \{f \in C_b(X, E) : \varphi(x)f(x) \in W \text{ for all } x \in X\}.
\]
If $A \subseteq X$ and $W \in \mathcal{W}$, then we write $N(A, W)$ to mean
\[
N(A, W) := N(\chi_A, W) = \{f \in C_b(X, E) : f(A) \subseteq W\},
\]
where $\chi_A$ is the characteristic function of $A$.

Lemma 1.1.4. Let $\varphi \in B(X)$ and $W \in \mathcal{W}$. Then
(a) If $V \in \mathcal{W}$ with $V \subseteq W$, then $N(\varphi, V) \subseteq N(\varphi, W)$.
(b) If $U, V \in \mathcal{W}$ with $U + V \subseteq W$, then $N(\varphi, U) + N(\varphi, V) \subseteq N(\varphi, W)$.
(c) For any $\lambda(\neq 0) \in \mathbb{K}$, $N(\varphi, \lambda W) = \lambda N(\varphi, W)$.
1. STRICT AND RELATED TOPOLOGIES ON $C_b(X, E)$

(d) $N(\varphi, W)$ is absorbing.

(e) If $W$ is balanced (resp. convex), then $N(\varphi, W)$ is also balanced (resp. convex).

Proof. (a) Let $f \in N(\varphi, V)$. Then, for any $x \in X$, $\varphi(x)f(x) \in V \subseteq W$ and so $f \in N(\varphi, W)$.

(b) Let $f \in N(\varphi, U)$ and $g \in N(\varphi, V)$. Then, for any $x \in X$, $\varphi(x)f(x) \in U$ and $\varphi(x)g(x) \in V$, and so

$$[\varphi(f + g)](x) = \varphi(x)f(x) + \varphi(x)g(x) \in U + V \subseteq W;$$

that is, $f + g \in N(\varphi, W)$. Hence $N(\varphi, V) + N(\varphi, V) \subseteq N(\varphi, W)$.

(c) If $\lambda \neq 0 \in \mathbb{K}$,

$$N(\varphi, \lambda W) = \{ f \in C_b(X, E) : \varphi(x)f(x) \in \lambda W \text{ for all } x \in X \}$$

$$= \lambda \{ \frac{1}{\lambda} f \in C_b(X, E) : \varphi(x)f(x) \in \lambda W \text{ for all } x \in X \}$$

$$= \lambda \{ \frac{1}{\lambda} f \in C_b(X, E) : \varphi(x)(\frac{1}{\lambda} f)(x) \in W \text{ for all } x \in X \}$$

$$= \lambda \{ g \in C_b(X, E) : \varphi(x)g(x) \in W \text{ for all } x \in X \}$$

$$= \lambda N(\varphi, W).$$

(d) Let $f \in C_b(X, E)$. Since $(\varphi f)(X)$ is bounded in $E$ and $W$ is absorbing, there exists $r > 0$ such that $(\varphi f)(X) \subseteq \lambda W$ for all $|\lambda| \geq r$. Hence, by (c),

$$\varphi f \in N(\varphi, \lambda W) = \lambda N(\varphi, W) \text{ for all } |\lambda| \geq r,$$

showing that $N(\varphi, W)$ is absorbing.

(e)(i) Suppose $W$ is balanced, and let $\lambda \in \mathbb{K}$ with $|\lambda| \leq 1$. Then $\lambda W \subseteq W$ and so, by (a) and (c),

$$\lambda N(\varphi, W) = N(\varphi, \lambda W) \subseteq N(\varphi, W).$$

Hence $N(\varphi, W)$ is balanced.

(ii) Suppose $W$ is convex, and let $t \in \mathbb{R}$ with $0 < t < 1$ and $f, g \in N(\varphi, W)$. Then $tW + (1 - t)W \subseteq W$, and so by (b) and (c),

$$tN(\varphi, W) + (1-t)N(\varphi, W) = N(\varphi, tW) + N(\varphi, (1-t)W) \subseteq N(\varphi, tW + (1-t)W) \subseteq N(\varphi, W).$$

Hence $N(\varphi, W)$ is convex.

We now describe a general method of defining linear topologies on $C_b(X, E)$. □
Definition. Let $S$ be any subset of $B(X)$. We define the $S$-topology on $C_b(X, E)$ to be the linear topology which has a subbase of neighborhoods of 0 consisting of all sets of the form $N(\varphi, W)$, where $\varphi \in S$ and $W \in \mathcal{W}$.

Definition. [Gil71] Let $\varphi, \varphi_1 \in B(X)$. Then $\varphi_1$ is said to dominate $\varphi$ if there exists a $t > 0$ such that $|\varphi(x)| \leq t|\varphi_1(x)|$ for all $x \in X$.

Lemma 1.1.5. [Gil71, Kh79]

Let $S$ and $S_1$ be two subsets of $B(X)$. If each element of $S$ is dominated by an element of $S_1$, then the $S$-topology on $C_b(X, E)$ is weaker than the $S_1$-topology.

Proof. Let $\mathcal{N}$ be any $S$-neighborhood of 0 in $C_b(X, E)$, and suppose

$$\mathcal{N} \supseteq \bigcap_{i=1}^{n} N(\varphi_i, W_i),$$

where $\varphi_1, ..., \varphi_n \in S$ and $W_1, ..., W_n \in \mathcal{W}$. For each $\varphi_i$ ($i = 1, ..., n$), choose a $t_i > 0$ and a $\psi_i \in S_1$ such that $|\varphi_i(x)| \leq t_i|\psi_i(x)|$ for all $x \in X$.

Let

$$\mathcal{N}_1 = \bigcap_{i=1}^{n} N(\psi_i, \frac{1}{t}W_i),$$

where $t = \max\{t_1, ..., t_n\}$. Then $\mathcal{N}_1$ is an $S_1$-neighborhood of 0 in $C_b(X, E)$. Further $\mathcal{N}_1 \subseteq \mathcal{N}$. [Let $g \in \mathcal{N}_1$. Then, for any $x \in X$ and $1 \leq i \leq n$,

$$\psi_i(x)g(x) \in \frac{1}{t}W_i;$$

since $|\varphi_i(x)| \leq t_i|\psi_i(x)| \leq t|\psi_i(x)|$ and $W_i$ is balanced,

$$\varphi_i(x)g(x) \in t\psi_i(x)W_i \subseteq W_i.$$

Therefore $g \in \bigcap_{i=1}^{n} N(\varphi_i, W_i) \subseteq \mathcal{N}$.] Hence $\mathcal{N}$ is an $S_1$-neighborhood of 0 in $C_b(X, E)$. □

Definition. Using the notion of an $S$-topology, we now introduce the strict topology, the $\sigma$-compact-open topology, and other related topologies on $C_b(X, E)$, as follows.
1. STRICT AND RELATED TOPOLOGIES ON $C_b(X, E)$

$S$ $S$-Topology

$B(X)$ uniform topology $u$

$B_o(X)$ strict Topology $\beta$

$B_{oo}(X)$ compact-open topology $k$

$B_{\sigma}(X)$ $\sigma$-compact-open topology $\sigma$

$B_{\sigma_o}(X)$ countable-open-topology $\sigma_o$

$B_p(X)$ pointwise topology $p$

**Remark 1.1.6.** It easily follows from Lemma 1.1.5 that:

1. The $u$-topology on $C_b(X, E)$ is the same as the $\{1\}$-topology, where $1 \in B(X)$ is the function identically 1 on $X$.

2. The topology $k$ (resp. $\sigma, \sigma_o, p$) on $C_b(X, E)$ is the linear topology which has a subbase of neighborhoods of 0 consisting of all sets of the form $N(K, W)$, where $K$ is any compact (resp. $\sigma$-compact, countable, finite) subset of $X$ and $W \in \mathcal{W}$.

3. Since $B_p(X) \subseteq B_{oo}(X) \subseteq B_o(X) \subseteq B(X)$ and $B_p(X) \subseteq B_{\sigma_o}(X) \subseteq B_{\sigma}(X) \subseteq B(X)$, it is easy to verify that $p \leq k \leq \beta \leq u$ and $p \leq \sigma_o \leq \sigma \leq u$.

Further, note that if $X$ is compact, then $k = u$; if $X$ is discrete, then $p = k$ (since every compact set in a discrete space is finite).

We mention that the topologies $p$ and $k$ can also be considered on the larger space $C(X, E)$.

The following lemma gives us a convenient form for a base of neighborhoods of 0 in $C_b(X, E)$ for each of the topologies defined above.

**Lemma 1.1.7.** [Kh79]

Let $S$ denote any one of the sets $B(X)$, $B_o(X)$, $B_{oo}(X)$, $B_{\sigma}(X)$, $B_{\sigma_o}(X)$, or $B_p(X)$. Then the $S$-topology on $C_b(X, E)$ has a base of neighborhoods of 0 consisting of all sets of the form $N(\varphi, W)$, where $\varphi \in S$ with $0 \leq \varphi \leq 1$ and $W \in \mathcal{W}$.

**Proof.** Let $\mathcal{N}$ be any $S$neighborhood of 0, and suppose

$$\mathcal{N} \supseteq \bigcap_{i=1}^{m} N(\varphi_i, W_i),$$

where $\varphi_1, \ldots, \varphi_m \in S$ and $W_1, \ldots, W_m \in \mathcal{W}$. Let $t = \max_{1 \leq i \leq m} \{\|\varphi_i\|\}$. If $t > 0$, choose a $W \in \mathcal{W}$ with $tW \subseteq \bigcap_{i=1}^{m} W_i$. Define

$$\varphi(x) = \max_{1 \leq i \leq m} \left\{ \frac{|\varphi_i(x)|}{t} \right\} \ (x \in X).$$
Then \( \varphi \in S \), \( 0 \leq \varphi \leq 1 \). Further, \( N(\varphi, W) \subseteq \mathcal{N} \). [Let \( g \in N(\varphi, W) \).]

Since, for each \( x \in X \) and \( 1 \leq i \leq m \), \( |\varphi_i(x)| \leq t\varphi(x) \) and \( \varphi(x)g(x) \in W_i \); we have

\[
\varphi_i(x)g(x) \in tW \subseteq W_i;
\]

that is, \( g \in \cap_{i=1}^m N(\varphi_i, W_i) \subseteq \mathcal{N} \). If \( t = 0 \), then each \( \varphi_i = 0 \) and so

\[
N(\varphi_i, W_i) = \{ f \in C_b(X, E) : 0f(x) \in W \text{ for all } x \in X \} = C_b(X, E).
\]

Hence \( \mathcal{N} = C_b(X, E) \), and so, if we take \( \varphi = 0 \), we have \( \mathcal{N} \supseteq N(\varphi, W) \) for any \( W \) in \( \mathcal{W} \). Thus the S-neghborhoods of 0 have a base of the required form.

We first study some basic properties of the topology \( \beta \) and its relation with the topologies \( k \) and \( u \). (For convenience, the properties of the topologies \( \sigma_o \) and \( \sigma \) will be given separately in a later theorem.)

**Theorem 1.1.8.** [Buc58, Kh79]

(a) \( p \leq k \leq \beta \leq u \).

(b) If \( X \) is completely regular, then:

(i) \( u = \beta \) iff \( X \) is compact.

(ii) \( \beta = k \) iff every \( \sigma \)-compact subset of \( X \) is relatively compact.

(c) \( u \) and \( \beta \) have the same bounded sets in \( C_b(X, E) \).

(d) \( \beta = k \) on \( u \)-bounded subsets of \( C_b(X, E) \).

(e) A sequence \( \{f_n\} \) in \( C_b(X, E) \) is \( \beta \)-convergent iff it is \( u \)-bounded and \( k \)-convergent.

**Proof.** (a) Since every element of \( B_o(X) \) is dominated by \( 1 \in B(X) \), it follows from Lemma 1.1.5 that the \( B_o(X) \)-topology is weaker than the \( \{1\} \)-topology, that is, \( \beta \leq u \). Similarly, since \( B_{oo}(X) \subseteq B_o(X) \), we have \( k \leq \beta \).

(b) (i) Suppose \( u \leq \beta \). Then, by Lemma 1.1.7, for any \( W \in \mathcal{W} \), there exist a \( \varphi \in B_o(X) \) with \( 0 \leq \varphi \leq 1 \) and a \( V \in \mathcal{W} \) such that

\[
N(\varphi, V) \subseteq N(1, W).
\]

If \( E \setminus W \neq \emptyset \), let \( c \in E \setminus W \) and choose \( t > 0 \) such that \( c \in tV \). If \( X \) is not compact, then \( X \setminus F \neq \emptyset \) for every compact set \( F \) in \( X \). Since \( \varphi \in B_0(X) \), the set \( K = \text{cl} - \{x \in X : \varphi(x) \geq 1/t\} \) is compact in \( X \). Let \( x_o \in X \setminus K \). By complete regularity, there exists a \( \psi \in C_b(X) \) such that

\[
0 \leq \psi \leq 1, \psi(x_o) = 1 \quad \text{and} \quad \psi(K) = 0.
\]

Let \( g = \psi \odot c \). Then \( g \in N(\varphi, V) \). [Let \( x \in X \). If \( x \in K \), then \( \psi(x) = 0 \) and so

\[
\varphi(x)g(x) = \varphi(x)\psi(x)c = 0 \in V;
\]
If \( x \notin K \), then \( t\varphi(x) < 1 \) and so
\[
\varphi(x)g(x) = \varphi(x)\psi(x)c \in \varphi(x)\psi(x)tV \subseteq V.]
\]

But \( g \notin N(1, W) \) (since \( g(x_o) = \psi(x_o)c = c \notin W \)), which is a contradiction. If \( W = E \), choose a \( W_0 \) in \( W \) such that \( W_0 \subseteq E \) and then argue as above with \( W_0 \) replacing \( W \). Thus \( X \) is compact.

Conversely, if \( X \) is compact, then \( k = u \) and so, by (a), \( \beta = u \).

(b) (ii) If every \( \sigma \)-compact subset of \( X \) is relatively compact, then it is easy to show that \( \beta \leq k \).

Conversely, let \( \beta \leq k \), and suppose that there is a set \( G = \bigcup_{n=1}^{\infty} K_n \) (\( K_n \) compact in \( X \)) which is not relatively compact. Then, for each compact set \( F \) in \( X \), \( G \setminus F \neq \emptyset \). Let
\[
\varphi = \sum_{n=1}^{\infty} \frac{1}{2^n} \chi_{K_n}.
\]

Then \( \varphi \in B_o(X) \). [Clearly \( \varphi \) is bounded. To show that \( \varphi \) vanishes at infinity, let \( \varepsilon > 0 \). Choose \( N \geq 1 \) such that \( \frac{1}{N} < \varepsilon \). Now \( K = \bigcup_{n=1}^{N} K_n \) is compact. If \( x \notin K \), then \( \chi_{K_n}(x) = 0 \) for all \( n = 1, 2, \ldots, N \) and so
\[
\varphi(x) = \sum_{n \geq N+1} \frac{1}{2^n} \chi_{K_n}(x) \leq \sum_{n \geq N+1} \frac{1}{2^n} = \frac{1}{2^N} < \frac{1}{N} < \varepsilon.
\]

Clearly, \( \varphi = 0 \) outside of \( G \). Since \( \beta \leq k \), for any balanced \( W \in W \), there exist a compact set \( K \) in \( X \) and a \( V \in W \) such that
\[
N(K, V) \subseteq N(\varphi, W).
\]

If \( E \setminus W \neq \emptyset \), let \( d \in E \setminus W \), and \( y_o \in G \setminus K \). Choose a \( \psi \in C_b(X) \) with
\[
0 \leq \psi \leq (1/\varphi(y_o)), \quad \psi(y_o) = 1/\varphi(y_o), \quad \text{and} \quad \psi(K) = 0.
\]

Let \( h = \psi \otimes d \). Then, since \( h(K) = 0 \subseteq V \), \( h \in N(K, V) \) but, since
\[
\varphi(y_o)h(y_o) = d \notin W,
\]
we have \( h \notin N(\varphi, W) \), a contradiction. If \( W = E \), choose a \( W_o \) in \( W \) such that \( W_o \subseteq E \) and then argue as above with \( W_o \) replacing \( W \).

(c) Since \( \beta \leq u \), it follows that every \( u \)-bounded subset of \( C_b(X, E) \) is \( \beta \)-bounded.

Conversely, suppose there is a set \( A \subseteq C_b(X, E) \) which is \( \beta \)-bounded but not \( u \)-bounded. Then there exist a sequence \( \{f_n\} \subseteq A \), \( \{x_n\} \subseteq X \), and a \( W \in W \) such that
\[
f_n(x_n) \notin n^2 W.
\]

Let \( \varphi(x) = 1/n \) if \( x = x_n \), and \( \varphi(x) = 0 \) if \( x \neq x_n \) \( (n = 1, 2, \ldots) \). Then, as seen in the proof of Lemma 1.1.3, \( \varphi \in B_o(X) \) but \( \varphi(x_n)f_n(x_n) \notin nW; \)
1. THE STRICT AND WEIGHTED TOPOLOGIES

that is, \( \{f_n\} \) and hence \( A \) is not \( \beta \)-bounded. This contradiction proves the result.

(d) Let \( A \) be a \( u \)-bounded subset of \( C_b(X, E) \). Since \( k \leq \beta \), it is sufficient to show that the identity mapping \( i : (A, k) \to (A, \beta) \) is continuous. Suppose \( f \in A \) and let \( \{f_\alpha\} \) be a net in \( A \) such that

\[
\lim_\alpha f_\alpha = f.
\]

Let \( \varphi \in B_\alpha(X) \) with \( 0 \leq \varphi \leq 1 \), and let \( W \in W \). We show that there exists an index \( \alpha_o \) such that

\[
\varphi(x)(f_\alpha(x) - f(x)) \in W \quad \text{for all } x \in X \text{ and } \alpha \geq \alpha_o.
\]

Choose a balanced \( V \in W \) such that \( V + V \subseteq W \). Since \( A \) is \( u \)-bounded, choose a \( t > 1 \) such that \( g(x) \in tV \) for all \( x \in X \) and \( g \in A \).

Since \( \varphi \in B_\alpha(X) \), there exists a compact set \( K \) in \( X \) such that \( \varphi(x) < \frac{1}{t} \) for \( x \notin K \).

Let \( \varphi_1 = \varphi \chi_K \). Then \( \varphi_1 \in B_{\alpha_o}(X) \) and so, since \( \lim_\alpha f_\alpha = f \), there exists an index \( \alpha_o \) such that

\[
\varphi_1(x)(f_\alpha(x) - f(x)) \in V \quad \text{for all } x \in X \text{ and } \alpha \geq \alpha_o.
\]

Let \( y \) be any point of \( X \). If \( y \in K \), then \( \varphi_1 = \varphi \), so that

\[
\varphi(y)(f_\alpha(y) - f(y)) \in V \subseteq W \quad \text{for all } \alpha \geq \alpha_o.
\]

If \( y \notin K \), then \( t\varphi(y) < 1 \)

\[
\varphi(y)(f_\alpha(y) - f(y)) \in \varphi(y)tV - \varphi(y)tV \subseteq V + V \subseteq W \quad \text{for all } \alpha.
\]

Thus \( \lim_\alpha f_\alpha = f \) and so \( i : (A, k) \to (A, \beta) \) is continuous; that is, \( \beta|A \subseteq k|A \), as required.

(e) Suppose \( \{f_n\} \) is \( \beta \)-convergent in \( C_b(X, E) \). Since a convergent sequence in a TVS is always bounded, it follows that \( \{f_n\} \) is \( \beta \)-bounded and hence \( u \)-bounded by (e). Moreover, since \( k \leq \beta \), \( \{f_n\} \) is \( k \)-convergent.

Conversely, suppose \( \{f_n\} \) is \( u \)-bounded in \( C_b(X, E) \) and \( \lim_{n \to \infty} f_n = f \), where \( f \) is in \( C_b(X, E) \). The \( \{f, f_n : n = 1, 2, \ldots\} \) is contained in a \( u \)-bounded subset, say \( A \), of \( C_b(X, E) \). By (d) \( \beta|A = k|A \), and so, it follows that \( \lim_{n \to \infty} f_n = f \).

The following remarks contain some counterexamples related to the above results.
1. Strict and Related Topologies on $C_b(X, E)$

**Remark 1.** Since $k \leq \beta$, every $\beta$-bounded set is $k$-bounded. But the converse is not true, in general; Hirschfeld ([Hir74], p.20) has given the following example of a $k$-bounded set which is not $\beta$-bounded.

**Example.** Let $X = \mathbb{N}$, so that $C_b(\mathbb{N}) = \ell_\infty$. Define $\varphi_n = n\chi_{[n, \infty)}$; that is,

$$
\varphi_n(m) = \begin{cases} 
  n & \text{if } m \geq n \\
  0 & \text{if } m < n
\end{cases} \quad (m \in X).
$$

Let $K \subseteq X = \mathbb{N}$ be any compact set. Then $K$ is finite, there exists an integer $n_o$ such that $\varphi_n|K = 0$ for all $n > n_o$. Since $K$ is arbitrary, $k$-$\lim n \varphi_n = 0$. So $\{\varphi_n\}$ is $k$-bounded. On the other hand, $\sup_n \|\varphi_n\| = \infty$ and so $\{\varphi_n\}$ is not $u$-bounded or equivalently $\{\varphi_n\}$ is not $\beta$-bounded.

**Remark 2.** It is easy to see that, if a net $\{f_\alpha\}$ in $C_b(X, E)$ is $u$-bounded and $k$-convergent, then $\{f_\alpha\}$ is $\beta$-convergent. However, the converse is not true; Gulick ([Gu72], p.166) has given an example of a $\beta$-convergent net which is not $u$-bounded as follows.

**Example.** Let $X = \Omega$, the space of ordinals less than the first uncountable ordinal $\omega$, with order topology. Let $\Omega_o \subseteq \Omega$ the collection of non-limit ordinals. Let $f_\lambda = n_\lambda \chi_{\{\lambda\}}$ where $\lambda$ is $n_\lambda(0 < n_\lambda < \infty)$ greater than a limit ordinal. The $f_\lambda \beta \rightarrow 0$. But $\{f_\lambda : \lambda \in \Omega \setminus \Omega_o\}$ is not $u$-bounded.

We now discuss the relation of $\beta$ with the Buck’s strict topology [Buc58]. Let $\beta'$ denote the $C_o(X)$-topology, which is the Buck’s original strict topology, on $C_b(X, E)$.

**Theorem 1.1.9.** [Kh79] If $X$ is locally compact, $\beta' = \beta$ on $C_b(X, E)$.

**Proof.** It is clear that $\beta' \leq \beta$, since $C_o(X) \subseteq B_o(X)$. Now, let $\varphi \in B_o(X)$. By Lemma 1.1.5, it is sufficient to show that there exists a function $\psi$ in $C_o(X)$ which dominates $\varphi$. For each $n$, the set

$$
\{x \in X : |\varphi(x)| \geq \frac{1}{2^n}\}
$$

has a compact closure $K_n$ say in $X$. Since $X$ is locally compact, there exist function $\psi_n \in C_\infty(X)$ such that

$$
0 \leq \psi_n \leq 1, \quad \psi_n(K_n) = 1, \quad \text{and } \text{supp } (\psi_n) \subseteq K_n.
$$

Let $\psi = \sum_{n=1}^{\infty} \frac{1}{2^n} \psi_n$. Clearly $K_n \subseteq K_{n+1}$ and $\psi \in C_o(X)$. To show $|\varphi| \leq \psi$ on $X$, let $y \in X$. If $y \in \bigcup_{n=1}^{\infty} K_n$, then $y \in K_{N+1} \setminus K_N$ for some $N \geq 1$, so
that

\[ \psi(y) \geq \sum_{n>N} \frac{1}{2^n} = \frac{1}{2^N} > |\varphi(y)|. \]

If \( y \notin \bigcup_{n=1}^{\infty} K_n \), then \( \varphi(y) = 0 \) but \( \psi(y) \geq 0 \). Thus \( |\varphi(y)| \leq \psi(y) \).

Consequently \( \beta \leq \beta' \). \( \square \)

**Remark.** If \( X \) is not locally compact, then \( C_o(X, E) \) may be the trivial vector space \( \{0\} \).

**Counter-example.** Let \( X = \mathbb{Q} \), the space of rationals and \( E = \mathbb{R} \). Then \( C_o(\mathbb{Q}, \mathbb{R}) = \{0\} \), as follows. Suppose \( \varphi(\neq 0) \in C_o(\mathbb{Q}, \mathbb{R}) \). Choose \( y \in \mathbb{Q} \) with \( \varphi(y) \neq 0 \). We may suppose \( \varphi(y) > 0 \). Choose \( a, b > 0 \) such that \( a < \varphi(y) < b \). Then the set

\[ U = \{ x \in \mathbb{Q} : a < \varphi(x) < b \} = \varphi^{-1}(a, b) \]

is non-empty (since \( y \in U \)) and open (since \( \varphi \) is continuous). Since \( \varphi \in C_o(\mathbb{Q}, \mathbb{R}) \), the set

\[ V = \{ x \in \mathbb{Q} : |\varphi(x)| > a \} = \{ x \in \mathbb{Q} : \varphi(x) > a \text{ or } \varphi(x) < -a \} \]

is relatively compact. Now, for any \( x \in U, \varphi(x) > a \) and so \( x \in V \); that is, \( U \subseteq V \). Then it follows that \( U \) is compact. Hence every point in \( U \) has a compact neighborhood. Upon translating \( U \) along \( \mathbb{Q} \), we arrive at the contradiction that \( \mathbb{Q} \) is locally compact. \( \square \)

**Theorem 1.1.10.** [Buc58, Kh79] Let \( E \) be any TVS. If \( X \) is locally compact, then:

(a) \( C_{oo}(X, E) \) is \( \beta \)-dense in \( C_b(X, E) \).
(b) \( C_{oo}(X, E) \) is \( k \)-dense in \( C(X, E) \).
(c) \( C_{oo}(X, E) \) is \( u \)-dense in \( C_b(X, E) \).
(d) \( C_o(X, E) \) is \( u \)-closed in \( C_b(X, E) \).

Conversely, if \( E \) is a non-trivial TVS, then each of the condition (a), (b) implies that \( X \) is locally compact; if, in addition, \( C_o(X, E) \neq \{0\} \), then (c) also implies that \( X \) is locally compact.

**Proof.** Suppose \( X \) is locally compact.

(a) Let \( f \in C_b(X, E) \). Let \( \varphi \in B_o(X) \), \( 0 \leq \varphi \leq 1 \), and \( W \in \mathcal{W} \). Let \( K \subseteq X \) be a compact set such that \( \varphi(x)f(x) \in W \) for \( x \notin K \). Since \( X \) is locally compact, choose a function \( \psi \in C_{oo}(X) \) such that \( 0 \leq \psi \leq 1 \) and \( \psi(K) = 1 \). Let \( g = \psi f \). Then \( g \in C_{oo}(X) \) and

\[
\varphi(x)[g(x) - f(x)] = \varphi(x)[\psi(x) - 1]f(x) \quad \left\{ \begin{array}{ll}
0 & \text{if } x \in K \\
\in (\psi(x) - 1)W & \text{if } x \notin K.
\end{array} \right.
\]
Thus $g - f \in N(\varphi, W)$, and so $f$ belongs to the $\beta$-closure of $C_\infty(X, E)$; that is, $C_\infty(X, E)$ is $\beta$-dense in $C_b(X, E)$.

(b) Let $f \in C(X, E)$. Let $K$ be a compact subset of $X$. Choose a $\psi \in C_\infty(X)$ such that $\psi = 1$ on $K$. Then $\psi f \in C_\infty(X, E)$ and

$$(\psi f)(x) - f(x) = 0 \text{ for all } x \in K.$$  

In particular, for any $W \in \mathcal{W}$, $\psi f - f \in N(K, W)$, and so $f$ belongs to the $k$-closure of $C_\infty(X, E)$ Thus $C_\infty(X, E)$ is $k$-dense in $C(X, E)$.

(c) Let $f \in C_o(X, E)$. To show that $f$ belongs to the $u$-closure of $C_\infty(X, E)$, let $W \in \mathcal{W}$ be balanced. Now $K = \{x \in X : f(x) \notin W\}$ is compact. Choose $\psi \in C_\infty(X)$ such that $0 \leq \psi \leq 1$ and $\psi = 1$ on $K$. Then clearly $g = \psi f \in C_\infty(X, E)$. Let $x \in X$. If $x \in K$, then

$$f(x) - g(x) = f(x) - f(x) = 0 \in W.$$  

If $x \in X \setminus K$, then

$$f(x) - g(x) = [\psi(x) - 1]f(x) \in [\psi(x) - 1]W \subseteq W.$$  

Hence $g - f \in N(X, W)$, Thus $C_\infty(X, E)$ is $u$-dense in $C_o(X, E)$.

(d) Let $f \in F(X, E)$ with $f \in C_\infty(X, E)$, Then there exists a sequence $\{f_n\} \subseteq C_o(X, E)$ such that $f_n \xrightarrow{u} f$. We need to show that $f \in C_o(X, E)$.

$f$ is continuous on $X$, as follows. Let $x_o \in X$ and $W \in \mathcal{W}$. We show that there exists a neighborhood $G$ of $x_o$ in $X$ such that, $f(y) - f(x_o) \in W$ for all $y \in G$. Choose a balanced $V \in \mathcal{W}$ such that $V + V + V \subseteq W$. Since $f_n \xrightarrow{u} f$, there exists $n_0 \geq 1$ such that

$$f_n(x) - f(x) \in V \text{ for all } n \geq n_0 \text{ and all } x \in X. \quad (1)$$

Since $f_{n_0}$ is continuous at $x_o$, there exists a neighborhood $G$ of $x_o$ in $X$ such that

$$f_{n_0}(y) - f_{n_0}(x_o) \in V \text{ for all } y \in G. \quad (2)$$

Then, for any $y \in G$, using (1) and (2),

$$f(y) - f(x_o) = (f(y) - f_{n_0}(y)) + (f_{n_0}(y) - f_{n_0}(x_o)) + (f_{n_0}(x_o) - f(x_o)) \in V + V + V \subseteq W.$$  

Hence $f$ is continuous on $X$. Thus $f \in C_b(X, E)$

We now need to verify that $f$ vanishes at infinity. Let $W \in \mathcal{W}$. Choose a balanced $V \in \mathcal{W}$ such that $V + V \subseteq W$. Since $f_n \xrightarrow{u} f$, there exists $k_0 \geq 1$ such that

$$f_n(x) - f(x) \in V \text{ for all } n \geq N_0 \text{ and } x \in X.$$
Since \( f_{k_0} \in C_0(X, E) \), there exists a compact set \( K \subseteq X \) such that
\[
f_{k_0}(x) \in V \text{ for all } x \in X \setminus K.
\]
Then, for any \( x \in X \setminus K \),
\[
f(x) = (f(x) - f_{k_0}(x)) + f_{k_0}(x) \in V + V \subseteq W.
\]
Therefore \( f \in C_o(X, E) \). Thus \( C_o(X, E) \) is \( u \)-closed in \( C_b(X, E) \).

Conversely, suppose \( E \) is a non-trivial TVS.

(i) Suppose \( C_oo(X, E) \) is \( \beta \)-dense in \( C_b(X, E) \) but that \( X \) is not locally compact. There there exists a \( y \in X \) which has no compact neighborhood. Consequently \( g(y) = 0 \) for all \( g \in C_oo(X, E) \). Now, let \( W \in W \) be balanced. Since \( E \) is non-trivial, we may assume \( W \neq E \) and choose \( a \in E \setminus W \). Consider the non-zero constant function \( h : X \to E \) function given by \( h(x) = a \) for all \( x \in X \). Then clearly \( h \in C_b(X, E) \). Now,
\[
N(\{y\}, W) = \{f \in C_b(X, E) : f(y) \in W\}
\]
is a \( p \)-neighborhood of \( 0 \) in \( C_b(X, E) \) and clearly, for any \( g \in C_oo(X, E) \),
\[
h(y) - g(y) = a - 0 \notin W;
\]
i.e., \( h - g \notin N(\{y\}, W) \). Therefore \( h \) does not belong to the \( p \)-closure of \( C_oo(X, E) \); since \( p \leq \beta \), \( h \) does not belong to the \( \beta \)-closure of \( C_oo(X, E) \). Therefore \( C_oo(X, E) \) is not \( \beta \)-dense in \( C_o(X, E) \), a contradiction. Thus \( X \) is locally compact.

(ii) Suppose \( C_oo(X, E) \) is \( k \)-dense in \( C(X, E) \) but that \( X \) is not locally compact. There there exists a \( y \in X \) which has no compact neighborhood. Consequently \( g(y) = 0 \) for all \( g \in C_oo(X, E) \). It follows (as in (i)) that, if \( h \) is any non-zero constant function in \( C(X, E) \), then \( h \) does not belong to the \( p \)-closure, and hence to the \( k \)-closure of \( C_oo(X, E) \) (since \( p \leq k \)); that is \( C_oo(X, E) \) is not \( k \)-dense in \( C(X, E) \).

(iii) Suppose \( C_o(X, E) \neq \{0\} \) and that \( C_oo(X, E) \) is \( u \)-dense in \( C_o(X, E) \) but that \( X \) is not locally compact. There there exists a \( y \in X \) which has no compact neighborhood. Consequently \( g(y) = 0 \) for all \( g \in C_oo(X, E) \). It follows (as in (i)) that, if \( h \) is any non-zero function in \( C_o(X, E) \), then \( h \) does not belong to the \( p \)-closure, and hence not to the \( u \)-closure of \( C_oo(X, E) \); that is \( C_oo(X, E) \) is not \( u \)-dense in \( C_o(X, E) \).

If \( X \) is not assumed to be locally compact, then a result related to the above theorem can be obtained by imposing a stronger condition on \( E \), as follows.

**Theorem 1.1.11.** [Kh95] Suppose \( E \) is a locally bounded TVS. Then \( C_b(X, E) \) is \( k \)-dense in \( C(X, E) \).
Proof. Let \( f \in C(X,E) \), and let \( K \) be a compact subset of \( X \) and \( W \in \mathcal{W} \). By hypothesis, we can choose a bounded set \( V \in \mathcal{W} \). Let \( S \) be a closed shrinkable neighborhood of 0 in \( E \) with \( S \subseteq V \) (see Section A.4). The Minkowski functional \( \rho_S \) of \( S \) is continuous and positively homogeneous, and so, for any \( r > 0 \), the function \( h_r : E \to E \) defined by

\[
h_r(a) = \begin{cases} \frac{a}{\rho_S(a)} & \text{if } a \in rS \\ a & \text{if } a \notin rS \end{cases}
\]

is continuous with \( h_r(E) \subseteq rS \). This implies that the function \( h_r \circ f \in C_b(X,E) \). [Clearly, \( h_r \circ f \) is continuous on \( X \). To show that \( h_r \circ f \) is bounded on \( X \), let \( U \in \mathcal{W} \). Choose \( \lambda > 0 \) with \( V \subseteq \lambda U \). Then

\[
h_r(f(x)) \subseteq h_r(E) \subseteq rS \subseteq rV \subseteq \lambda U.
\]

Choose \( t \geq 1 \) with \( f(K) \subseteq tS \). Then \( h_t \circ f \in C_b(X,E) \). Further, for any \( x \in K \), \( f(x) \in tS \) and so

\[
h_t(f(x)) - f(x) = f(x) - f(x) = 0 \in W.
\]

Thus \( C_b(X,E) \) is \( k \)-dense in \( C(X,E) \). \( \square \)

We next consider the properties of \( \sigma \) and \( \sigma_o \) topologies on \( C_b(X,E) \).

**Lemma 1.1.12.** [KR91] Suppose that \( A, B \subseteq X \) and that \( V, W \in \mathcal{W} \) are such that \( N(A,V) \subseteq N(B,W) \). Then:

(i) If \( W \neq E \), then \( B \subseteq \overline{A} \).

(ii) If \( B \) is non-empty, then \( V \subseteq W \).

**Proof.** (i) Suppose \( B \not\subseteq \overline{A} \), and let \( x \in B \setminus \overline{A} \). Since \( X \) is completely regular, there exists a \( \varphi \) in \( C_b(X) \) such that \( \varphi(x) = 1 \) and \( \varphi(\overline{A}) = \{0\} \). Then clearly \( \text{supp } (\varphi) \cap \overline{A} = \emptyset \). Choose \( a \in E \setminus W \) and let \( f = \varphi \circ a \). Then \( f(A) = \{0\} \subseteq V \), and so \( f \in N(A,V) \); but \( f(x) = a \notin W \) and so \( f \notin N(B,W) \). This contradiction implies that \( B \subseteq \overline{A} \).

(ii) Suppose \( V \not\subseteq W \). Choose \( b \in V \setminus W \). and let \( g = \chi_X \otimes b \). If \( A = \emptyset \), then \( g(A) = \emptyset \subseteq V \); if \( A \neq \emptyset \), then \( g(A) = \{b\} \subseteq V \). Hence \( g \in N(A,V) \). Since \( B \neq \emptyset \), \( g(B) = \{b\} \notin W \) and so \( g \notin N(B,W) \), a contradiction. \( \square \)

**Theorem 1.1.13.** [KR91] Let \( p, k, \beta, \sigma, u \) and \( \sigma_o \) be the topologies on \( C_b(X,E) \) as defined above. Then

(a) \( p \leq k \leq \beta \leq \sigma \leq u \) and \( p \leq \sigma_o \leq \sigma \).

(b) \( \sigma = u \) iff \( X = \overline{A} \) for some \( \sigma \)-compact subset \( A \) of \( X \).

(c) \( k = \sigma \) and \( \sigma = \beta \) iff every \( \sigma \)-compact subset of \( X \) is relatively compact.

(d) \( \sigma_o = u \) iff \( X \) is separable.

(e) \( \sigma_o \leq k \) iff every countable subset of \( X \) is relatively compact.
1. THE STRICT AND WEIGHTED TOPOLOGIES

(f) $\sigma_o$, $\sigma$ and $u$ have the same bounded sets in $C_b(X, E)$. 

Proof. (a) Clearly, $p \leq \sigma_o \leq \sigma \leq u$ (as noted in Remark 1.1.6) and, by Theorem 1.1.8(a), $p \leq k \leq \beta \leq u$. Hence, we only need to verify that $\beta \leq \sigma$. Let $\varphi \in B_o(X)$, with $0 \leq \varphi \leq 1$, and $W \in \mathcal{W}$. Then there exists a sequence $\{K_n\}$ of compact sets in $X$ such that

$$\varphi(x) < 1/n \text{ for } x \notin K_n,$$

which implies that $\varphi = 0$ outside of the set $A = \bigcup_{n=1}^{\infty} K_n$ and $N(A, W) \subseteq N(\varphi, W)$, giving $\beta \leq \sigma$.

(b) If $X = \overline{A}$ for some $\sigma$-compact subset $A$ of $X$, then it is clear that $u \leq \sigma$.

Conversely, suppose that $u \leq \sigma$ and let $W \in \mathcal{W}$, with $W \neq E$. Then there exist a $\sigma$-compact subset $A$ of $X$ and a $V \in \mathcal{W}$ such that

$$N(A, V) \subseteq N(X, W).$$

By Lemma 1.1.12, $X = \overline{A}$, as required.

(c) If every $\sigma$-compact subset of $X$ is relatively compact, then clearly $\sigma \leq k$.

Conversely, suppose that $\sigma \leq k$ and let $A$ be any $\sigma$-compact subset of $X$. If $W \in \mathcal{W}$ and $W \neq E$, then there exist a compact set $B$ and a $V \in \mathcal{W}$ such that

$$N(B, V) \subseteq N(A, W).$$

By Lemma 1.1.12, $A \subseteq \overline{B}$, which implies that $A$ is relatively compact. The second part of the assertion follows from part (b) and Theorem 1.1.8(b)(ii).

(d) Since a separable set is the closure of a countable set, the $\sigma_o$-topology is equivalent to the topology of uniform convergence on the separable subsets of $X$. Thus, if $X$ is countable, it is clear that $u \leq \sigma_o$.

Conversely, if $u \leq \sigma_o$, then, for any $W \in \mathcal{W}$, $W \neq E$, there exist a separable set $A$ and a $V \in \mathcal{W}$ such that

$$N(A, V) \subseteq N(X, W).$$

By Lemma 1.1.12, $X = \overline{A}$ and so $X$ is separable.

(e) This is similar to that of part (c).

(f) Since $\sigma_o \leq u$, every $u$-bounded set is $\sigma_o$-bounded.

Conversely, suppose that there exists a $\sigma_o$-bounded set $H$ which is not $u$-bounded. Then there exist sequences $\{f_n\} \subseteq H$, $\{x_n\} \subseteq X$ and a $W \in \mathcal{W}$ such that

$$f_n(x_n) \notin nW.$$
Let \( A = \{x_n\} \). Then \( H \) is not absorbed by \( N(A, W) \), which is a contradiction. □

Recall that a TVS \( E \) is called ultrabornological if every bounded linear map from \( E \) into any TVS \( F \) is continuous; \( E \) is called ultrabarrelled if every closed linear map from \( E \) into any complete metrizable TVS \( F \) is continuous; \( E \) is called quasi-ultrabarrelled if every bounded linear map from \( E \) into any complete metrizable TVS \( F \) is continuous. Every metrizable TVS is ultrabornological and every complete metrizable TVS is ultrabarrelled. Further, every ultrabornological and every ultrabarrelled TVS is quasi-ultrabarrelled.

**Theorem 1.1.14.** [Buc58, Kh79, KR91] Let \( t \) denote any one of the topologies \( \sigma_o, \beta \) or \( \sigma \) on \( C_b(X,E) \).

(a) If \( t \) is ultrabornological, then \( t = u \) on \( C_b(X,E) \).

(b) If \( E \) is metrizable, then the following are equivalent:
   (i) \( t = u \).
   (ii) \( t \) is metrizable.
   (iii) \( t \) is ultrabornological.

Proof. (a) The identity map \( i : (C_b(X,E), t) \rightarrow (C_b(X,E), u) \) is linear and bounded and so, since \( t \) is ultrabornological, \( i \) is continuous. Hence \( u \leq t \).

(b) If \( E \) is metrizable, then \( (C_b(X,E), u) \) is also metrizable. Therefore (i) \( \Rightarrow \) (ii) \( \Rightarrow \) (iii) is clear and (iii) \( \Rightarrow \) (i) follows from part (a). □

**Remarks.** (1) If \( E \) is a complete metrizable TVS, so is \( (C_b(X,E), u) \) and therefore in Theorem 1.1.14 (b), we have (i) \( \Rightarrow \) (iv) \( \Rightarrow \) (v), where:
   (iv) \( t \) is ultrabarrelled
   (v) \( t \) is quasi-ultrabarrelled.

(2) In view of Theorem 1.1.14(a), the topologies \( \sigma_o, \beta \) and \( \sigma \) are not ultrabornological or metrizable (even in the case of real-valued function). In fact, if \( X \) is not pseudocompact, then \( \beta \neq u \) on \( C_b(X) \).

The following theorem which gives an alternate form of \( \beta \)-neighborhoods of 0 in \( C_b(X,E) \) is due to Katsaras [Kat81].

**Theorem 1.1.15.** [Kat81] \( \beta \) has a base of neighborhoods of 0 in \( C_b(X,E) \) consisting of all sets of the form

\[
N = N(K_n, a_n, W) = \bigcap_{i=1}^n \{f \in C_b(X,E) : f(K_n) \subseteq a_n W\}, \tag{*}
\]

where \( \{K_n\} \) is a sequence of compact subsets of \( X \), \( \{a_n\} \) a sequence of positive numbers with \( a_n \rightarrow \infty \) and \( W \in \mathcal{W} \).
Proof. Let \( \mathcal{N} \) be of the form (\( \ast \)) with \( W \) balanced. Then
\[
\varphi := \sup_{n \geq 1} \left\{ \frac{1}{a_n} \chi_{K_n} \right\} \in B_b(X).
\]
Moreover, \( N(\varphi, W) \subseteq \mathcal{N} \), as follows. Let \( f \in N(\varphi, W) \). Then
\[
\frac{1}{a_n} \chi_{K_n}(x)f(x) = \varphi(x)f(x) \in W \text{ for all } x \in X,
\]
and so \( f(K_n) \subseteq a_n W \) for all \( n \geq 1 \). Therefore \( f \in \mathcal{N}(K_n, a_n, W) \). Hence \( \mathcal{N} \) is a \( \beta \)-neighborhood of 0 in \( C_b(X, E) \).

Conversely, let \( \mathcal{N}_o \) be a \( \beta \)-neighborhood of 0 in \( C_b(X, E) \). There exists a balanced neighborhood \( W \) of 0 in \( E \) and \( \varphi \in B_b(X) \) with \( 0 \leq \varphi \leq 1 \) such that
\[
\mathcal{N}_1 = N(\varphi, W) \subseteq \mathcal{N}_o.
\]
For \( n \geq 1 \), there exists a compact subset \( K_n \) of \( X \) such that
\[
\{ x \in X : g(x) \geq \frac{1}{n} \} \subseteq K_n.
\]
Set
\[
\mathcal{N}_2 = \mathcal{N}(K_n, \frac{n}{2}, W) = \bigcap_{n=1}^{\infty} \{ f \in C_b(X, E) : f(K_n) \subseteq \frac{n}{2} W \}.
\]
Then \( \mathcal{N}_2 \subseteq \mathcal{N}_1 \), as follows. Let \( f \in \mathcal{N}_2 \). Then \( f(K_n) \subseteq \frac{n}{2} W \) for all \( n \geq 1 \).

Let \( x \in X \). If \( \varphi(x) \neq 0 \), then there exists \( n \geq 2 \) such that
\[
\frac{1}{n} < \varphi(x) < \frac{1}{n-1} \text{ and so } x \in K_n \setminus K_{n+1}.
\]
Then, since \( n < 2(n-1) \),
\[
f(x) \in \frac{n}{2} W \text{ and } \varphi(x) < \frac{1}{n-1} < \frac{2}{n},
\]
Therefore, \( \varphi(x)f(x) \in \frac{n}{2(n-1)} W \subseteq W \), and so \( f \in \mathcal{N}_1 \). Thus \( \mathcal{N}_2 \subseteq \mathcal{N}_o \), as required.

Corollary 1.1.16. [Kat81] If \( \mathcal{N} \) is a \( \beta \)-neighborhood of 0 in \( C_b(X, E) \), then there exists a balanced \( W \in \mathcal{W} \) with the following property: For each \( d > 0 \), there exist a compact set \( K \subseteq X \) and a positive number \( \delta > 0 \) such that
\[
\{ f \in C_b(X, E) : f(X) \subseteq dW, f(K) \subseteq \delta W \} \subseteq \mathcal{N}.
\]
Proof. Since \( N \) is a \( \beta \)-neighborhood of 0 in \( C_b(X, E) \), there exist \( \varphi \in B_o(X) \), \( 0 \leq \varphi \leq 1 \), and \( W \in \mathcal{W} \) such that \( N(\varphi, W) \subseteq N \). For each \( n \geq 1 \), let \( K_n = \{ x \in X : \varphi(x) \geq \frac{1}{n} \} \).

Then each \( K_n \) is compact, \( K_{n-1} \subseteq K_n \), \( \varphi = 1 \) on \( K_1 \),

\[
\frac{1}{n} \leq \varphi(x) < \frac{1}{n-1} \quad \text{for } x \in K_n \setminus K_{n-1} (n \geq 2),
\]

and \( \varphi = 0 \) outside \( \bigcup_{n=1}^{\infty} K_n \). Let now \( d > 0 \) be given. Choose an integer \( N \) such that \( N \geq 2d \). Let \( K = \bigcup_{n=1}^{N} K_n = K_N \), \( \delta = \frac{1}{2} \). We now verify that

\[
\{ f \in C_b(X, E) : f(X) \subseteq dW, f(K) \subseteq \delta W \} \subseteq N(\varphi, W).
\]

Let \( f \in C_b(X, E) \) with \( f(X) \subseteq dW, f(K) \subseteq \delta W \). Let \( x \subseteq X \). Suppose \( x \in K \). If \( x \in K_1 \), \( \varphi(x) = 1 \), and so

\[
\varphi(x)f(x) \in \delta W = \frac{1}{2}W \subseteq W.
\]

If \( x \in K_j \setminus K_{j-1} (2 \leq j \leq N) \), and so

\[
\varphi(x)f(x) \in \frac{1}{2(j-1)}W \subseteq W.
\]

Now, let \( x \in \bigcup_{n=1}^{\infty} K_n \setminus K_N \). Then \( x \in K_p \setminus K_{p-1} \) for some \( p \geq N+1 \). Then

\[
\varphi(x) < \frac{1}{p-1} \leq \frac{1}{N} < \frac{1}{d},
\]

and so

\[
\varphi(x)f(x) \in \frac{1}{d}dW = W.
\]

Next, if \( x \notin \bigcup_{n=1}^{\infty} K_n \), then \( \varphi = 0 \) and so \( \varphi(x)f(x) = 0 \in W \). Thus, in each case, \( f \in N(\varphi, W) \). \( \square \)
2. Weighted Topology on $CV_b(X, E)$

Let $X$ be a completely regular Hausdorff space, $E$ a Hausdorff TVS with a base $W$ of balanced neighborhoods of $0$ and $V$ a Nachbin family on $X$. The purpose of this section is to consider the weighted topology on the weighted function space $CV_b(X, E)$ and study its various properties. In particular, we discuss the relation of weighted topology with the uniform, strict, compact-open and pointwise topologies.

Recall that, if $X$ is a topological space, then a function $\varphi : X \to \mathbb{R}$ is said to be upper (resp. lower) semicontinuous on $X$ if, for each $r \in \mathbb{R}$, the set $\{x \in X : \varphi(x) < r\}$ (resp. $\{x \in X : \varphi(x) > r\}$) is open. Clearly, $\varphi : X \to \mathbb{R}$ is continuous iff it is both upper and lower semicontinuous.

Further, the characteristic function $\chi_A$ of a subset $A$ of $X$ is upper (resp. lower) semicontinuous iff $A$ is closed (resp. open). Every upper (resp. lower) semicontinuous function assumes its supremum (resp. infimum) on a compact set. In particular, every non-negative upper semicontinuous function on a compact set is bounded.

**Definition.** A Nachbin family $V$ on $X$ is a set of non-negative upper semicontinuous functions on $X$ such that, given $u, v \in V$ and $\lambda > 0$, there exists a $w \in V$ such that $\lambda u, \lambda v \leq w$ (pointwise). Let $CV_b(X, E)$ (resp. $CV_o(X, E)$) denote the subspace consisting of those $f \in C(X, E)$ such that $vf$ is bounded (resp. vanishes at infinity) for all $v \in V$. If $U$ and $V$ are two Nachbin families on $X$ and, for every $u \in U$, there is a $v \in V$ such that $u \leq v$ (pointwise), then we write $U \leq V$.

**Lemma 1.2.1.** [Pro71b, Kh85b] Let $U$ and $V$ be two Nachbin families on $X$ with $U \leq V$. Then $CV_b(X, E) \subseteq CU_b(X, E)$ and $CV_o(X, E) \subseteq CU_o(X, E)$.

**Proof.** The first inclusion is immediate from the definition. Let $f \in CV_o(X, E)$, and let $u \in U$ and $G \in W$. Choose a $v \in V$ with $u \leq v$. There exists a compact set $K \subseteq X$ such that

$$v(x)f(x) \in G \quad \text{for all} \quad x \notin K.$$ 

Let $y \notin K \cap \text{supp}(u)$. If $y \notin K$, then $u(y)f(y) \in G$. If $y \notin \text{supp}(u)$, then $u(y) = 0$, and so again $u(y)f(y) \in G$. Hence $f \in CU_o(X, E)$. □

We now list some important classes of Nachbin families on $X$. For any subset $A$ of $X$, $\chi_A$ will denote its characteristic function.

1. $K_f^+(X) = \{\lambda \chi_F : \lambda > 0 \text{ and } F \subseteq X, F \text{ finite}\}$.
2. $K_F^+(X) = \{\lambda \chi_K : \lambda > 0 \text{ and } F \subseteq X, K \text{ compact}\}$. 
2. WEIGHTED TOPOLOGY ON $CV_b(X, E)$

(3) $S^+_{oo}(X)$, the set of all non-negative upper semi-continuous functions on $X$ which have compact support,

(4) $S^+_o(X)$, the set of all non-negative upper semi-continuous functions on $X$ which vanish at infinity.

(5) $K^+(X)$, the set of all non-negative constant functions on $X$.

Clearly,

$$K^+_f(X) \subseteq K^+_c(X) \subseteq S^+_{oo}(X) \subseteq S^+_o(X) \subseteq K^+(X).$$

If $V$ is a Nachbin family on $X$ with $V > 0$, then it is easily seen that $K^+_f(X) \subseteq V$.

**Lemma 1.2.2.** [Pro71b, Kh85b] If $V = S^+_o(X)$, then

$$CV_b(X, E) = CV_o(X, E) = C_b(X, E).$$

**Proof.** Clearly, $CV_o(X, E) \subseteq CV_b(X, E)$ and $C_b(X, E) \subseteq CV_b(X, E)$. Since every $v \in S^+_o(X)$ is bounded on a compact set, it follows that

$$CV_b(X, E) \subseteq CV_o(X, E).$$

To show that $CV_b(X, E) \subseteq C_b(X, E)$, let $f \in CV_b(X, E)$. Suppose $f$ is not bounded. Then there exist a $G \in W$ and a sequence $\{x_n\} \subseteq X$ such that

$$f(x_n) \not\in 2^{2n}G \text{ for all } n \geq 1.$$  

Let $\varphi = \sum_{n=1}^{\infty} \frac{1}{2^n} \chi_{\{x_n\}}$. Then $\varphi \in S^+_o(X)$ and, for any $n \geq 1$,

$$\varphi(x_n)f(x_n) \not\in 2^nG.$$  

This implies that $\varphi f$ is not bounded, a contradiction. Hence $f \in C_b(X, E)$. □

We also note that:

(1) If $V = K^+(X)$, then

$$CV_b(X, E) = C_b(X, E) \text{ and } CV_o(X, E) = C_o(X, E).$$

(2) If $V = K^+_c(X)$ or $V = S^+_{oo}(X)$, then

$$CV_b(X, E) = CV_o(X, E) = C(X, E).$$

(3) If $V = K^+_f(X)$, then also $CV_b(X, E) = CV_o(X, E) = C(X, E)$.

**Definition.** [Pro71b, Kh85b] Let $V$ be a Nachbin family on $X$. The **weighted topology** $\omega_V$ on $CV_b(X, E)$ is defined as the linear topology which has a base of neighborhoods of $0$ consisting of all sets of the form

$$N(v, G) = \{f \in CV_b(X, E) : (vf)(X) \subseteq G\},$$

where $v \in V$ and $G \in W$. 

The assumption \( V > 0 \) implies that \( p \leq \omega \). Recall that \( p \leq k \) on \( C(X, E) \) and \( k \leq \beta \leq u \) on \( C_b(X, E) \).

**Theorem 1.2.3.** [Pro71b, Kh85b] If \( V = S^+_\varphi(X) \), then \( \omega = \beta \), the strict topology on \( CV_b(X, E) = C_b(X, E) \).

**Proof.** It is clear that \( \omega \leq \beta \), since \( S^+_\varphi(X) \subseteq B^o(X) \). Now, let \( \varphi \in B_o(X) \). By Lemma 1.1.5, it is sufficient to show that there exists a function \( \psi \) in \( S^+_\varphi(X) \) which dominates \( \varphi \). For each \( n \), the set
\[
\{ x \in X : |\varphi(x)| \geq \frac{1}{2^n} \}
\]
has compact closure, \( K \) say, in \( X \). Let \( \psi = \sum_{n=1}^\infty \frac{1}{2^n} \chi_{K_n} \). Clearly \( K_n \subseteq K_{n+1} \) and \( \psi \in S^+_\varphi(X) \).

To show \( |\varphi| \leq \psi \) on \( X \), let \( y \in X \). If \( y \in \bigcup_{n=1}^\infty K_n \), then \( y \in K_{n+1} \setminus K_N \) for some \( N \geq 1 \), so that
\[
\psi(y) \geq \sum_{n>N} \frac{1}{2^n} = \frac{1}{2^N} > |\varphi(y)|.
\]
If \( y \notin \bigcup_{n=1}^\infty K_n \), then \( \varphi(y) = 0 \) but \( \psi(y) \geq 0 \). Thus \( |\varphi(y)| \leq \psi(y) \). Consequently \( \beta \leq \omega \), as required. \( \square \)

**Remark.** Similarly, as above, we note that:

(1) If \( V = K^+(X) \), then \( \omega \) is the uniform topology \( u \) on \( CV_b(X, E) = C_b(X, E) \).

(2) If \( V = K^+_c(X) \), then \( \omega \) is the compact-open topology \( k \) on \( CV_b(X, E) = C(X, E) \).

(3) If \( V = K^+_f(X) \), then \( \omega \) is the pointwise topology \( p \) on \( CV_b(X, E) = C(X, E) \).

**Lemma 1.2.4.** Let \( U \) and \( V \) be Nachbin families on \( X \) with \( U \leq V \). Then \( \omega_U \leq \omega_V \) on \( CV_b(X, E) \).

**Proof.** This is clear.

**Theorem 1.2.5.** [Pro71b, Kh85b] Let \( V \) be a Nachbin family on \( X \) such that \( \chi_c(X) \leq V \) and all \( v \in V \) vanish at infinity. Then

(a) \( k \leq \omega \leq \beta \leq u \) on \( C_b(X, E) \).

(b) \( k \) and \( \omega \) coincide on \( u \)-bounded subsets of \( C_b(X, E) \).

**Proof.** (a) This follows immediately from Lemma 1.2.4.
We claim that \( CV \) family \( V \) is a Hausdorff TVS, there exists some shrinkable neighborhood \( E \) a contradiction. □

Let \( \{ x \in X : v(x) \geq \frac{1}{t} \} \) and \( s = \sup \{ v(x) : x \in K \} \).

There exists an \( h \in A \) such that \( (h - f)(K) \subseteq \frac{1}{s} \). Then, for any \( y \in X \),

\[
v(y)[h(y) - f(y)] \in \begin{cases} 
  v(y) \frac{1}{s} H \subseteq G & \text{if } y \in K \\
  v(y)(rH - rH) \subseteq G & \text{if } y \notin K;
\end{cases}
\]

that is, \( h - f \in N(v, G) \). Hence \( f \) belongs to the \( \omega_V \)-closure of \( A \), and so \( \omega_V \leq k \) on \( A \).

The following example shows that \( CV_b(X, E) \) may be trivial for relatively nice \( X \).

**Example.** Let \( X = \mathbb{Q} \), the set of all rationals with the natural topology. This is of course a metrizable space. Consider on \( X \) the Nachbin family \( V = C^+(X) \) consisting of all non-negative continuous functions.

We claim that \( CV_b(X, E) \) is reduced to \( \{ 0 \} \) for every non-trivial TVS \( E \).

Indeed, assume that, for a given TVS \( E \), \( CV_b(X, E) \neq \{ 0 \} \), and let \( f(\neq 0) \in CV_b(X; E) \). Then \( f(x_o) \neq 0 \) for some \( x_o \in X \). Since \( E \) is a Hausdorff TVS, there exists some shrinkable neighborhood \( G \in \mathcal{W} \) so that \( \rho_G(f(x_o)) \neq 0 \). With no loss of generality, we assume that \( \rho_G(f(x_o)) = 1 \). Since \( \rho_G \circ f : X \to \mathbb{R} \) is continuous at \( x_o \), taking \( \varepsilon = \frac{1}{2} \), there exists \( \delta > 0 \) such that if \( |x - x_o| < \delta \),

\[
|\rho_G(f(x)) - \rho_G(f(x_o))| < \frac{1}{2};
\]

in particular, \( \rho_G(f(x)) > \rho_G(f(x_o)) - \frac{1}{2} = \frac{1}{2} \). For an irrational \( t \in \mathbb{P} \) with \( |t - x_o| < \delta \), the function \( v_t : X \to \mathbb{R}^+ \) given by

\[
v_t(x) = \frac{1}{|t - x|}, x \in X,
\]

belongs to \( V = C^+(X) \) and then, since \( f \in CV_b(X; E) \), \( v_t \) must verify

\[
\sup \{ v_t(x)\rho_G(f(x)) : x \in X \} < +\infty.
\]

But

\[
\sup \{ v_t(x)\rho_G(f(x)) : x \in X \} \geq v_t(x_o)\rho_G(f(x_o)) = \frac{1}{|t - x_o|} \to \infty \text{ as } t \to x_o,
\]
a contradiction.] □
Section 1.1. A general method for defining linear topologies on $C_b(X, E)$, called the $S$-topologies, is considered first. Using the notion of an $S$-topology, the uniform topology $u$, strict Topology $\beta$, compact-open topology $k$, $\sigma$-compact-open topology $\sigma$, countable-open-topology $\sigma_o$, and pointwise topology $p$ are introduced. See also ([KN63], Section 8, p. 68-82).

The topologies $k$ and $p$ on both $C(X)$ and $C(X, Y)$, $Y$ a uniform space, have been extensively studied over the past sixty years and significant contributions have been made in this field, among others, by Fox [Fox45], Arens [Are46a], Myers [My46], Hewitt [Hew48], Warner [war58], Wheeler [Whe76], Schmets [Schm83] and McCoy [Mc80]. In recent years, the $C_p$-theory has been actively investigated by several mathematicians (see the monographs by Arkhangel’skii [Ark92] and Tkachuk [Tka11], and their references).

The notion of the strict topology $\beta$ on $C_b(X, E)$ was first introduced by Buck [Buc58] in 1958 in the case of $X$ a locally compact space and $E$ a locally convex TVS. (In fact, this was also considered earlier in 1952 by Buck himself [Buc52] on $C_b(X)$ for the special case of $X$ a group; it was there called the "strict topology" because of its resemblance to a topology used by Beurling [Beu45].) Since then a large number of papers have appeared in the literature concerned with extending the results contained in Buck’s paper; see the Preface and the list of references.

Another useful topology on $C_b(X, E)$ is the $\sigma$-compact-open topology $\sigma$ which was introduced by Gulick [Gu72] and studied further by Gulick and Schmets [GuSc72]. Most of these investigations have been concerned with generalizing the space $X$ and taking $E$ to be the scalar field or a locally convex space. This section includes some results from [Kh79, KR91, Kat81], where $E$ is a Hausdorff TVS but not necessarily locally convex.

Section 1.2. This section is devoted to the study of weighted topology $\omega_V$ on the weighted function space $CV_b(X, E)$, as given in [Kh85, Kh87]. In particular, the relation of weighted topology with the uniform, strict, compact-open and pointwise topologies is also given.

The fundamental work on weighted spaces had been mainly done by Nachbin [Nac65, Nac67]. Further investigations have been made by Summers [SumW69, SumW72], Prolla [Pro71, Pro72, Pro77], Bierstedt [Bie73] and other authors (see the monograph [Pro77] and survey article [Col76] for more references). Most of these investigations have been
concerned with generalizing the space $X$ and taking $E$ to be the scalar field or a locally convex space. Later, Khan [Kh85, Kh87], Nawrocki [Naw89], Prolla [Pro93], Khan and co-authors [KT97, KT02, KMT05, KO05] and Manhas and Singh [MS98] have studied various aspects of the strict, weighted and other related topologies, where $E$ is a Hausdorff TVS. These results are given in later chapters.
CHAPTER 2

Completeness in Function Spaces

In this chapter, we study various completeness properties of the uniform strict, weighted and other related topologies.
1. Completeness in the uniform topology

In this section, we consider completeness of the spaces \((B(X, E), u), (C_b(X, E), u)\) and \((C_0(X, E), u)\). Let \(X\) be a topological space and \(E\) a TVS with a base \(W\) of neighborhoods of 0. Let \(B(X, E)\) denote the vector space of all bounded functions \(f: X \to E\). The uniform topology \(u\) on \(B(X, E)\) is defined as the linear topology which has a base of neighborhoods of 0 consisting of all sets of the form \(N(X, W) = \{f \in B(X, E) : f(X) \subseteq W\}\), where \(W\) over \(W\). If \(E = (E, d)\) is metrizable, then \((B(X, E), u)\) and \((C_b(X, E), u)\) are also metrizable with respect to the metric given by:

\[\rho(f, g) = \sup_{x \in X} d(f(x), f(x)), \quad f, g \in B(X, E) \text{ or } C_b(X, E)\].

**Theorem 2.1.1.** [KN63]

Let \(X\) be a topological space and \(E\) a complete metrizable TVS. Then:

(a) \((B(X, E), u)\) is complete, hence an F-space.
(b) \((C_b(X, E), u)\) is complete, hence an F-space.
(c) If \(X\) is locally compact, then \((C_0(X, E), u)\) is complete, hence an F-space.

**Proof.** (a) Let \(\{f_n\}\) be a \(u\)-Cauchy sequence in \(C_b(X, E)\). Since \(p \leq u\), \(\{f_n\}\) is a \(p\)-Cauchy net in \(C(X, E)\); that is, for each \(x \in X\), \(\{f_n(x)\}\) is a Cauchy net in \(E\). Since \(E\) is complete, for each \(x \in X\), \(f(x) = \lim_{n \to \infty} f_n(x)\) exists in \(E\). We need to show that \(f_n \xrightarrow{u} f\) and that \(f\) is continuous and bounded.

Now, \(f_n \xrightarrow{u} f\), as follows. Let \(W \in \mathcal{W}\) be closed. Since \(E\) is complete, for each \(x \in X\), \(f(x) = \lim_{n \to \infty} f_n(x)\) exists in \(E\). We need to show that \(f_n \xrightarrow{u} f\) and that \(f\) is continuous and bounded.

Thus, \(f_n(x) - f(x) \in W\) for all \(n \geq n_0\) and \(x \in X\).

Fix any \(n \geq n_0\). Since \(W\) is closed and, for each \(x \in X\), \(f_n(x) \rightarrow f(x)\), we have \(f_n(x) - f(x) \in W\) for \(x \in X\).

Thus \(f_n(x) - f(x) \in W\) for all \(n \geq n_0\) and all \(x \in X\); i.e. \(f_n - f \in N(X, W)\) for all \(n \geq n_0\).

This proves that \(f_n \xrightarrow{u} f\).
Finally, $f$ is bounded as follows. Let $W \in \mathcal{W}$. Choose a balanced $V \in \mathcal{W}$ such that $V + V \subseteq W$. Since $f_n \stackrel{u}{\longrightarrow} f$, there exists an $n_0 \geq 1$ such that

$$f_n(x) - f(x) \in V \quad \text{for all } n \geq n_0 \text{ and } x \in X.$$ 

Since $f_{n_0}$ is bounded, we can choose a constant $r > 1$ such that

$$f_{n_0}(x) \subseteq rV \quad \text{for all } x \in X.$$ 

Then, for any $x \in X$,

$$f(x) = [f(x) - f_{n_0}(x)] + f_{n_0}(x) \in V + rV \subseteq rV + rV = r(V + V) \subseteq rW.$$ 

Hence $f$ is bounded on $X$. Then $(C_b(X, E), u)$ is complete.

(b) Let $\{f_n : n \in I\}$ be a $u$-Cauchy net in $C_b(X, E)$. Since $p \leq u$, $\{f_n\}$ is a $p$-Cauchy net in $C_b(X, E)$; that is, for each $x \in X$, $\{f_n(x)\}$ is a Cauchy net in $E$. Since $E$ is complete, for each $x \in X$, $f(x) = \lim_n f_n(x)$ exists in $E$. We need to show that $f_n \stackrel{u}{\longrightarrow} f$ and that $f$ is continuous and bounded.

As shown in part (a), $f$ is bounded and that $f_n \stackrel{u}{\longrightarrow} f$. Therefore, we only need to show that $f$ is continuous on $X$. Let $x_0 \in X$ and $W \in \mathcal{W}$. We show that there exists a neighborhood $G$ of $x_0$ in $X$ such that

$$f(y) - f(x_0) \in W \quad \text{for all } y \in G.$$ 

Choose a balanced $V \in \mathcal{W}$ such that $V + V + V \subseteq W$. Since $f_n \stackrel{u}{\longrightarrow} f$, there exists and $n_0 \in I$ such that

$$f_n(x) - f(x) \in V \quad \text{for all } n \geq n_0 \text{ and all } x \in X. \quad (1)$$ 

Since $f_{n_0}$ is continuous at $x_0$, there exists a neighborhood $G$ of $x_0$ in $X$ such that

$$f_{n_0}(y) - f_{n_0}(x_0) \in V \quad \text{for all } y \in G. \quad (2)$$ 

Then, for any $y \in G$, using (1) and (2),

$$f(y) - f(x_0) = (f(y) - f_{n_0}(y)) + (f_{n_0}(y) - f_{n_0}(x_0)) + (f_{n_0}(x_0) - f(x_0))$$

$$\in V + V + V \subseteq W.$$ 

Hence $f$ is continuous on $X$. Thus $f \in C_b(X, E)$, and so $(C_b(X, E), u)$ is complete.

(c) Let $\{f_n\}$ be a $u$-Cauchy sequence in $C_0(X, E)$. Since $p \leq u$, $\{f_n\}$ is a $p$-Cauchy sequence in $C_0(X, E)$; that is, for each $x \in X$, $\{f_n(x)\}$ is a Cauchy sequence in $E$. Since $E$ is complete, for each $x \in X$, $f(x) = \lim_{n \to \infty} f_n(x)$ exists in $E$. Then as in part (a), $f$ is continuous and that $f_n \stackrel{u}{\longrightarrow} f$. We now need to verify that $f \in C_0(X, E)$. Let $W \in \mathcal{W}$.
Choose a balanced $V \in \mathcal{W}$ such that $V + V \subseteq W$. Since $f_n \xrightarrow{u} f$, there exists $N_0 \geq 1$ such that

$$f_n(x) - f(x) \in V \quad \text{for all } n \geq N_0 \text{ and } x \in X.$$ 

Since $f_{k_0} \in C_0(X, E)$, there exists a compact set $K \subseteq X$ such that

$$f_{k_0}(x) \in V \quad \text{for all } x \in X \setminus K.$$ 

Then, for any $x \in X \setminus K$,

$$f(x) = [f(x) - f_{k_0}(x)] + f_{k_0}(x) \in V + V \subseteq W.$$ 

Therefore $f \in C_0(X, E)$. Thus $(C_0(X, E), u)$ is complete. \qed

**Theorem 2.1.2.** Let $X$ be a locally compact Hausdorff space and $E$ any TVS. Then:

(a) $(C_0(X, E), u)$ is closed in $C_b(X, E)$.

(b) If, in addition, $E$ is complete, then $(C_0(X, E), u)$ is also complete.

**Proof.** (a) Let $f \in F(X, E)$ with $f \in C_0(X, E)^u$. Then there exists a sequence $\{f_n\} \subseteq F(X, E)$ such that $f_n \xrightarrow{u} f$. We need to show that $f \in C_0(X, E)$.

To show that $f$ is continuous on $X$. Let $x_0 \in X$ and $W \in \mathcal{W}$. We show that there exists a neighborhood $G$ of $x_0$ in $X$ such that

$$f(y) - f(x_0) \in W \quad \text{for all } y \in G.$$ 

Choose a balanced $V \in \mathcal{W}$ such that $V + V + V \subseteq W$. Since $f_n \xrightarrow{u} f$, there exists and $n_0 \in \mathcal{I}$ such that

$$f_n(x) - f(x) \in V \quad \text{for all } n \geq n_0 \text{ and } x \in X. \quad (1)$$ 

Since $f_{n_0}$ is continuous at $x_0$, there exists a neighborhood $G$ of $x_0$ in $X$ such that

$$f_{n_0}(y) - f_{n_0}(x_0) \in V \quad \text{for all } y \in G. \quad (2)$$ 

Then, for any $y \in G$, using (1) and (2),

$$f(y) - f(x_0) = (f(y) - f_{n_0}(y)) + (f_{n_0}(y) - f_{n_0}(x_0)) + (f_{n_0}(x_0) - f(x_0)) \in V + V + V \subseteq W.$$ 

Hence $f$ is continuous on $X$. Thus $f \in C_b(X, E)$.

Next, $f$ is bounded on $X$, as follows. Let $W \in \mathcal{W}$. Choose a balanced $V \in \mathcal{W}$ such that $V + V \subseteq W$. Since $f_n \xrightarrow{u} f$, there exists an $n_0 \geq 1$ such that

$$f_n(x) - f(x) \in V \quad \text{for all } n \geq n_0 \text{ and } x \in X.$$ 

Since $f_{n_0}$ is bounded, we can choose a constant $r > 1$ such that

$$f_{n_0}(x) \subseteq rV \quad \text{for all } x \in X.$$
Then, for any $x \in X$,

$$f(x) = [f(x) - f_{n_0}(x)] + f_{n_0}(x) \in V + rV 
\subseteq rV + rV = r(V + V) \subseteq rW.$$ 

Hence $f$ is bounded on $X$.

We now need to verify that $f$ vanishes at infinity. Let $W \in \mathcal{W}$. Choose a balanced $V \in \mathcal{W}$ such that $V + V \subseteq W$. Since $f_n \to f$, there exists $N_0 \geq 1$ such that

$$f_n(x) - f(x) \in V \text{ for all } n \geq N_0 \text{ and } x \in X.$$ 

Since $f_{k_0} \in C_0(X, E)$, there exists a compact set $K \subseteq X$ such that

$$f_{k_0}(x) \in V \text{ for all } x \in X \setminus K.$$ 

Then, for any $x \in X \setminus K$,

$$f(x) = [f(x) - f_{k_0}(x)] + f_{k_0}(x) \in V + V \subseteq W.$$ 

Therefore $f \in C_0(X, E)$. Thus $(C_0(X, E), u)$ is closed in $C_b(X, E)$.

(b) If $E$ is complete, then, by Theorem 2.1.1 (a), $(C_b(X, E), u)$ is complete. Hence, $C_0(X, E)$, being a closed subset of the complete space $C_0(X, E)$, is also complete. \hfill \square

Note. The above theorem gives an alternate proof of Theorem 2.1.1(c).

Recall that a TVS $F$ is called quasi-complete (or boundedly complete) if every bounded closed subset $\mathcal{H}$ of $F$ is complete. Clearly completeness implies quasi-completeness. Further, every closed subset of a quasi-complete TVS is quasi-complete.

**Theorem 2.1.3.** Let $E$ be a quasi-complete TVS. Then:

(a) $(C_b(X, E), u)$ and $(B(X, E), u)$ and are quasi-complete.

(b) If $X$ is locally compact, then $(C_0(X, E), u)$ is quasi-complete.

**Proof.** (a) Let $\mathcal{H}$ be a closed and bounded set on $(C_b(X, E), u)$, and let $\{f_\alpha\}$ be a $u$-Cauchy net in $\mathcal{H}$. Clearly, for each $x \in X$, $\{f(x) : f \in \mathcal{H}\}$ is a bounded set in $E$; in particular, for each $x \in X$, $\{f_\alpha(x)\}$ is a bounded net in $E$. Further, since $p \leq u$, for each $x \in X$, $\{f_\alpha(x)\}$ is a Cauchy net in $E$. Since $E$ is quasi-complete, for each $x \in X$, $f(x) := \lim_\alpha f_\alpha(x)$ exists in $E$, and let

$$f(x) := \lim_\alpha f_\alpha(x), \ x \in X.$$
\( f_\alpha \xrightarrow{u} f : \) Let \( W \in \mathcal{W} \) be closed. Since \( \{ f_\alpha \} \) is \( u \)-Cauchy, there exists \( \alpha_o \in I \) such that

\[
f_\alpha(x) - f_\gamma(x) \in W \text{ for all } \alpha, \gamma \geq \alpha_o \text{ and } x \in X.
\]

Fix any \( \alpha \geq \alpha_o \). Since \( W \) is closed and, for each \( x \in X \), \( f_\gamma(x) \rightarrow f(x) \), we have

\[
f_\alpha(x) - f(x) \in W \text{ for all } x \in X.
\]

Thus \( f_\alpha - f \in \mathcal{N}(X, W) \) for all \( \alpha \geq \alpha_o \). This proves that \( f_\alpha \xrightarrow{u} f \). Finally, since \( \{ f_\alpha \} \subseteq \mathcal{H} \) and \( \mathcal{H} \) is \( u \)-closed, \( f \in \mathcal{H} \). This shows that \( \mathcal{H} \) is complete; hence \((C_b(X, E), u)\) is quasi-complete. By the same argument, \((B(X, E), u)\) is also quasi-complete.

(b) It is similar to part (a). \( \square \)
2. Completeness in the Strict Topology

We consider completeness, sequential completeness and quasi-completeness of the $\beta, \sigma, \sigma_o$ on $C_b(X, E)$ and $\omega_V$ topologies on $CV_b(X, E)$ and of $k$ topology on $C(X, E)$.

Recall that a topological space $(X, \tau)$ is called:

(i) a $k$-space if, for any $U \subseteq X$, $U$ is closed in $X$ whenever $U \cap K$ is closed in $K$ for each compact $K \subseteq X$;

(ii) a $k_\mathbb{R}$-space if a function $f : X \to \mathbb{R}$ is continuous on $X$ whenever $f \mid K$ is continuous for every compact subset $K$ of $X$;

Every locally compact space and every first countable (in particular, metric) space is a $k$-space. If $X$ is a $k$-space and $Y$ any topological space, then a function $f : X \to Y$ is continuous on $X$ iff $f \mid K$ is continuous for every compact subset $K$ of $X$; hence every $k$-space is a $k_\mathbb{R}$-space.

**Theorem 2.2.1.** [Buck58, Kh79] Let $X$ be a $k$-space and $E$ a complete TVS. Then:

(a) $(C_b(X, E), \beta)$ is complete.

(b) $(C(X, E), k)$ is complete.

**Proof.** (a) Let \{ $f_\alpha : \alpha \in I$ \} be a $\beta$-Cauchy net in $C_b(X, E)$.

- Since $p \leq \beta$, \{ $f_\alpha$ \} is a $p$-Cauchy net in $C(X, E)$; that is, for each $x \in X$, \{ $f_\alpha(x)$ \} is a Cauchy net in $E$.
- Since $E$ is complete, for each $x \in X$, $f(x) := \lim_\alpha f_\alpha(x)$ exists in $E$.
- We need to show that $f_\alpha \xrightarrow{\beta} f$ and that $f$ is continuous and bounded. Now, $f_\alpha \xrightarrow{\beta} f$, as follows.
- Let $\varphi \in B_o(X), 0 \leq \varphi \leq 1$, and let $W \in \mathcal{W}$ be closed.
- Since \{ $f_\alpha$ \} is $\beta$-Cauchy, there exists $\alpha_o \in I$ such that
  \[ \varphi(x)[f_\alpha(x) - f_\gamma(x)] \in W \text{ for all } \alpha, \gamma \geq \alpha_o \text{ and } x \in X. \]

Fix any $\alpha \geq \alpha_o$. Since $W$ is closed and, for each $x \in X$, $f_\gamma(x) \to f(x)$, we have

\[ \varphi(x)[f_\alpha(x) - f(x)] \in W \text{ for } x \in X. \]

Thus $\varphi(x)[f_\alpha(x) - f(x)] \in W$ for all $\alpha \geq \alpha_o$ and all $x \in X$; i.e.

\[ f_\alpha - f \in N(\varphi, W) \text{ for all } \alpha \geq \alpha_o. \]

This proves that $f_\alpha \xrightarrow{\beta} f$. Next, $f$ is continuous as follows. Since $X$ is a $k$-space, it suffices to show that, for any compact $K \subseteq X$, $f|K$ is continuous.

Let $x_o \in K$ and $W \in \mathcal{W}$. We show that there exists a neighborhood $G$ of $x_o$ in $X$ such that

\[ f(y) \in f(x_o) + W \text{ for all } y \in G \cap K. \]
Choose a balanced $V \in \mathcal{W}$ such that $V + V + V \subseteq W$. Since $f_\alpha \xrightarrow{\beta} f$ and $k \leq \beta$, $f_\alpha \xrightarrow{k} f$ and so there exists an $\alpha_o \in I$ such that $f_\alpha(x) - f(x) \in V$ for all $\alpha \geq \alpha_o$ and all $x \in K$. \hfill (1)

Since $f_{\alpha_o}$ is continuous at $x_o$, there exists a neighborhood $G$ of $x_o$ in $X$ such that

$$f_{\alpha_o}(y) \in f_{\alpha_o}(x_o) + V \quad \text{for all } y \in G.$$ \hfill (2)

Then, for any $y \in G \cap K$, using (1) and (2),

$$f(y) - f(x_o) = [f(y) - f_{\alpha_o}(y)] + [f_{\alpha_o}(y) - f_{\alpha_o}(x_o)] + [f_{\alpha_o}(x_o) - f(x_o)] \in V + V + V \subseteq W.$$

Finally, $f$ is bounded as follows. Suppose $f$ is unbounded. Then there exists a $W \in \mathcal{W}$ such that $f(X)$ is not contained in $nW$ for all $n \geq 1$. Hence, for each $n \geq 1$, there exists $x_n \in X$ such that $f(x_n) \notin nW$. Choose a balanced $V \in \mathcal{W}$ such that $V + V \subseteq W$. Let

$$\psi = \sum_{n=1}^{\infty} \frac{1}{n} \chi\{x_n\}.$$  

Then $\psi \in B_o(X)$, and so, since $f_\alpha \xrightarrow{\beta} f$, there exists an $\alpha_o \in I$ such that $\psi(x)[f_\alpha(x) - f(x)] \in V$ for all $\alpha \geq \alpha_o$ and $x \in X$. In particular, since $\psi(x_n) = \frac{1}{n}$,

$$f_{\alpha_o}(x_n) - f(x_o) \in nV \quad \text{for all } n \geq 1.$$

Since $f_{\alpha_o}$ is bounded, there exists an integer $m > 1$ such that $f_{\alpha_o}(X) \subseteq mV$. Then

$$f(x_m) = [f(x_m) - f_{\alpha_o}(x_m)] + f_{\alpha_o}(x_m) \in mV + mV \subseteq mW,$$

a contradiction. Hence $f$ is bounded.

(b) Let $\{f_\alpha\}$ be a $k$-Cauchy net in $C(X, E)$, and let

$$f(x) := \lim_{\alpha} f_\alpha(x), \quad x \in X.$$  

Using the arguments of part (a) with replacing $\varphi$ by $\chi_K$, $K$ a compact set in $X$, it easily follows that $f_\alpha \xrightarrow{k} f$ and $f$ is continuous. Hence $C(X, E)$ is $k$-complete. \hfill \Box

**Theorem 2.2.2.** [KR91] Let $X$ be a $k$-space and $E$ a complete TVS. Then

(a) $(C_b(X, E), \sigma)$ is complete.

(b) If $k \leq \sigma_o$, then $(C_b(X, E), \sigma_o)$ is complete.
Proof. (a) Let \( \{f_\alpha\} \) be a \( \sigma \)-Cauchy net in \( (C^b(X, E)) \) and, for each \( x \in X \), let
\[
f(x) := \lim_{\alpha} f_\alpha(x), \quad x \in X.
\]
As in the proof of Theorem 2.2.1, since \( X \) is a \( k \)-space and \( k \leq \sigma \), it is easily seen that \( f_\alpha \overset{\sigma}{\to} f \) and that \( f \) is continuous on \( X \). Suppose \( f \) is not bounded. Then there exist a \( W \in \mathcal{W} \) and a sequence \( \{x_n\} \) in \( X \) such that
\[
f\{x_n\} \notin nW \quad \text{for all } n \geq 1.
\]
Let \( A = \{x_n\} \) and choose a \( V \in \mathcal{W} \) such that \( V + V \subseteq W \). There exists an index \( \alpha_o \) such that
\[
f_\alpha(x) - f_\alpha(x) \in V \quad \text{for all } \alpha \geq \alpha_o \text{ and } x \in A.
\]
Let \( \lambda \) be a positive number such that \( f_\alpha(X) \subseteq \lambda V \). Then, for \( n > \lambda \),
\[
f(x_n) = [f(x_n) - f_\alpha(x_n)] + f_\alpha(x_n) \in V + \lambda V \subseteq nW,
\]
a contradiction. Hence \( (C^b(X, E), \sigma) \) is complete.

(b) The proof is almost identical to the one given for (a). \( \square \)

**Theorem 2.2.3.** [KR91] Let \( E \) be a sequentially complete TVS. Then \( (C^b(X, E), \sigma) \) and \( (C^b(X, E), \sigma_o) \) are sequentially complete.

Proof. Let \( \{f_n\} \) be a \( \sigma \)-Cauchy sequence in \( C^b(X, E) \), and let
\[
f(x) := \lim_{\alpha} f_\alpha(x), \quad x \in X.
\]
As in the proof of Theorem 2.2.2, we see that \( f_n \overset{\sigma}{\to} f \) and that \( f \) is bounded. Suppose that \( f \) is not continuous at some \( x_o \in X \). Then, without loss of generality, we may assume that there exist a net \( \{x_\lambda : \lambda \in I\} \) in \( X \), with \( x_\lambda \to x_o \), and a \( W \in \mathcal{W} \) such that
\[
f(x_\lambda) - f(x_o) \notin W \quad \text{for all } \lambda \in I.
\]
Choose a \( V \in \mathcal{W} \) such that \( V + V + V \subseteq W \). The functions \( f_n \) \( (n = 1, 2, ...) \) are continuous and so, for each positive integer \( n \), there exists an index \( \lambda_n \) such that
\[
f_n(x_\lambda) - f_n(x_o) \in V \quad \text{for all } \lambda \geq \lambda_n.
\]
Let \( A = \{x_o, x_{\lambda_1}, x_{\lambda_2}, \ldots\} \), a \( \sigma \)-compact set. Choose an integer \( m \) such that
\[
f_n - f \in N(A, V) \quad \text{for all } n \geq m.
\]
Then, for \( n \geq m \),
\[
f(x_\lambda) - f(x_o) = [f(x_\lambda) - f_n(x_\lambda)] + [f_n(x_\lambda) - f_n(x_o)]
+ [f_n(x_o) - f(x_o)]
\]
a contradiction to (1). Hence \((C_b(X, E), \sigma)\) is sequentially complete. The sequential completeness of \((C_b(X, E), \sigma_o)\) may be proved in the same way. \(\Box\)

**Theorem 2.2.4.** [KR91] Let \(E\) be a complete locally bounded TVS. If \((C_b(X, E), \sigma)\) (resp. \((C_b(X, E), \sigma_o)\)) is quasi-complete, then it is complete.

**Proof.** Suppose that \((C_b(X, E), \sigma)\) is quasi-complete, and let \(\{f_\alpha\}\) be a \(\sigma\)-Cauchy net in \(C_b(X, E)\). Then, as in Theorem 2.1.3, there exists a bounded \(E\)-valued function \(f\) on \(X\) such that \(f_\alpha \sigma \rightarrow f\). We show that \(f\) is continuous, as follows. Let \(V \in \mathcal{W}\) be bounded. There exists a closed shrinkable neighborhood \(S\) of 0 in \(E\) such that \(S \subseteq V\); the Minkowski functional \(\rho_S\) is continuous (§ A.4). Since \(V\) is bounded, we may choose a real number \(\lambda \geq 1\), such that \(V + V \subseteq \lambda S\), and, since \(f\) is bounded, we may assume that

\[
f(X) \subseteq \frac{r}{\lambda}S \quad \text{for some } r \geq 1.
\]

We define a function \(h_r : E \rightarrow E\) by

\[
h_r(x) = \begin{cases} x & \text{if } x \in rS, \\ \frac{r}{\rho_S(x)} x & \text{if } x \in E \setminus rS. \end{cases}
\]

Since \(\rho_S\) is continuous and \(\rho_S(x) = r\) if \(x \in \partial(rS)\), it follows from Map Gluing Theorem (§ A.1) that \(h_r\) is continuous. Now \(h_r(E) \subseteq rS\) and so the net \(\{h_r \circ f_\alpha\}\) is \(\sigma\)-bounded. Furthermore, \(\{h_r \circ f_\alpha\}\) is \(\sigma\)-Cauchy, as follows. Let \(A\) be \(\sigma\)-compact and let \(W \in \mathcal{W}\) be any neighborhood of 0 in \(E\). Choose \(t \geq 1\) such that \(V + V \subseteq tW\). There exists an index \(\alpha_o\) such that

\[
f_\alpha - f \in N(A, \frac{1}{t\lambda}V) \quad \text{for all } \alpha \geq \alpha_o.
\]

This implies that \(f_\alpha(A) \subseteq rS\) for \(\alpha \geq \alpha_o\). Hence, for any \(x \in A\) and \(\alpha, \gamma \geq \alpha_o\),

\[
h_r(f_\alpha(x)) - h_r(f_\gamma(x)) = f_\alpha(x) - f_\alpha(x) \in \frac{1}{t\lambda}(V + V) \subseteq W.
\]

Since \((C_b(X, E), \sigma)\) is quasi-complete there exists a function \(g\) in \(C_b(X, E)\) such that

\[
h_r \circ f_\alpha \sigma \rightarrow g.
\]

The above argument also shows that \(h_r \circ f_\alpha \sigma \rightarrow f\), and so \(f = g\). Thus \(f\) is continuous.
Assuming quasi-completeness of \((C_b(X, E), \sigma_o)\), its completeness may be proved by using a similar argument as above. □
3. Completion of \((C_b(X, E), \beta)\)

We now consider the completion of \((C_b(X, E), \beta)\), where \(X\) is not necessarily a \(k\)-space and \(E\) is not necessarily complete. Following Katsaras [Kat81], let

\[ B_{ck}(X, E) = \{ f \in B(X, E) : f|K \text{ is continuous for each compact } K \subseteq X \} \]

On \(B_{ck}(X, E)\) we consider the linear topology \(\beta_c\) which has as a base of neighborhoods of 0 consisting of all sets of the form

\[ \{ f \in B_{ck}(X, E) : (\varphi f)(X) \subseteq W \}, \]

where \(W\) is a neighborhood of 0 in \(E\) and \(\varphi \in B_o(X)\). Clearly the topology induced on \(C_b(X, E)\) by \(\beta_c\) is the strict topology \(\beta\).

**Theorem 2.3.1.** [Kat81] If \(E\) is complete, then \((B_{ck}(X, E), \beta_c)\) is also complete.

**Proof.** Let \((f_\alpha)\) be a \(\beta_c\)-Cauchy net in \(B_{ck}(X, E)\). Since \(p \leq \beta_c\) and \(E\) is complete, there exists an \(E\)-valued function \(f\) on \(X\) such that \(f(x) = \lim_\alpha f_\alpha(x)\) for each \(x \in X\). Now, by the argument similar to the one used in the proof of Theorem 2.2.1, it is easy to verify that \(f \in B_{ck}(X, E)\) and \(\beta_c\)-\(\lim_\alpha f_\alpha = f\).

**Theorem 2.3.2.** [Kat81] If \((C_b(X, E), \beta)\) is complete, then \(E\) is complete.

**Proof.** Let \(\{s_\alpha\}\) be a Cauchy net in \(E\). For each \(\alpha\), let \(f_\alpha : X \to E\) be the constant function given by \(f_\alpha(x) = s_\alpha, \quad x \in X\). Clearly, the net \(\{f_\alpha\}\) is Cauchy in \((C_b(X, E), \beta)\). Since \((C_b(X, E), \beta)\) is complete, there exists an \(f \in C_b(X, E)\) such that \(f_\alpha \to f\) in the topology \(\beta\). Hence, for any \(x \in X\), \(\lim_\alpha s_\alpha = \lim_\alpha f_\alpha(x) = f(x)\). \(\square\)

**Definition.** Let \(\hat{E}\) be the unique (up to a topological isomorphism) Hausdorff completion of \(E\). We may consider \(E\) as a dense subspace of \(\hat{E}\). Let \(\hat{\beta}_c\) denote the linear topology on \(B_{ck}(X, \hat{E})\) which has as a base of neighborhoods of 0 consisting of all sets of the form

\[ \hat{U}(\varphi, W) = \{ f \in B_{ck}(X, \hat{E}) : (\varphi f)(X) \subseteq W \}, \]

where \(\varphi \in B_o(X)\) and \(W\) is a neighborhood of 0 in \(\hat{E}\).

Using Theorem 2.3.1, we have the following.

**Theorem 2.3.3.** [Kat81] With the above notations, the completion of \((C_b(X, E), \beta)\) is the \(\hat{\beta}_c\)-closure of \(C_b(X, E)\) in \(B_{ck}(X, \hat{E})\) with the induced topology.
Theorem 2.3.4. Let $B_{ck}(X, \hat{E})$ and $\hat{\beta}_c$ be as above and suppose that, for every compact subset $K$ of $X$, the space $C(K) \otimes \hat{E}$ is $u$-dense in $C(K, \hat{E})$. Then $(B_{ck}(X, \hat{E}), \hat{\beta}_c)$ is the completion of $(C_b(X, E), \beta)$.

Proof. Let $f \in B_{ck}(X, \hat{E})$. We need to show that $f$ belongs to the $\hat{\beta}_c$-closure of $C_b(X, E)$. Consider any $\hat{\beta}_c$-neighborhood $\hat{U}(\varphi, W)$ of 0 in $B_{ck}(X, \hat{E})$, where $\varphi \in B_0(X), 0 \leq \varphi \leq 1$, and $W$ is a neighborhood of 0 in $\hat{E}$. Choose an open balanced neighborhood $V$ of 0 in $\hat{E}$ such that $V + V + V + V \subseteq W$. Let $\lambda > 1$ be such that $f(X) \subseteq \lambda V$. There exists a compact subset $K$ of $X$ such that

$$\varphi(x) < \frac{1}{\lambda} \text{ for all } x \in X \setminus K.$$ 

Since $f|K \in C(K, \hat{E})$, there exists $h \in C(K) \otimes \hat{E}$ such that $(h - f)(K) \subseteq \frac{1}{\lambda}V$. We have

$$h(K) \subseteq f(K) + \frac{1}{\lambda}V \subseteq \lambda V + V \subseteq \lambda V + \lambda V.$$ 

There exists an extension $h_1 \in C_b(X) \otimes \hat{E}$, $h_1 = h$ on $K$, $h_1(X) \subseteq \lambda V + \lambda V$. Since $E$ is dense in $\hat{E}$, there exists $g \in C_b(X) \otimes E$ such that $(g - h_1)(X) \subseteq V$. Now

$$(h_1 - f)(X) \subseteq h_1(X) - f(X) \subseteq \lambda V + \lambda V + \lambda V.$$ 

Let $x \in X$. If $x \in K$,

$$\varphi(x)[g(x) - f(x)] = \varphi(x)[g(x) - h_1(x)] + \varphi(x)[h_1(x) - f(x)] \in \varphi(x)V + V \subseteq V + V \subseteq W.$$ 

If $x \in X \setminus K$,

$$\varphi(x)[g(x) - f(x)] = \varphi(x)[g(x) - h_1(x)] + \varphi(x)[h_1(x) - f(x)] \in \varphi(x)V + \frac{1}{\lambda}[\lambda V + \lambda V + \lambda V] \subseteq W.$$ 

Therefore $g - f \in \hat{U}(\varphi, W)$. Thus, by the above theorem, $(B_{ck}(X, \hat{E}), \hat{\beta}_c)$ is the completion of $(C_b(X, E), \beta)$.

Corollary 2.2.5. [Kat81] $(B_{ck}(X, \hat{E}), \hat{\beta}_c)$ coincides with the completion of $(C_b(X, E), \beta)$ in each of the following cases:

1. $X$ has finite covering dimension.
(2) Every compact subset of $X$ has finite covering dimension.

(3) $E$ is a complete metrizable TVS with a basis.

(4) $E$ is locally convex.

(5) $E$ is a complete TVS and has the approximation property.

Proof. The proof follows from the above theorem and a later result, corollary of the Stone-Weierstrass theorem (section 4.1), by which each of the conditions (1)-(5) implies that, for every compact subset $K$ of $X$, the space $C(K) \otimes \hat{E}$ is $u$-dense in $C(K, \hat{E})$. 
4. Completeness in the Weighted Topology

We now consider completeness in the weighted topology. Let $V$ be a Nachbin family on $X$, and let $FV_o(X, E)$ denote the set of all $E$-valued functions $f$ on $X$ such that, for each $v \in V$, $vf$ is bounded and vanishes at infinity. We may define the weighted topology $\omega_V$ on $FV_o(X, E)$ in the same way as in Section 1.2:

**Definition.** $\omega_V$ is the linear topology $FV_o(X, E)$ which has a base of neighborhoods of 0 consisting of all sets of the form $N(v, G) = \{ f \in FV_o(X, E) : (vf)(x) \subseteq G \}$, where $v \in V$ and $G \in \mathcal{W}$. Clearly, $(FV_o(X, E), \omega_V)$ is Hausdorff if $V > 0$.

**Theorem 2.4.1.** [Pro71b, Kh85b] Let $V$ be a Nachbin family on $X$ with $V > 0$, and suppose $E$ is complete. Then $(FV_o(X, E), \omega_V)$ is complete; hence $(CV_o(X, E), \omega_V)$ is complete iff it is closed in $(FV_o(X, E), \omega_V)$.

**Proof.** Let $\{ f_\alpha \}$ be an $\omega_V$-Cauchy net in $FV_o(X, E)$. Since $V > 0$, $\rho \leq \omega_V$ and so $\{ f_\alpha \}$ is $\rho$-Cauchy. Since $E$ is complete, for each $x \in X$, $f(x) = \lim_\alpha f_\alpha(x)$ exists, and it is easily shown that $f_\alpha \xrightarrow{\omega_V} f$. Let $v \in V$, $G \in \mathcal{W}$. Choose a balanced $H \in \mathcal{W}$ with $H + H \subseteq G$. There exists an index $\alpha_0$ such that

$$v(x)(f_\alpha(x) - f(x)) \in H \text{ for all } \alpha \geq \alpha_0 \text{ and } x \in X.$$ 

Choose $r \geq 1$ with $(vf_\alpha)(x) \subseteq rH$. Then, for any $x \in X$,

$$v(x)f(x) = v(x)(f(x) - f_\alpha(x)) + v(x)f_\alpha(x) \in H + rG \subseteq rG,$$

and so $vf$ is bounded. Next, with $v$, $G$, $H$, and $\alpha_0$ as above, choose a compact set $K \subseteq X$ such that $v(x)f_\alpha(x) \in H$ for all $x \not\in K$. Then, for $x \not\in K$, we have $v(x)f(x) \in G$. Thus $f \in FV_o(X, E)$.

**Theorem 2.4.2.** [Pro71b, Kh85b] Suppose $E$ is complete, and $U$ and $V$ are Nachbin families on $X$ with $U \leq V$ and $V > 0$.

(a) If $CU_o(X, E)$ is closed in $(FU_o(X, E), \omega_U)$, then $(CV_o(X, E), \omega_V)$ is complete.

(b) If $(FU_o(X, E), \omega_U)$ is complete (quasi-complete), so is $(CU_o(X, E), \omega_V)$.

**Proof.** (a) In view of Theorem 2.4.1, we need to show that $CV_o(X, E)$ is $\omega_V$-closed in $FV_o(X, E)$. Let $f$ belong to the $\omega_V$-closure of $CV_o(X, E)$ in $FV_o(X, E)$. There exists a net $\{ f_\alpha \} \subseteq CV_o(X, E)$ with $f_\alpha \xrightarrow{\omega_V} f$. Since $U \leq V$, $\omega_U \leq \omega_V$ and so $f_\alpha \xrightarrow{\omega_U} f$. Since $CV_o(X, E) \subseteq CU_o(X, E)$ and
$CU_o(X,E)$ is $\omega_V$-closed in $FU_o(X,E)$, we have $f \in FU_o(X,E)$. Hence $f$ is continuous and so $f \in CV_o(X,E)$.

(b) If $(CU_o(X,E), \omega_U)$ is complete, then it follows immediately from (a) that $(CV_o(X,E), \omega_V)$ is complete. Suppose now that $(CU_o(X,E), \omega_V)$ is quasi-complete, and let $\{f_\alpha\}$ be a bounded Cauchy net in $(CU_o(X,E), \omega_V)$.

By Theorem 2.4.1, there exists an $f \in (FU_o(X,E))$ such that $f_\alpha \xrightarrow{\omega_V} f$. Since $U \leq V$, $\{f_\alpha\} \subseteq FV_o(X,E)$ such that $f_\alpha \xrightarrow{\omega_U} f$. Clearly, $\{f_\alpha\}$ is $\omega_U$-bounded. Then, by hypothesis, $f \in CU_o(X,E)$. Hence $f \in CV_o(X,E)$.

□

**Corollary 2.4.3.** [Pro71b, Kh85b] If $X$ is a $k$-space and $E$ is complete, then $(CB(X,E), \beta)$ is complete.

*Proof.* Let $V = S^+_o(X)$. Then $\omega_V = \beta$ on $CV_o(X,E) = CB(X,E)$. Further, for each $x \in X$, $\chi_{\{x\}} \in V$, and so $V > 0$. Take $U = K^+_c(X)$. Then $(CU_o(X,E), \omega_U) = (C(X,E), k)$ which is complete since $X$ is a $k$-space (see Theorem 2.2.1). So, by Theorem 2.4.2, $(CB(X,E), \beta)$ is complete.
5. Notes and Comments

Section 2.1. We present some classical results on completeness of the spaces $(B(X, E), u)$, $(C_b(X, E), u)$ and $(C_o(X, E), u)$, as given in [KN63].

Section 2.2. Various completeness properties (such as completeness, sequential completeness and quasi-completeness) of function spaces with the topologies $u, \beta, \sigma$ and $k$ are considered. These are taken from [Kh79, KR91] which generalize earlier results given in [Buc58, GulD72].

Section 2.3. This includes some results, due to Katsaras [Kat81], on the completion of $(C_b(X, E), \beta)$, where $X$ is not necessarily a $k$-space and $E$ is not necessarily complete.

Section 2.4. Completeness in the weighted spaces $CV_b(X, E)$ and $CV_o(X, E)$ is considered. These were obtained by Summers [SumW69] for $E$ a scalar field, by Prolla [Pro71b] for $E$ a locally convex TVS and by Khan [Kh85] for $E$ any TVS.
CHAPTER 3

The Arzela-Ascoli type Theorems

In this chapter we study Arzela-Ascoli type theorems for the compact-open, strict and weighted topologies. The abstract version of the Arzela-Ascoli theorem was first given by Myers [My46] which was later improved by Gale [Gal50]. The name *k*-space also first appeared in the paper [Gal50]. We include here some related results from [KN63, Kh79, RS88, KO05].
1. Arzela-Ascoli Theorem for \( k \) and \( \beta \) Topologies

In this section, we present Arzela-Ascoli type theorems for the compact-open and strict topologies.

**Definition.** Let \( X \) be a topological space and \( E \) a TVS with a base \( \mathcal{W} \) of neighborhoods of 0. A subset \( A \) of \( C(X, E) \) is said to be equicontinuous at \( x_o \in X \) if, for each \( W \in \mathcal{W} \), there exists a neighborhood \( G(x_o) \) of \( x_o \) such that
\[
f(x) - f(x_o) \in W \quad \text{for all} \quad x \in G(x_o) \quad \text{and} \quad f \in A.
\]

\( A \) is said to be equicontinuous on \( X \) if it is equicontinuous at each point of \( X \).

The following version of the Arzela-Ascoli type theorem for the compact-open topology is given in [KN63, p. 81].

**Theorem 3.1.1.** [KN63, p. 81] Let \( X \) be a Hausdorff \( k \)-space and \( E \) a Hausdorff TVS, and let \( A \subseteq C(X, E) \).

\( A \) is \( k \)-compact iff the following conditions are satisfied.

(i) \( A \) is \( k \)-closed;
(ii) \( A(x) = \{ f(x) : f \in A \} \) is relatively compact in \( E \) for each \( x \in X \);
(iii) \( A \) is equicontinuous on each compact subset of \( X \).

For the readers’ benefit and interest, we present here a well-known version of above theorem in the more general setting of uniform spaces. Its proof is adapted from [Eng88, p. 440-443; Will70, § 43].

Recall that, if \( Y \) is a non-empty set, then, for any \( A, B \subseteq Y \times Y \), we define
\[
A^{-1} = \{ (y, x) : (x, y) \in A \},
\]
\[
A \circ B = \{ (x, y) \in Y \times Y : \exists z \in Y \text{ such that } (x, z) \in A \text{ and } (z, y) \in B \}.
\]

If \( A = B \), we shall write \( A \circ A = A^2 \). Note that,
\[
(x, z) \in A, (z, y) \in B \Rightarrow (x, y) \in A \circ B.
\]

A subset \( \mathcal{H} \) of subsets of \( Y \times Y \) is called a uniformity (or a uniform structure) on \( Y \) if

\( (u_1) \Delta(Y) = \{ (x, x) : x \in Y \} \subseteq A \) for all \( A \in \mathcal{H} \).
\( (u_2) A \in \mathcal{H} \implies A^{-1} \in \mathcal{H} \), i.e. each \( A \in \mathcal{H} \) is symmetric.
\( (u_3) \) If \( A \in \mathcal{H} \), \( \exists \) some \( B \in \mathcal{H} \) such that \( B^2 \subseteq A \).
\( (u_4) \) If \( A, B \in \mathcal{H} \), \( \exists \) some \( C \in \mathcal{H} \) such that \( C \subseteq A \cap B \).
\( (u_5) \) If \( A \in \mathcal{H} \) and \( A \subseteq B \), then \( B \in \mathcal{H} \).
In this case, the pair \((Y, \mathcal{H})\) is called a uniform space. The members of \(\mathcal{H}\) are called vicinities of \(\mathcal{H}\). For any \(x \in Y\) and \(A \in \mathcal{H}\), let

\[ N_A(x) = \{ y \in Y : (x, y) \in A \}. \]

Then, for each \(x \in Y\), the collection \(\{ N_A(x) : A \in \mathcal{H} \}\) satisfies all the conditions of a neighborhood system at \(x\). Consequently, there exists a unique topology \(\tau_\mathcal{H}\) on \(Y\) such that, for each \(x \in Y\), the collection \(\{ N_A(x) : A \in \mathcal{H} \}\) forms a local base at \(x\). In this case, the topology \(\tau_\mathcal{H}\) on \(Y\) is said to be the topology induced by (or derived from) the uniformity \(\mathcal{H}\). For any net \(\{y_\alpha\} \subseteq (Y, \mathcal{H})\), we say that \(y_\alpha \to y \in Y\) if, for each \(V \in \mathcal{H}\), there exists an \(\alpha_0 \in I\) such that \((y_\alpha, y) \in V\) for all \(\alpha \geq \alpha_0\).

Every TVS \((E, \tau)\) is a uniform space. [In fact, let \(W\) be a base of neighborhoods of 0 in \(E\). For each \(V \in W\), let \(A_V = \{(x, y) \in E \times E : x - y \in V\}\). Then \(\mathcal{H} = \{A_V : V \in W\}\) is a uniformity on \(E\).]

**Definition.** Let \(X\) be a Hausdorff space and \((Y, \mathcal{H})\) a uniform space, and let \(\mathcal{S} = \mathcal{S}(X)\) be a certain collection of subsets of \(X\). For any \(K \in \mathcal{S}(X)\) and \(D \in \mathcal{H}\), let

\[ M(K, D) = \{(f, g) \in C(X, Y) \times C(X, Y) : (f(x), g(x)) \in D \text{ for all } x \in K\}. \]

Then the collection \(\{M(K, D) : K \in \mathcal{S}(X) \text{ and } D \in \mathcal{H}\}\) form a subbase for a uniformity, called the uniformity of uniform convergence on the sets in \(\mathcal{S}(X)\) induced by \(\mathcal{H}\). The resultant topology \(\tau_\mathcal{S}\) on \(C(X, Y)\) is called the topology of uniform convergence on the sets in \(\mathcal{S}(X)\).

1. If \(\mathcal{S}(X) = \{X\}\), then the \(\tau_\mathcal{S}\)-topology on \(C(X, Y)\) is called the uniform topology and is denoted by \(u\).
2. If \(\mathcal{S}(X)\) is the collection of all compact subsets of \(X\), then the \(\tau_\mathcal{S}\)-topology on \(C(X, Y)\) is called the compact-open topology and is denoted by \(k\).
3. If \(\mathcal{S}(X)\) is the collection of all finite (or singleton) subsets of \(X\), then the \(\tau_\mathcal{S}\)-topology on \(C(X, Y)\) is called the pointwise topology and is denoted by \(p\).

Clearly \(p \leq k \leq u\) on \(C(X, Y)\). and \(k = u\) if \(X\) is compact.

**Alternate Definition.** [Edw65, p. 33] Let \(X\) be a Hausdorff space and \((Y, \mathcal{H})\) a uniform space having a base \(\mathcal{B}\) of vicinities of \(\mathcal{H}\), and let
$S = \mathcal{S}(X)$ be a collection of subsets of $X$. For any $f \in C(X,Y)$, $K \in \mathcal{S}(X)$ and $A \in \mathcal{B}$ (or $A \in \mathcal{H}$), let

$$N(f, K, A) = \{ g \in C(X,Y) : (g(x), f(x)) \in A \text{ for all } x \in K \}.$$ 

Then the collection $\{ N(f, K, A) : f \in C(X,Y), K \in \mathcal{S}(X), A \in \mathcal{B} \}$ form a subbase for a topology $\tau_S$ on $C(X,Y)$, called the topology of uniform convergence on the sets in $\mathcal{S}(X)$.

**Definition.** [Eng88, p. 442] Let $X$ be a Hausdorff space and $(Y, \mathcal{H})$ a uniform space. A subset $A$ of $C(X,Y)$ is said to be equicontinuous at $x_o \in X$ if, for each $D \in \mathcal{H}$, there exists a neighborhood $G(x_o)$ of $x_o$ such that

$$(f(x), f(x_o)) \in D \text{ for all } x \in G(x_o) \text{ and } f \in A.$$ 

$A$ is said to be equicontinuous on $X$ if it is equicontinuous at each point of $X$.

**Lemma 3.1.2.** Let $X$ be a Hausdorff space and $(Y, \mathcal{H})$ a uniform space, and let $A \subseteq C(X,Y)$. If $A$ is equicontinuous on $X$, then $\overline{A}^p$ is also equicontinuous on $X$.

**Proof.** Let $x_o \in X$ and $D \in \mathcal{H}$. We show that there exists a neighborhood $G(x_o)$ of $x_o$ such that

$$(g(y), g(x_o)) \in D \text{ for all } y \in G(x_o) \text{ and } g \in \overline{A}^p.$$  \hspace{1cm} (1)

Choose a symmetric $V \in \mathcal{H}$ with $V \circ V \circ V \subseteq D$. Since $A$ is equicontinuous on $X$, there exists a neighborhood $G(x_o)$ of $x_o$ such that

$$(f(y), f(x_o)) \in V \text{ for all } y \in G(x_o) \text{ and } f \in A.$$  \hspace{1cm} (2)

Let $f \in \overline{A}^p$, and let $\{ f_\alpha : \alpha \in I \}$ be a net in $A$ such that $f_\alpha \overset{p}{\rightarrow} f$. Fix any $y \in G(x_o)$. Choose $\alpha_0 \in I$ such that

$$(f_\alpha_0(y), f(y)) \in V \text{ and } (f_\alpha_0(x_o), f(x_o)) \in V.$$  \hspace{1cm} (3)

Now, by (2) and (3),

$$(f(y), f_\alpha_0(y)), (f_\alpha_0(y), f_\alpha_0(x_o)), (f_\alpha_0(x_o), f(x_o)) \in V;$$

and so

$$(f(y), f(x_o)) \in V \circ V \circ V \subseteq D.$$ 

This proves (1), Thus $\overline{A}^p$ is equicontinuous on $X$. \hspace{1cm} \square

**Lemma 3.1.3** Let $X$ be a Hausdorff space and $(Y, \mathcal{H})$ a uniform space, and let $A \subseteq C(X,Y)$. If $A$ is equicontinuous on each compact subset of $X$, then:

(a) The topologies $k$ and $p$ coincide on $A$.

(b) $\overline{A}^k = \overline{A}^p$. 
Proof. The proofs of (a) and (b) are similar and so we shall only give the proof of (b).

Since $p \subseteq k$, $\mathcal{A}^k \subseteq \mathcal{A}^\beta$ and so we need to prove that $\mathcal{A}^\beta \subseteq \mathcal{A}^k$. Let $f \in \mathcal{A}^\beta$ and let \{ $f_\alpha : \alpha \in I$ \} be a net in $\mathcal{A}$ such that $p\text{-}\lim f_\alpha = f$. Let $K \subseteq X$ be compact and $D \in \mathcal{H}$. Choose a symmetric $V \in \mathcal{H}$ with $V \circ V \circ V \subseteq D$. Since $p\text{-}\lim f_\alpha = f$, for each $x \in K$, there exists $\alpha_x \in I$ such that
\[(f_\alpha(x), f(x)) \in V \text{ for all } \alpha \geq \alpha_x. \tag{1}\]
Since $\mathcal{A}$ is equicontinuous subset on $K$, by Lemma 3.1.2, $\mathcal{A}^k$ is also equicontinuous on $K$. Hence there exists a neighborhood $G(x)$ of $x$ such that
\[(g(y), g(x)) \in V \text{ for all } y \in G(x) \cap K \text{ and } g \in \mathcal{A}^\beta. \tag{2}\]
Choose a finite subcover \{ $G(x_i) : i = 1, \ldots, m$ \} of \{ $G(x) : x \in K$ \} for $K$. Choose $\alpha_0 \in I$ such that $\alpha_0 \geq \alpha_{x_i}$, $i = 1, \ldots, m$. Let $y \in K$. Then $y \in G(x_q) \cap K$ for some $q$ ($1 \leq q \leq m$). Hence, for all $\alpha \geq \alpha_0$, it follows from (1) and (2) that
\[(f_\alpha(y), f_\alpha(x_q)), (f_\alpha(x_q), f(x_q)), (f(x_q), f(y)) \in V; \tag{3}\]
consequently,
\[(f_\alpha(y), f(y)) \in V \circ V \circ V \subseteq D. \tag{4}\]
This implies that $(f_\alpha, f) \in M(K, D)$ for all $\alpha \geq \alpha_0$. Hence $f \in \mathcal{A}^k$. \hfill $\square$

We now establish a version of the Arzela-Ascoli theorem for $(C(X,Y), k)$.

**Theorem 3.1.4.** Let $X$ be a Hausdorff $k$-space and $(Y, \mathcal{H})$ a Hausdorff uniform space, and let $\mathcal{A} \subseteq C(X,Y)$. Then $\mathcal{A}$ is $k$-compact iff the following conditions are satisfied.

(i) $\mathcal{A}$ is $k$-closed;

(ii) $\mathcal{A}(x) = \{ f(x) : f \in \mathcal{A} \}$ is relatively compact in $Y$ for each $x \in X$;

(iii) $\mathcal{A}$ is equicontinuous on each compact subset of $X$.

**Proof.** ($\Rightarrow$) Suppose $\mathcal{A}$ is $k$-compact in $C(X,Y)$. Then conditions (i) - (iii) hold, as follows.

(i) Since $Y$ is Hausdorff, $C(X,Y), k)$ is also Hausdorff. So $\mathcal{A}$, being a compact subset of the Hausdorff space, is $k$-closed.

(ii) Let $x$ be any point in $X$. Consider the evaluation map $\delta_x : C(X,Y) \to Y$ given by
\[\delta_x(f) = f(x), \quad f \in C(X,Y).\]
Now $\delta_x$ is $p$-continuous, as follows. For any $f \in C(X, Y)$ and $D \in \mathcal{H}$, let

$$N(f, \{x\}, D) = \{g \in C(X, Y) : (g(x), f(x)) \in D\}.$$  

It is clear that $N(f, \{x\}, D)$ is a $p$-neighborhood of $f$ in $C(X, Y)$ and

$$\delta_x(N(f, \{x\}, D)) = \{g(x) : (g(x), f(x)) \in D\} \subseteq N_D(f(x)).$$  

Hence $\delta_x$ is $p$-continuous. Since $p \leq k$ and $\mathcal{A}$ is $k$-compact, it follows that $\mathcal{A}$ is $p$-compact. Thus $\delta_x(\mathcal{A}) = \mathcal{A}(x)$ is relatively compact in $Y$.

(iii) Suppose $K$ is any compact subset of $X$, let $x_o \in K$ and $D \in \mathcal{H}$. Choose a symmetric $V \in \mathcal{H}$ with $V \circ V \circ V \subseteq D$. Now, for each $f \in \mathcal{A}$, there exists a neighborhood $G_1(x_o, f)$ of $x_o$ such that

$$(f(x), f(x_o)) \in D \text{ for all } x \in G_1(x_o, f).$$

Since $K$, being compact and Hausdorff, is regular, there exists a closed neighborhood $G(x_o, f)$ of $x_o$ in $K$ such that $G(x_o, f) \subseteq G_1(x_o, f) \cap K$. Let

$$N(f, G(x_o, f), V) = \{g \in C(X, Y) : (g(x), f(x)) \in V \text{ for all } x \in G(x_o, f)\},$$

a $k$-open neighborhood of $f$. Since $\mathcal{A}$ is $k$-compact, its open cover

$$\{N(f, G(x_o, f), V) : f \in \mathcal{A}\}$$

has a finite subcover $\{N(f_i, G(x_o, f_i), V), i = 1, ..., m\}$. Let $f \in \mathcal{A}$, so that $f \in N(f_j, G(x_o, f_j))$ for some $j$ ($1 \leq j \leq m$). Then, for any $x \in \bigcap_{i=1}^m G(x_o, f_i)$,

$$(f(x), f_j(x)), (f_j(x), f_j(x_o)), (f_j(x_o), f(x_o)) \in V,$$

and so

$$(f(x), f(x_o)) \in V \circ V \circ V \subseteq D.$$

So $\mathcal{A}$ is equicontinuous at $x_o$ and hence on $K$.

$[\Leftarrow]$ Suppose that $\mathcal{A}$ satisfies conditions (i) - (iii). Since $\mathcal{A}$ is equicontinuous on each compact subset of $X$, it follows from Lemmas 3.1.2 and 3.1.3 that $\mathcal{A}^p$ is equicontinuous, $k = p$ on $\mathcal{A}^p$ and $\mathcal{A}^p = \mathcal{A}^k$. Since $\mathcal{A}$ is $k$-closed, to show that $\mathcal{A}$ is $k$-compact, it suffices to show that $\mathcal{A}$ is $p$-compact. We may regard $C(X, Y)$ as being embedded in the product space $\prod_{x \in X} Y_x$ where each $Y_x = Y$ via the correspondence $f \to (f(x))_{x \in X}$. The product topology on $\prod_{x \in X} Y_x$ induces on $C(X, Y)$ the topology $p$. In particular, we may regard $\mathcal{A}$ as being embedded in $\prod_{x \in X} \mathcal{A}(x)$. By (ii) and the Tychonoff theorem, $\prod_{x \in X} \mathcal{A}(x)$ is compact. Hence it suffices to show that $\mathcal{A}$ is $p$-closed in $\prod_{x \in X} \mathcal{A}(x)$.

Let $g$ belong to the $p$-closure of $\mathcal{A}$ in $\prod_{x \in X} \mathcal{A}(x)$. Since $X$ is a $k$-space, to show that $g$ is continuous on $X$, it is sufficient to show that $g$
is continuous on any compact subset $K$ of $X$. Let $x_o \in K$ and $D \in \mathcal{H}$. Choose a symmetric $V \in \mathcal{H}$ with $V \circ V \circ V \subseteq W$. Since $\mathcal{A}$ is equicontinuous on $X$, there exists a neighborhood $G(x_o)$ of $x_o$ such that

$$(f(y), f(x_o)) \in V \text{ for all } y \in G(x_o) \cap K \text{ and } f \in \mathcal{A}.$$ Since $g$ belongs to the $p$-closure of $\mathcal{A}$ in $\prod_{x \in X} \overline{\mathcal{A}(x)}$, for each $x \in X$, there exists $f_x \in \mathcal{A}$ such that

$$(f_x(x), g(x)) \in V.$$ Then, for any $y \in G(x_o) \cap K$,

$$(g(y), f(y)), (f(y), f(x_o)), (f(x_o), g(x_o)) \in V.$$ and so

$$(g(y), g(x_o)) \in V \circ V \circ V \subseteq D.$$ Therefore $g \in C(X,Y)$. Hence $g \in \overline{\mathcal{A}}^p$. But $\overline{\mathcal{A}}^p = \overline{\mathcal{A}}^k = \overline{\mathcal{A}}$ (since $\mathcal{A}$ is $k$-closed). So $g \in \mathcal{A}$, showing that $\mathcal{A}$ is $p$-closed in $\prod_{x \in X} \overline{\mathcal{A}(x)}$. □

Note. In the case of $Y = E$, a TVS, the above theorem reduces to Theorem 3.1.1.

We now present an analogue of the above theorem for the $\beta$-topology on $C_b(X,E)$ with $E$ a TVS. Here we use the terminology of Chapter 1.

**Theorem 3.1.5.** [Kh79] Let $X$ be a Hausdorff $k$-space and $E$ a Hausdorff TVS. Then a subset $\mathcal{A}$ of $C_b(X,E)$ is $\beta$-compact iff the following conditions hold:

(i) $\mathcal{A}$ is $\beta$-closed and $\beta$-bounded;
(ii) $\mathcal{A}(x)$ is relatively compact in $E$ for each $x \in X$;
(iii) $\mathcal{A}$ is equicontinuous on each compact subset of $X$.

**Proof.** Suppose $\mathcal{A}$ is $\beta$-compact in $C_b(X,E)$. Then conditions (i) - (iii) hold, as follows.

(i) Since $E$ is Hausdorff, $C_b(X,E), \beta)$ is also Hausdorff. So $\mathcal{A}$, being a compact subset of the Hausdorff space, is $\beta$-closed. Further, $\mathcal{A}$ is bounded since a compact set in a TVS is always bounded.

(ii) Let $x$ be any point in $X$. Consider the map $\delta_x : C_b(X,E) \rightarrow E$ given by

$$\delta_x(f) = f(x), \ f \in C_b(X,E).$$ Now $\delta_x$ is $p$-continuous, as follows. For any $f \in C_b(X,E)$ and $W \in W$. Let

$$N(f, \{x\}, W) = \{g \in C_b(X,E) : g(x) - f(x) \in W\}.$$ It is clear that $N(f, \{x\}, W)$ is a $p$-neighborhood of $f$ in $C_b(X,E)$ and

$$\delta_x(N(f, \{x\}, W)) \subseteq f(x) + W.$$
Hence $\delta_x$ is $p$-continuous. Since $p \leq \beta$ and $A$ is $\beta$-compact, it follows that $A$ is $p$-compact. Thus $\delta_x (A) = A(x)$ is relatively compact in $E$.

(iii) Since $k \leq \beta$ and $A$ is $\beta$-compact, it follows by Theorem 3.1.1 that $A$ is equicontinuous on each compact subset of $X$.

Conversely, suppose that a subset $A$ of $C_b(X,E)$ satisfies conditions (i) - (iii). Since $A$, being $\beta$-bounded, is $u$-bounded, the topologies $\beta$ and $k$ coincide on $A$ (Theorem 1.1.8, (c), (d)). Thus, to show that $A$ is $\beta$-compact, it is only necessary to show that $A$ is $k$-compact. Now, by using the same argument as the one use to prove Theorem 1.1.8 (c), we can show that the $\beta$ and $k$ closures of $A$ are the same. Consequently, $A$ is $k$-closed. In view of Theorem 3.1.1, this fact together with conditions (ii) and (iii) imply that $A$ is $k$-compact. \qed
2. Arzela-Ascoli Theorem for Weighted Topology

In this section, we present the Arzela-Ascoli type theorems which characterize precompact subsets of weighted functions spaces $CV_o(X, E)$ and $CV_{pe}(X, E)$ in the non-locally convex setting. When the range space $E$ happens to be quasi-complete, these characterizations become criteria for relative compactness.

As in section 1.2, let $X$ be a completely regular Hausdorff space, $E$ is non-trivial Hausdorff TVS with a base $W$ of closed balanced neighborhoods of 0 and $V$ a Nachbin family of non-negative upper semicontinuous function on $X$. We also assume that $V > 0$; i.e., for each $x \in X$, there exists a $v \in V$ with $v(x) > 0$. Further, let

$$CV_{pc}(X, E) = \{ f \in C(X, E) : (vf)(X) \text{ is precompact in } E \text{ for all } v \in V \}.$$ 

Clearly, $CV_o(X, E) \subseteq CV_{pc}(X, E) \subseteq CV_b(X, E)$. The weighted topology $\omega_V$ on $CV_b(X, E)$ has a base of neighborhoods of 0 consisting of all sets of the form

$$N(v, G) = \{ f \in CV_b(X, E) : (vf)(X) \subseteq G \},$$

where $v \in V$ and $G \in W$.

Note that:

1. If $V = K^+(X)$, then $CV_{pc}(X, E) = C_b(X, E)$ and $\omega_V$ is the uniform topology $u$.
2. If $V = S_0^+(X)$, then $CV_{pc}(X, E) = C_b(X, E)$ and $\omega_V$ is the strict topology $\beta_o$.
3. If $V = K_e^+(X)$, then $CV_{pc}(X, E) = CV_b(X, E) = CV_o(X, E) = C(X, E)$ and $\omega_V$ is the compact-open topology $k$.
4. If $V = K_f^+(X)$, then $CV_{pc}(X, E) = CV_b(X, E) = CV_o(X, E) = C(X, E)$ and $\omega_V$ is the pointwise topology $p$.

The assumption $V > 0$ implies that $p \leq \omega_V$. Recall that $p \leq k$ on $C(X, E)$ and $k \leq \beta_o \leq u$ on $C_b(X, E)$.

**Definition.** Let $E$ and $F$ be TVSs. For any collection $G$ of subsets of $E$, $CL_G(E, F)$ denotes the subspace of $CL(E, F)$ consisting of those $T$ which are bounded on the members of $G$ together with the topology $t_G$ of uniform convergence on the elements of $G$. This topology has a base of neighborhoods of 0 consisting of all sets of the form

$$U(D, W) := \{ T \in CL(E, F) : T(D) \subseteq W \},$$

where $D \in G$ and $W$ is a neighborhood of 0 in $F$ (see § A.3).
Definition. If $V$ is a Nachbin family on $X$, a net $\{x_\alpha : \alpha \in I\}$ is called a $V$-net if it is contained in $S_{v,1} := \{x \in X : v(x) \geq 1\}$ for some $v \in V$. We hence get the classical $V_\mathbb{R}$-spaces introduced by Bierstedt [Bie75b]: $X$ is said to be a $V_\mathbb{R}$-space if a function $f : X \to \mathbb{R}$ is continuous whenever, for each $v \in V$, the restriction of $f$ to $S_{v,1}$ is continuous. If $V = K^+(X)$, then $X$ is a $V_\mathbb{R}$-space means $X$ is a $k_\mathbb{R}$-space.

Definition. For any $x \in X$, let $\delta_x : CV_b(X, E) \to E$ denote the evaluation map $\delta_x(f) = f(x)$, $f \in CV_b(X, E)$. Clearly, $\delta_x \in CL(CV_b(X, E), E)$. Next, we define $\Delta : X \to CL(CV_b(X, E), E)$ as the evaluation map given by

$$\Delta(x) = \delta_x, \ x \in X.$$

Let the subscript $pc$ in $CL_{pc}(CV_b(X, E), E)$ stand for the topology of uniform convergence on precompact subsets of $CV_b(X, E)$.

Lemma 3.2.1. [RS88, KO05] The evaluation map

$$\Delta : X \to CL_{pc}(CV_b(X, E), E)$$

is continuous iff every precompact subset of $CV_b(X, E)$ is equicontinuous.

Proof. ($\Rightarrow$) Suppose $\Delta : X \to CL_{pc}(CV_b(X, E), E)$ is continuous, and let $\mathcal{A}$ be a precompact subset of $CV_b(X, E)$. To show that $\mathcal{A}$ is equicontinuous, let $x_0 \in X$ and $W \in \mathcal{W}$. Since $\Delta$ is continuous at $x_0$, there exists an open neighborhood $G(x_0)$ of $x_0$ in $X$ such that $\Delta(G(x_0)) \subseteq \Delta(x_0) + U(\mathcal{A}, W)$; that is,

$$\delta_x(f) - \delta_{x_0}(f) \in W \text{ for all } x \in G(x_0) \text{ and } f \in \mathcal{A}.$$

Hence $f(G) \subseteq f(x_0) + W$ for all $f \in \mathcal{A}$, and so $\mathcal{A}$ is equicontinuous.

($\Leftarrow$) Suppose that every precompact subset of $CV_b(X, E)$ is equicontinuous. To show that $\Delta : X \to CL_{pc}(CV_b(X, E), E)$ is continuous, let $x_0 \in X$ and let $\mathcal{A}$ be a precompact subset of $CV_b(X, E)$ and $W$ a balanced set in $\mathcal{W}$. Since $\mathcal{A}$, being precompact, is equicontinuous (by hypothesis), there exists an open neighborhood $G(x_0)$ of $x_0$ in $X$ such that $f(G) \subseteq f(x_0) + W$ for all $f \in \mathcal{A}$; that is

$$\delta_x - \delta_{x_0} \in U(\mathcal{A}, W) \text{ for all } x \in G.$$

Hence $\Delta(G(x_0)) \subseteq \Delta(x_0) + U(\mathcal{A}, W)$. □

Theorem 3.2.2. [RS88, KO05] Let $X$ be a $V_\mathbb{R}$-space. Then every precompact subset of $CV_b(X, E)$ is equicontinuous.

Proof. In view of Lemma 3.2.1, it suffices to show that the evaluation map $\Delta : X \to CL_{pc}(CV_b(X, E), E)$ is continuous. Since $X$ is a $V_\mathbb{R}$-spaces, we only need to show that $\Delta$ is continuous on each $S_{v,1} = \{x \in X :$
Let \( v \in V \) and \( x \in S_{v,1} \), and let \( A \) be a precompact subset of \( CV_0(X, E) \) and \( W \in \mathcal{W} \). Choose a balanced \( H \in \mathcal{W} \) with \( H + H + H \subseteq W \). Since \( A \) is precompact, there exist \( h_1, ..., h_n \in A \) such that

\[
A \subseteq \bigcup_{i=1}^{n} (h_i + N(v, H)).
\]

Since each \( h_i \) is continuous, there exists a neighborhood \( G_i \) of \( x \) in \( X \) such that

\[
h_i(y) - h_i(x) \in H \quad \text{for all} \quad y \in G_i, \quad i = 1, ..., n.
\]

Let \( G = \bigcap_{i=1}^{n} G_i \). Now, if \( y \in G \cap S_{v,1} \) and \( f \in A \), then \( f = h_i + g \) for some \( 1 \leq i \leq 1 \) and \( g \in N(v, H) \). Hence

\[
\delta_y(f) - \delta_x(f) = h_i(y) + g(y) - h_i(x) - g(x) = h_i(y) - h_i(x) + \frac{1}{v(y)}v(y)g(y) - \frac{1}{v(x)}v(x)g(x) \leq H + \frac{1}{v(y)}H - \frac{1}{v(x)}H \subseteq H + H - H \subseteq W;
\]

that is, \( \delta_y - \delta_x \in U(A, W) \) for all \( y \in G \cap S_{v,1} \). Hence \( \Delta(G \cap S_{v,1}) \subseteq \Delta(x) + U(A, W) \).

**Theorem 3.2.3.** [RS88, KO05] Let \( X \) be a \( V_\mathbb{R} \)-space. A subset \( A \) of \( CV_0(X, E) \) is precompact if and only if the following conditions hold.

(i) \( A \) is equicontinuous.

(ii) \( A(x) = \{ f(x) : f \in A \} \) is precompact in \( E \) for each \( x \in X \).

(iii) \( vA \) vanishes at infinity on \( X \) for each \( v \in V \); i.e., given \( v \in V \) and \( W \in \mathcal{W} \), there exists a compact set \( K \subseteq X \) such that \( v(y)f(y) \in W \) for all \( f \in A \) and \( y \in X \setminus K \).

**Proof.** \((\Rightarrow)\) Suppose \( A \) is a precompact subset of \( CV_0(X, E) \). We verify (i)-(iii), as follows.

(i) Since \( X \) is a \( V_\mathbb{R} \)-space, by Theorem 3.2.2, \( A \) is equicontinuous.

(ii) Since \( V > 0 \), \( p \leq \omega_V \) and so \( A \) is \( p \)-precompact. Hence, for each \( x \in X \), \( A(x) \) is precompact in \( E \).

(iii) Let \( v \in V \) and \( W \in \mathcal{W} \). Choose a balanced \( H \in \mathcal{W} \) with \( H + H \subseteq W \). Since \( A \) is precompact, there exist \( h_1, ..., h_n \in A \) such that

\[
A \subseteq \bigcup_{i=1}^{n} (h_i + N(v, H)).
\]
Put $K = \bigcup_{i=1}^{n} \{ y \in X : v(y)h_i(y) \notin H \}$. Then $K$ is compact (since each $h_i \in CV_o(X, E)$). Now let $f \in \mathcal{A}$ and $y \in X \setminus K$. By (1), there exists $i \in \{1, ..., n\}$ such that $f \in h_i + N(v,H)$. Hence

$$v(y)f(y) = v(y)[f(y) - h_i(y)] + v(y)h_i(y) \in H + H \subseteq W.$$ 

Thus $v\mathcal{A}$ vanishes at infinity on $X$.

$(\Rightarrow)$ Suppose (i) - (iii) hold. Since $CV_o(X, E) \subseteq C(X, E)$, by (i), $\mathcal{A}$ is an equicontinuous subset of $C(X, E)$. Further, since, by (ii), $\mathcal{A}$ is $p$-precompact, it follows that $\mathcal{A}$ is a precompact subset of $(C(X, E), k)$ (cf. Lemma 3.1.2). To show that $\mathcal{A}$ is precompact in $CV_o(X, E)$, let $v \in V$ and $W \in \mathcal{W}$. Choose a balanced $H \in \mathcal{W}$ such that $H + H \subseteq W$.

(2) Since $v$ is upper-semicontinuous, $\|v\|_K = \sup\{v(y) : y \in K\} < \infty$. Since $\mathcal{A}$ is precompact in $(C(X, E), k)$, there exist $h_1, ..., h_n \in \mathcal{A}$ such that

$$\mathcal{A} \subseteq \bigcup_{i=1}^{n} (h_i + N(\chi_K, 1/\|v\|_K + 1)H)).$$

We claim that

$$\mathcal{A} \subseteq \bigcup_{i=1}^{n} (h_i + N(v, W)).$$

Let $f \in \mathcal{A}$ and $y \in X$. If $y \in K$, by (3), there exists $j \in \{1, ..., n\}$ such that

$$v(y)[f(y) - h_j(y)] \in v(y)\frac{1}{\|v\|_K + 1}H \subseteq H.$$ 

If $y \in X \setminus K$, then, for any $i \in \{1, ..., n\}$, (2) gives

$$v(y)[f(y) - h_i(y)] = v(y)f(y) - v(y)h_i(y) \in H - H \subseteq W.$$ 

This establishes our claim, and so $\mathcal{A}$ is precompact in $CV_o(X, E)$.

Following [RS88], we shall use the following notation: for any $\mathcal{A} \subseteq CV_{pc}(X, E)$, $v \in V$ and $W \in \mathcal{W}$, let

$$T_x(\mathcal{A}, v, W) = \{ y \in X : v(y)f(y) - v(x)f(x) \in W \text{ for all } f \in \mathcal{A} \}, \; x \in X.$$

**Theorem 3.2.4.** [RS88, KO05] Let $X$ be a $V_{\mathbb{R}}$-space. Then, for any $\mathcal{A} \subseteq CV_{pc}(X, E)$, the following are equivalent:

(a) $\mathcal{A}$ is precompact.

(b) (i) $\mathcal{A}$ is equicontinuous;

(ii) $\mathcal{A}(x)$ is precompact in $E$ for each $x \in X$;
(iii) given \( v \in V \) and \( W \in \mathcal{W} \), there exists a compact set \( K \subseteq X \) such that \( \{ T_x(\mathcal{A}, v, W) : x \in K \} \) covers \( X \).

(c) (i) \( v, \mathcal{A}(X) = \{ v(x) f(x) : x \in X, f \in \mathcal{A} \} \) is precompact in \( E \) each \( v \in V \);

(ii) given \( v \in V \) and \( W \in \mathcal{W} \), there exists a finite set \( F \subseteq X \) such that \( \{ T_x(\mathcal{A}, v, W) : x \in F \} \) covers \( X \).

(d) (i) \( \mathcal{A}(x) \) is precompact in \( E \) for each \( x \in X \);

(ii) given \( v \in V \) and \( W \in \mathcal{W} \), there exists a finite set \( F \subseteq X \) such that \( \{ T_x(\mathcal{A}, v, W) : x \in F \} \) covers \( X \).

Proof. (a) \( \Rightarrow \) (b) Since \( \mathcal{A} \) is a precompact subset of \( CV_{pc}(X, E) \), just as in the proof of Theorem 3.2.3, (b)(i) follow from Theorem 3.2.2 and (b)(ii) follows from the fact that \( V > 0 \). To prove (b)(iii), let \( v \in V \) and \( W \in \mathcal{W} \). Choose a balanced \( H \in \mathcal{W} \) such that \( H + H + H \subseteq W \). Since \( \mathcal{A} \) is precompact, there exist \( h_1, ..., h_n \in \mathcal{A} \) such that \( \mathcal{A} \subseteq \bigcup_{i=1}^{n} [h_i + N(v, H)] \). (1)

Now, consider the function \( h : X \rightarrow E^n \) defined by

\[
h(x) = (h_1(x), h_2(x), \ldots, h_n(x)).
\]

This is a continuous function such that \( (vh)(X) \) is precompact, for it is contained in the product \( \prod_{i=1}^{n} (vh_i)(X) \) which is precompact. Hence for the neighborhood \( H^n \), there exists a finite subset \( F \) of \( X \) such that

\[
(vh)(X) = \bigcup_{x \in F} ((vh)(x) + H^n).
\]

This gives

\[
X \subseteq \bigcup_{x \in F} T_x(\{h_1, \ldots, h_n\}, v, H). \quad (2)
\]

We now show that \( \{ T_x(\mathcal{A}, v, W) : x \in F \} \) covers \( X \). Fix \( y \in X \). By (2), \( y \in T_x(\{h_1, \ldots, h_n\}, v, H) \) for some \( x \in F \) and so

\[
v(y)h_i(y) = v(x)h_i(x) \in H \quad \text{for all } i = 1, \ldots, n. \quad (3)
\]

Given \( f \in \mathcal{A} \), by (1), there exists \( i \in \{1, ..., n\} \) such that \( f - h_i \in N(v, H) \); i.e.,

\[
v(z)(f(z) - h_i(z)) \in H \quad \text{for all } z \in X. \quad (4)
\]

So, by (3) and (4),

\[
v(y)f(y) - v(x)f(x) = (v(y)f(y) - v(y)h_i(y)) + (v(y)h_i(y) - v(x)h_i(x)) + (v(x)h_i(x) - v(x)h(x)
\]
\[ y \in T_x(\mathcal{A}, v, W); \text{ i.e., } (b)(iii) \text{ holds.} \]

\((b) \Rightarrow (c)\) Suppose \((b)\) holds. We first note that \((b)(i)\) and \((b)(ii)\) together imply (as in the proof of Theorem 3.2.3) that \(\mathcal{A}\) is a precompact subset of \((C(X, E), k)\). We now verify \((c)(i)\) and \((c)(ii)\).

\((c)(i)\) Let \(v \in V\) and \(W \in W\). Choose a balanced \(H \in W\) such that \(H + H + H + H \subseteq W\).

By \((b)(iii)\), there exists a compact \(K \subseteq X\) such that
\[
\{T_x(\mathcal{A}, v, H) : x \in K\} \text{ covers } X. \tag{1}
\]

Since \(\mathcal{A}\) is precompact in \((C(X, E), k)\), there exist \(h_1, ..., h_n \in \mathcal{A}\) such that
\[
A \subseteq \bigcup_{i=1}^{n}[h_i + N(\chi_K, 1/\|v\|_K + 1)H]. \tag{2}
\]

Moreover, since each \(vh_i(K)\) is precompact in \(E\), there exist \(\{x_{ij} : 1 \leq j \leq n_i\} \subseteq K\) such that
\[
vh_i(K) \subseteq \bigcup_{j=1}^{n_i}[v(x_{ij})h_i(x_{ij}) + H]. \tag{3}
\]

Now, fix any \(y \in X\) and \(f \in \mathcal{A}\). By \((1)\), \(y \in T_x(\mathcal{A}, v, H)\) for some \(x \in K\) and so
\[
v(y)f(y) - v(x)f(x) \in H \text{ for all } f \in \mathcal{A}. \tag{1a}
\]

By \((2)\), there exists \(i \in \{1, ..., n\}\) such that
\[
(f - h_i)(K) \subseteq \frac{1}{\|v\|_K + 1}H. \tag{2a}
\]

By \((3)\), there exists \(j \in \{1, ..., n\}\) such that
\[
v(x)h_i(x) - v(x_{ij})h_i(x_{ij}) \in H. \tag{3a}
\]

By \((1a)\), \((2a)\), \((3a)\)
\[
v(y)f(y) - v(x_{ij})h_i(x_{ij}) \in v(x)h_i(x) - v(x_{ij})h_i(x_{ij}) + v(x)(f(x) - h_i(x)]
+ [v(x)h_i(x) - v(x_{ij})h_i(x_{ij})]
\in H + v(x)\frac{1}{\|v\|_K + 1}H + H \subseteq W. \tag{4}
\]

i.e. \(v\mathcal{A}(X) \subseteq \bigcup_{i=1}^{n} \bigcup_{j=1}^{n_i}[v(x_{ij})h_i(x_{ij}) + W]\) and \(v\mathcal{A}(X)\) is precompact in \(E\).
Let \( v \in V \) and \( W \in \mathcal{W} \), and suppose these be same as in the proof of (c)(i). Also, let \( h_i \) and \( x_{ij} \) be same as above. Let \( F = \bigcup_{i=1}^{n} \{ x_{ij} : j = 1, \ldots, n_i \} \). Then, for any fixed \( y \in X \) and \( f \in \mathcal{A} \), (3a) and (4) give
\[
v(y)f(y) - v(x_{ij})f(x_{ij}) = [v(y)f(y) - v(x_{ij})h_i(x_{ij})] + [v(x_{ij})h_i(x_{ij}) - v(x_{ij})f(x_{ij})] \\
\quad \in (H + H + H) + H \subseteq W;
\]
that is, \( y \in T_x(\mathcal{A}, v, W) \) with \( x \in F \). Hence \( \{ T_x(\mathcal{A}, v, W) : x \in F \} \) covers \( X \).

(c) \Rightarrow (d) Since \( V > 0 \), \( p \leq \omega_V \) and so, for any \( v \in V \), \( v\mathcal{A}(X) \) is precompact implies that \( \mathcal{A}(x) \) is precompact in \( E \) for each \( x \in X \).

(d) \Rightarrow (a) Suppose (d) holds. Fix any \( v \in V \) and \( W \in \mathcal{W} \). Choose a balanced \( H \in \mathcal{W} \) with \( H + H + H \subseteq W \). By (d)(ii) there exists a finite set \( F \subseteq X \) such that \( \{ T_x(\mathcal{A}, v, H) : x \in F \} \) covers \( X \). By (d)(i), \( \mathcal{A} \) is \( p \)-precompact, and so there exists \( h_i, \ldots, h_n \in \mathcal{A} \) such that
\[
\mathcal{A} \subseteq \bigcup_{i=1}^{n} (h_i + N(\chi_{F, 1} \|v\|_F + 1)H).
\]
(1)
We claim that \( \mathcal{A} \subseteq \bigcup_{i=1}^{n} (h_i + N(v, W)) \). Fix any \( f \in \mathcal{A} \). By (1), there exists \( i \in \{1, \ldots, n\} \) such that
\[
(f - h_i)(F) \subseteq \frac{1}{\|v\|_F + 1}H.
\]
Then, for any \( y \in X \), \( y \in T_x(\mathcal{A}, v, H) \) for some \( x \in F \) and so
\[
v(y)f(y) - v(y)h_i(y) = [v(y)f(y) - v(x)f(x)] + v(x)[f(x) - h_i(x)] \\
\quad + [v(x)h_i(x) - v(y)h_i(y)] \\
\quad \subseteq H + v(x)\frac{1}{\|v\|_F + 1}H - H \\
\quad \subseteq H + H + H \subseteq W.
\]
This proves our claim; i.e., (a) holds.

In the particular cases of \( V = K^+_c(X) \), \( K^+(X) \) and \( S^+_0(X) \), we mention that Theorems 3.2.3 and 3.2.4 give further improvements of Theorems 3.1.3 and 3.1.4 and are stated as follows:

**Corollary 3.2.5.** Let \( X \) be a \( k_\mathbb{R} \)-space and \( E \) a quasi-complete TVS. A subset \( \mathcal{A} \) of \( (C(X, E), k) \) is relatively compact iff the following conditions hold:

(i) \( \mathcal{A} \) is equicontinuous on each compact subset of \( X \),
(ii) $A(x)$ is relatively compact in $E$ for each $x \in X$.

Corollary 3.2.6. Let $X$ be a locally compact space and $E$ a quasi-complete TVS. A subset $A$ of $(C_o(X,E), u)$ is relatively compact iff the following conditions hold:

(i) $A$ is equicontinuous,
(ii) $A(x)$ is relatively compact in $E$ for each $x \in X$,
(iii) $A$ uniformly vanishes at infinity on $X$; i.e., for any $W \in \mathcal{W}$, there exists a compact set $K \subseteq X$ such that $f(y) \in W$ for all $f \in A$ and $y \in X\setminus K$.

Corollary 3.2.7. Let $X$ be a $k_R$-space and $E$ a quasi-complete TVS. A subset $A$ of $(C_b(X,E), \beta_o)$ is relatively compact iff the following conditions hold:

(i) $A$ is equicontinuous on each compact subset of $X$,
(ii) $A(X)$ is relatively compact in $E$,
(iii) $A$ is uniformly bounded (i.e., $A(X)$ is bounded in $E$).

Corollary 3.2.8. Let $E$ be a quasi-complete TVS. A subset $A$ of $(C_{pc}(X,E), u)$ is relatively compact iff the following conditions hold:

(i) $A(X)$ is relatively compact in $E$,
(ii) given $W \in \mathcal{W}$, there exists a finite open cover $\{K_i : i = 1, \ldots, n\}$ of $X$ such that, for any $i \in \{1, \ldots, n\}$ and $x, y \in K_i$,
$$f(x) - f(y) \in W \text{ for all } f \in A.$$
Section 3.1. Theorem 3.1.1. is a version of the Arzela-Ascoli Theorem for the \( k \)-topology, given in the book of Kelley and Namioka [KN63, p. 81] without a proof. For the readers’ benefit and interest, we present a well-known version of this theorem (Theorem 3.1.4) in the more general setting of uniform spaces. Its proof is adapted from [Eng88, p. 440-443; Will70, § 43]. Theorem 3.1.5 is an analogue of the Arzela-Ascoli theorem for the \( \beta \)-topology, as given in [Kh79]. We mention that the abstract version of the Arzela-Ascoli theorem was first given by Myers [My46] which was later improved by Gale [Gal50]. The name \( k \)-space also first appeared in the paper [Gal50].

Section 3.2. Here the Arzéla-Ascoli type theorems which characterize precompact subsets of weighted functions spaces \( CV_o(X, E) \) and \( CV_{pc}(X, E) \) are taken from [RS88, KO05]. When the range space \( E \) happens to be quasi-complete, these characterizations become criteria for relative compactness. Theorem 3.2.2. is an analogue of Bierstedt [Bie75b], Ruess and Summers ([RS88], Lemma 2.3) and Oubbi ([Ou02], Proposition 9).
CHAPTER 4

The Stone-Weierstrass type Theorems

In this chapter, we are primarily concerned with presenting some generalizations of the classical Stone-Weierstrass theorem to $C_b(X, E)$ and $CV_b(X, E)$ endowed with various topologies for $E$ a non-locally convex TVS. We also consider Weierstrass polynomial approximation, characterization of maximal submodules and the approximation property for vector-valued functions. Both $X$ and $E$ are always assumed to be Hausdorff.
1. Approximation in the Strict Topology

Various versions of the Stone-Weierstrass theorem for the spaces \((C_b(X,E),\beta)\), \((C_b(X,E),u)\) and \((C(X,E),k)\) are considered in this section.

Recall that M.H. Stone ([St37], p. 466) obtained the following generalization of the Weierstrass approximation theorem of 1885 (see [St48]; [Simm62], p.157-167).

**Theorem A.** (Classical Stone-Weierstrass theorem for \(C(X) = C(X,\mathbb{C})\).) Let \(X\) be a compact space. Suppose a subset \(A\) of \(C(X)\) satisfies

1. \(A\) is a self-adjoint subalgebra,
2. \(A\) separates points of \(X\), i.e., for any \(x \neq y\) in \(X\), there exists an \(f \in A\) such that \(f(x) \neq f(y)\),
3. \(A\) does not vanish on \(X\), i.e., for each \(x \in X\) there exists a function \(g \in A\) such that \(g(x) \neq 0\).

Then \(A\) is \(u\)-dense in \(C(X)\).

**Note.** If \(X\) is locally compact, then the above theorem holds for \(C_o(X)\) in place of \(C(X)\).

We now consider the case of vector-valued functions. If \(E\) is a vector space, then \(C(X,E)\) need not be an algebra but is a \(C_b(X)-\)module. An apparent analogue of Theorem A may be stated in the following form.

**Theorem B.** Let \(X\) be a compact space and \(E\) a normed space (or a TVS). Suppose a subset \(A\) of \(C(X,E)\) satisfies

1. \(A\) is a \(C(X)-\)submodule,
2. \(A\) separates points of \(X\),
3. \(A\) does not vanish on \(X\).

Then \(A\) is \(u\)-dense in \(C(X,E)\).

Unfortunately, Theorem B is not true in general as we now show.

**Counter-example:** [Buc58] For any fixed \(x_0 \in X\) and \(M\) a proper closed vector subspace of a TVS \(E\), let

\[ S(x_0, M) = \{ f \in C_b(X,E) : f(x_0) \in M \}. \]

Then \(A = S(x_0, M)\) satisfies conditions (i), (ii), (iii) as follows. Clearly, \(A\) is a \(C(X)-\)submodule of \(C(X,E)\). To verify (ii), let \(x \neq y \in X\). Since \(X\), being a compact Hausdorff space, is normal, there exists a function \(\varphi \in C_b(X)\) such that \(0 \leq \varphi \leq 1\), \(\varphi(x) = 0\) and \(\varphi(y) = 1\). For any \(a(\neq 0)\) in \(M\), \((\varphi \otimes a)(x_0) = \varphi(x_0)a \in M\) and so \(\varphi \otimes a \in A\). Further \((\varphi \otimes a)(x) \neq (\varphi \otimes a)(y)\). To verify (iii), consider any \(b(\neq 0)\) in \(M\). If \(1\) is the unit function in \(C_b(X)\), then \(1 \otimes b\) is a non-zero constant function in
A. Clearly, $A = S(x_0, M)$ is a proper $u$-closed subspace of $C(X, E)$ and so cannot be $u$-dense in $C(X, E)$. □

Thus one must look for an appropriate replacement of conditions (i), (ii) and (iii) in Theorem B. Kaplansky ([Kap51], §3) had considered similar conditions and obtained some non-commutative generalizations of the classical Stone-Weierstrass theorem, replacing $E$ by a family $\{E_x : x \in X\}$ of $C^*$-algebras having units. But in our setting, the following theorem for the strict topology $\beta$ on $C_b(X, E)$ was established by R.C. Buck [Buc58].

**Theorem 4.1.1. (Stone-Weierstrass theorem for $(C_b(X, E), \beta)$)**

Let $X$ be a locally compact metrizable space and $E$ a finite dimensional normed space. Suppose a subset $A$ of $C_b(X, E)$ satisfies

1. $A$ is a $C_b(X)$-submodule,
2. for each $x \in X$, the set $A(x) = \{g(x) : g \in A\}$ is dense in $E$.

Then $A$ is $\beta$-dense in $C_b(X, E)$.

The above result was later extended to the case of $X$ any locally compact space and $E$ a locally convex TVS by Todd [Tod65] and Wells [Wel65], independently. In this chapter we first present a Stone-Weierstrass type theorem, due to Khan [Kh79], with $E$ any TVS but under the additional condition of finite covering dimension on $X$ (see Section A.1).

The following lemma on “partition of unity”, due to Nachbin ([Nac67], Lemma 2, p. 69) plays a crucial role in the proof of Theorem 4.1.3 and also in some later theorems.

**Lemma 4.1.2. (Partition of Unity)** Let $X$ be a completely regular space, and let $M \subseteq C(X, K)$ be a $C_b(X, K)$-submodule. Let $K$ be a compact subset of $X$ such that

(∗) $M$ does not vanish on $K$.

Then, given any open cover $\{U_1, ..., U_n\}$ of $K$, there exist $\varphi_1, ..., \varphi_n \in M$, such that

$$0 \leq \varphi_i \leq 1, \varphi_i = 0 \text{ outside } U_i, \sum_{i=1}^{n} \varphi_i = 1 \text{ on } K, \text{ and } \sum_{i=1}^{n} \varphi_i \leq 1 \text{ on } X.$$ 

We mention that if $M = C_b(X, K)$, (∗) clearly holds.

**Proof.** By (∗), for each $t \in K$, there is an $f_t \in M$ such that $f_t(t) \neq 0$. We may assume that $f_t \geq 0$. [Indeed, let $\gamma : K \to \mathbb{K}$ be defined by

$$\gamma(x) = \begin{cases} 0 & \text{if } x = 0, \\ |x|^2/x & \text{if } 0 < |x| < 1, \\ |x|/x & \text{if } |x| \geq 1. \end{cases}$$

By (∗), there is an $f \in M$ such that $f(t) \neq 0$. Then $f = f \gamma$ and $f \gamma(t) \neq 0$. Thus $f_t = f \gamma_t \in M$ as required.
Clearly, $\gamma$ is bounded, continuous and satisfies $\gamma(x) \neq 0$ for $x \neq 0$ and $\gamma(x)x \geq 0$ for $x \in \mathbb{K}$. Set $\phi = \gamma \circ f_t \in C_b(X, \mathbb{K})$; then $\phi(x) \neq 0$ and $\phi f_t \geq 0$. We replace $f_t$ by $\phi f_t \in \mathcal{M}$ to achieve $f_t \geq 0$. We may also assume that $f_t = 0$ outside some $U_t$. [In fact, choose some $U_t$ containing $t$ and $\psi \in C(X, \mathbb{K})$ such that $0 \leq \psi \leq 1$, $\psi(t) = 1$ and $\psi = 0$ outside $U_t$.]

By compactness of $K$, there exists a finite collection $\{f_1, ..., f_m\} \subseteq \{f_t : t \in K\} \subseteq \mathcal{M}$ such that

$$f_j \geq 0 \text{ on } X, \ f_j = 0 \text{ outside some } U_j \ (1 \leq j \leq m) \text{ and } \sum_{j=1}^{m} f_j(x) > 0 \text{ for all } x \in K.$$ 

For each $i = 1, ..., n$, define $J_i = \{ j : f_j = 0 \text{ outside } U_i, 1 \leq j \leq m \}$, and let $g_i = \sum_{j \in J_i} f_j$. Then $g_i \in \mathcal{M}$,

$$g_i \geq 0, \ g_i = 0 \text{ outside } U_i \ (1 \leq i \leq n) \text{ and } \sum_{i=1}^{n} g_i(x) > 0 \text{ for all } x \in K.$$ 

By compactness of $K$, there is an $h \in C_b(X, \mathbb{K})$ such that $0 \leq h \leq 1/\sum_{i=1}^{n} g_i$ on $X$, and that $h = 1/\sum_{i=1}^{n} g_i$ on $K$. Setting $\varphi_i = hg_i$, we clearly have

$$0 \leq \varphi_i \leq 1, \ \varphi_i = 0 \text{ outside } U_i, \sum_{i=1}^{n} \varphi_i = 1 \text{ on } K \text{ and } \sum_{i=1}^{n} \varphi_i \leq 1 \text{ on } X. \quad \Box$$

**Theorem 4.1.3.** [Kh79] Let $X$ be a completely regular space of finite covering dimension and $E$ a TVS. If $\mathcal{A}$ is a $C_b(X)$-submodule of $C_b(X, E)$ such that, for each $x \in \mathcal{A}$, $\mathcal{A}(x)$ is dense in $E$, then $\mathcal{A}$ is $\beta$-dense in $C_b(X, E)$.

**Proof.** Suppose $X$ has covering dimension of order $n$, and let $f \in C_b(X, E)$. Let $\varphi \in B_b(X)$, $0 \leq \varphi \leq 1$, and $W \in \mathcal{W}$. There exists a balanced $V \in \mathcal{W}$ such that $V + V + ... + V ((n+2)-terms) \subseteq W$. Let $K$ be a compact subset of $X$ such that $\varphi(x)f(x) \in V$ for $x \notin K$. For each $x \in X$, choose a function $g_x$ in $\mathcal{A}$ and an open neighborhood $G(x)$ of $x$ such that $g_x(y) - f(y) \in V$ for all $y \in G(x)$.
The sets in \( \{ G(x) : x \in K \} \) form an open cover of \( K \), and so there exists a finite open cover, \( \{ G(x_j) : j = 1, ..., m \} \) say, of \( K \). The sets in \( U = \{ X \setminus K, G(x_j) : j = 1, ..., m \} \) form a finite open cover of \( X \), and so, by hypothesis, there exists an open cover \( B \) of \( X \) such that \( B \) is a refinement of \( U \) and any point of \( X \) belong to at most \( n + 1 \) members of \( B \). Since \( K \) is compact, a finite number of members of \( B \), \( U_1, ..., U_r \) say, will cover \( K \). Moreover, since \( B \) is a refinement of \( U \), for each \( 1 \leq i \leq r \), there exists a \( j_i \), \( 1 \leq m \), such that \( U_i \subseteq G(x_{j_i}) \). By Lemma 4.1.2, there exist \( \varphi_1, ..., \varphi_r \in C_b(X) \) such that \( 0 \leq \varphi_i \leq 1 \), \( \varphi_i = 0 \) outside of \( U_i \), \( \sum_{i=1}^{r} \varphi_i(x) = 1 \) for \( x \in K \), and \( \sum_{i=1}^{r} \varphi_i(x) \leq 1 \) for \( x \in X \). We define an \( E \)-valued function \( g \) on \( X \) by

\[
g(x) = \sum_{i=1}^{r} \varphi_i(x) g_{x_{j_i}}(x), \quad x \in X,
\]

where \( g_{x_{j_i}} \) is the function in \( A \) chosen as indicated earlier. Then \( g \in A \).

Let \( y \) be any point in \( X \). If \( I_y = \{ i : y \in U_i \} \), then \( I_y \) has at most \( (n + 1) \)-members and \( \varphi_i(y) = 0 \) if \( i \notin I_y \). Consequently, if \( y \in K \), then

\[
\varphi(y)(g(y) - f(y)) = \varphi(y) \left\{ \sum_{i=1}^{r} \varphi_i(x)(g_{x_{j_i}}(y) - f(y)) \right\} = \varphi(y) \left\{ \sum_{i \in I_y} \varphi_i(y)(g_{x_{j_i}}(y) - f(y)) \right\} \in V + V + ... + V \quad \text{(at most (n+1)-times)} \subseteq W.
\]

If \( y \notin K \), we have

\[
\varphi(y)(g(y) - f(y)) = \varphi(y) \sum_{i=1}^{r} \varphi_i(y)[g_{x_{j_i}}(y) - f(y)] + \left\{ \sum_{i=1}^{r} \varphi_i(y) - 1 \right\} \varphi(y)f(y) \quad (*)
\]

\[
\in V + ... + V \quad \text{(at most (n+1)-times)} + V \subseteq W.
\]

Thus \( g - f \in N(\varphi, W) \), and so \( f \) belongs to the \( \beta \)-closure of \( A \); that is, \( A \) is \( \beta \)-dense in \( C_b(X, E) \), as required. \( \square \)

An equivalent version of Theorem 4.1.3 is as follows.
Theorem 4.1.4. [Buc58, Kh79] Let $X$ and $E$ be as in Theorem 4.1.3, and let $\mathcal{A}$ be a $C_b(X)$-submodule of $C_0(X,E)$ and $f \in C_b(X,E)$. Then the following conditions are equivalent:

1. $f$ belongs to the $\beta$-closure of $\mathcal{A}$.
2. For each $x \in X$, $f(x) \in \mathcal{A}(x)$.

Proof. (1) $\Rightarrow$ (2) Suppose that $f$ belongs to the $\beta$-closure of $\mathcal{A}$, and let $x$ be any point in $X$. Let $\{f_\alpha\}$ be a net in $\mathcal{A}$ such that $f = \beta$-lim $\alpha f_\alpha$. Since $p \leq \beta$, $f = p$-lim $\alpha f_\alpha$, and so, in particular, $f_\alpha(x) \rightarrow f(x)$. Since $\{f_\alpha(x)\} \subseteq \mathcal{A}(x)$, it follows that $f(x) \in \overline{\mathcal{A}(x)}$.

(2) $\Rightarrow$ (1) Suppose that, for each $x \in X$, $f(x) \in \overline{\mathcal{A}(x)}$. If $N(\varphi,W)$ is any $\beta$-neighborhood of 0 in $C_b(X,E)$, then, in the same way as in Theorem 4.1.3, we can find a function $h$ in $\mathcal{A}$ such that $h - f \in N(\varphi,W)$. Thus $h$ belongs to the $\beta$-closure of $\mathcal{A}$.

We next state two versions of the Stone-Weierstrass theorem for the $u$ and $k$ topologies. Their proofs are omitted as they are similar to that of Theorem 4.1.3.

Theorem 4.1.5. [Kh80] Let $X$ be a locally compact space of finite covering dimension and $E$ a Hausdorff TVS. If $\mathcal{A}$ is a $C_0(X)$-submodule of $C_0(X,E)$ such that, for each $x \in X$, $\mathcal{A}(x)$ is dense in $E$, then $\mathcal{A}$ is $u$-dense in $C_0(X,E)$.

Theorem 4.1.6. [Kh95] Let $X$ be a completely regular space such that every compact subset of $X$ has a finite covering dimension, and let $E$ be a TVS. If $\mathcal{A}$ is a $C(X)$-submodule of $C(X,E)$ such that, for each $x \in X$, $\mathcal{A}(x)$ is dense in $E$, then $\mathcal{A}$ is $k$-dense in $C(X,E)$.

Corollary 4.1.7. Let $X$ be a completely regular space and $E$ a TVS.

(a) If $X$ has finite covering dimension, then $C_b(X) \otimes E$ is $\beta$-dense in $C_b(X,E)$.

(b) If $X$ is locally compact and has finite covering dimension, then $C_0(X) \otimes E$ is $u$-dense in $C_0(X,E)$.

(c) If each compact subset of $X$ has finite covering dimension, then $C(X) \otimes E$ is $E$ is $k$-dense in $C(X,E)$.

Proof. (a) Since $C_b(X)$ contains the unit function 1, it is easy to see that $(C_b(X) \otimes E)(x) = E$ for each $x \in X$. It follows from the Theorem 4.1.3 that $C_b(X) \otimes E$ is $\beta$-dense in $C_b(X,E)$.

(b) Its proof is similar to part (a) by using Theorem 4.1.5.

(c) Its proof is similar to part (a) by using Theorem 4.1.6.

Remark 4.1.8. If $E$ is a locally convex TVS, then, with slight modification in the proofs, Theorems 4.1.3-4.1.6 and Corollary 4.1.7 hold without the hypothesis of finite covering dimension.
Let $C_{pc}(X, E)$ (resp. $C_{rc}(X, E)$) denote the subspace of $C_b(X, E)$ consisting of those functions $f$ such that $f(X)$ is precompact (resp. relatively compact). Note that if $E$ is complete, then $C_{pc}(X, E) = C_{rc}(X, E)$.

Recall that a TVS $E$ is said to have the approximation property if the identity map on $E$ can be approximated uniformly on precompact sets by continuous and linear maps of finite rank; $E$ is said to be admissible if the identity map on $E$ can be approximated uniformly on compact sets by continuous maps of finite rank. Every locally convex TVS and every $F$-space with a basis (e.g. $\ell_p$, $0 < p < 1$) is admissible.

**Theorem 4.1.9.** [Shu72, Kh79, Kh95] Let $X$ be a completely regular space and $E$ a TVS.

(a) $E$ is admissible iff for all topological spaces $X$, $C_b(X) \otimes E$ is $u$-dense in $C_{rc}(X, E)$.

(b) If $E$ has the approximation property, then $C_b(X) \otimes E$ is $u$-dense in $C_{pc}(X, E)$.

(c) If $X$ is a normal space of finite covering dimension and $E$ any TVS, then $C_b(X) \otimes E$ is $u$-dense in $C_{rc}(X, E)$.

(d) If $E$ is an admissible TVS, then

(i) $C(X) \otimes E$ is $k$-dense in $C(X, E)$.

(ii) $C_b(X) \otimes E$ is $\beta$-dense in $C_b(X, E)$.

*Proof.* (a) Suppose $E$ is admissible, and let $f \in C_{rc}(X, E)$ and $W \in W$ a balanced set. Since $\overline{f(X)}$ is compact in $E$, there exists a continuous map $u : f(X) \to E$ with range contained in a finite dimensional subspace of $E$ such that $u(a) - a \in W$ for all $a \in f(X)$. Then $h = u \circ f \in C_b(X) \otimes E$ and $h(x) - f(x) \in W$ for all $x \in X$.

Conversely, let $K \subseteq E$ be a compact set and $W \in W$. Since, by hypothesis, $C_b(K) \otimes E$ is $u$-dense in $C_b(K, E)$, there exists some $v = \sum_{i=1}^{n} v_i \otimes a_i \in C_b(K) \otimes E$ such that $v(a) - a \in W$ for all $a \in K$. Note that the range of $v$ is contained in the finite dimensional subspace spanned by $\{a_1, \ldots, a_n\}$. Thus $E$ is admissible.

(b) Suppose $E$ has the approximation property, and let $f \in C_{pc}(X, E)$ and $W \in W$ a balanced set. Since $\overline{f(X)}$ is precompact in $E$, there exists a continuous linear map $u : f(X) \to E$ with range contained in a finite dimensional subspace of $E$ such that $u(a) - a \in W$ for all $a \in f(X)$. Then $h = u \circ f \in C_b(X) \otimes E$ and $h(x) - f(x) \in W$ for all $x \in X$.

(c) Since $X$ is normal, its Stone-Cech compactification $\beta X$ also has finite covering dimension [GJ60]. So, by Corollary 4.1.7, $C_b(\beta X) \otimes E$ is $u$-dense in $C_b(\beta X, E)$. Note that each function in $C_{rc}(X, E)$ has a continuous extension to all of $\beta X$. Hence $C_b(X) \otimes E$ and $C_{rc}(X, E)$ are
linearly isomorphic to $C_b(\beta X) \otimes E$ and $C_b(\beta X, E)$, respectively, and so the result follows.

(d) (i) Let $f \in C(X, E)$, and let $K$ be a compact subset of $X$ and $W \in \mathcal{W}$ a balanced set. By hypothesis, there exists a continuous map $u : f(K) \to E$ with range contained in a finite dimensional subspace of $E$ such that $u(f(x)) - f(x) \in W$ for all $x \in K$. We can write

$$u \circ f = \sum_{i=1}^{m} (u_i \circ f) \otimes a_i,$$

where $u_i \circ f \in C(K)$ and $a_i \in E$.

By the famous Tietze extension Theorem, there exist $h_1, ..., h_m \in C(X)$ such that $h_i = u_i \circ f$ on $K$. Let $g = \sum_{i=1}^{m} h_i \otimes a_i$. Then $g \in C(X) \otimes E$ and $g - f \in N(K, W)$.

(d) (ii) Let $f \in C_b(X, E)$, and let $\varphi \in B_0(X)$ and $W \in \mathcal{W}$. Choose an open balanced neighborhood $V \in \mathcal{W}$ such that $V + V + V \subseteq W$. Choose $r > \|\varphi\|$ with $f(x) \subseteq rV$, and let

$$K = \{x \in X : \varphi(x) \geq 1/r\}.$$

Then $f(K)$ is a compact subset of $E$ and so, by hypothesis, there exists a continuous map $u : f(K) \to E$ with range contained in a finite dimensional subspace of $E$ such that

$$u(f(x)) - f(x) \in (1/r)G \quad \text{for all } x \in K.$$

We can write

$$u \circ f = \sum_{i=1}^{n} (u_i \circ f)(x) a_i \quad (u_i \circ f \in C(K) \text{ and } a_i \in E).$$

Again there exist $\theta_i (1 \leq i \leq n)$ in $C_b(X)$ such that $\theta_i = u_i \circ f$ on $K$. Let $h = \sum_{i=1}^{n} \theta_i \otimes a_i$. Then

$$K \subseteq h^{-1}(rV + rV) = F \quad \text{(say)},$$

which is open in $X$, and so there exists a $\theta \in C_b(X)$ with $0 \leq \theta \leq 1, \theta = 0$ on $X \setminus F$. Let $g = \theta h$. Then $g \in C_b(X) \otimes E$ and $g = h = u \circ f$ on $K$. Further, $g(x) \subseteq rV + rV$. It is now easily verified that $g - f \in N(\varphi, W)$. \qed

Next, we give a brief account of an interesting and more general version of the Stone-Weierstrass theorem due to Prolla [Pro93b]. Most of the preceding versions of this theorem describe the closures of “$C_b(X)$-submodules” of $C_b(X, E)$ under various topologies. Prolla [Pro93b] has established a result that describes the $u$-closure of certain “subsets” of $C_b(X, E)$ which are not necessarily linear subspaces (see also [Pro93a,
1. APPROXIMATION IN THE STRICT TOPOLOGY

Its proof uses two results of Jewitt ([Je63], Lemma 2 and Theorem 1) and a “lemma” of Prolla ([Pro93b], Lemma 3) on “approximating the characteristic functions of open neighborhoods”. (This lemma is an analogue of Lemma 1.3 of Machado’s proof [Mac77] of the Stone-Weierstrass theorem, and of Lemma 1 of Brosowski and Deutsch [BD81]. We shall not reproduce the proofs here since they can also be found in the book ([Pro93a], Chapter 11).

We first introduce some notations and terminology. Let $X$ be a compact space and $E$ a TVS. (In this case $C(X, E) = C_b(X, E)$ and $\beta = u.$)

Let $I = [0, 1]$, and let $C(X, I)$ denote the subset of $C(X, \mathbb{R})$ consisting of those functions from $X$ into $I$. A subset $M$ of $C(X, I)$ is said to have property $(V)$ [Je63] if

1. $\varphi \in M$ implies $1 - \varphi \in M$;
2. $\varphi, \theta \in M$ implies $\varphi \theta \in M$.

If $A \subseteq C(X, E)$, then a function $\varphi \in C(X, I)$ is called a convex multiplier of $A$ if

$$\varphi f + (1 - \varphi)g \in A \quad \text{for every pair } f, g \in A.$$ Let $M_A$ denote the set of all multipliers of a subset $A$ of $C(X, E)$. Clearly, $\varphi \in M_A$ implies $1 - \varphi \in M_A$. Further, if $\varphi, \theta \in M$, then the identity

$$(\varphi \theta)f + (1 - \varphi \theta)g = \varphi(\theta f + (1 - \theta)g) + (1 - \varphi)g$$
for $f, g \in A$ shows that $\varphi \theta \in M_A$. Hence $M_A$ has property $V$. We mention that property $(V)$ is considered as a hypothesis in Theorem 1 of [Je63].

The main results of [Pro93a] are stated as follows.

**Theorem 4.1.10.** Let $X$ be a compact space of finite covering dimension and $E$ a TVS. Let $A$ be a subset of $C(X, E)$ such that set $M_A$ of all multipliers of $A$ separates the points of $X$. Let $f \in C(X, E)$ be given. The following are equivalent:

1. $f$ belongs to $u$-closure of $A$.
2. For each $x \in X$, $f(x) \in \overline{A(x)}$.

**Corollary 4.1.11.** Let $X$ and $E$ be as given above, and let $A$ be a subset of $C(X, E)$ such that

(a) the set $M_A$ of multipliers separates points of $X$;
(b) for each $x \in X$, $A(x)$ is dense in $E$.

Then $A$ is $u$-dense in $C(X, E)$.

**Corollary 4.1.12.** Let $X$ and $E$ be as given above. Then $C(X) \otimes E$ is $u$-dense in $C(X, E)$. 
2. Approximation in the Weighted Topology

We next consider the weighted approximation problem. Let $V$ be a Nachbin family on $X$, $\mathcal{M} \subseteq C(X)$ a $C_b(X)$-submodule, and $\mathcal{A} \subseteq CV_o(X, E)$ an $\mathcal{M}$-submodule. The ‘weighted approximation problem’, formulated originally by Nachbin [Nac65], consists in finding a characterization of the $\omega_V$-closure of $\mathcal{A}$ in $CV_o(X, E)$.

For $\mathcal{M} \subseteq C(X)$ or $C(X, E)$, we shall denote

$$z(\mathcal{M}) = \{x \in X : g(x) = 0 \text{ for all } g \in \mathcal{M}\}.$$

**Theorem 4.2.1.** [Pr71a, Kh85] Let $X$ be a completely regular space of finite covering dimension, $E$ a TVS, $V > 0$ a Nachbin family on $X$, and $\mathcal{M} \subseteq C(X)$ a $C_b(X)$-submodule. Let $\mathcal{A} \subseteq CV_o(X, E)$ be an $\mathcal{M}$-submodule such that $z(\mathcal{M}) \subseteq z(\mathcal{A})$, and let $f \in CV_o(X, E)$. Then $f$ belongs to the $\omega_V$-closure of $\mathcal{A}$ iff, for each $x \in X$, $f(x) \in \overline{\mathcal{A}(x)}$.

**Proof.** ($\Rightarrow$) Suppose $f$ belongs to the $\omega_V$-closure of $\mathcal{A}$. Since $V > 0$, $\rho \leq \omega_V$ and so $f$ belongs to the $\rho$-closure of $\mathcal{A}$; that is, or each $x \in X$, $f(x) \in \overline{\mathcal{A}(x)}$. ($\Leftarrow$) Suppose $X$ has covering dimension $n$, and let $v \in V$ and $G \in \mathcal{W}$. Choose a balanced $H \in \mathcal{W}$ with

$$H + H + \ldots + H((n + 2)\text{-terms}) \subseteq G.$$ 

Then $K = \{x \in X : v(x)f(x) \notin H\}$ is compact. By hypothesis, for each $x \in K$, we can choose a $g_x \in \mathcal{A}$ such that $f(x) - g_x(x) \in H$. Note that $g_x(x) \neq 0$ and so, since $z(\mathcal{M}) \subseteq z(\mathcal{A})$, $K \subseteq X\setminus z(\mathcal{M})$. Choose open neighborhoods $S'(x)$ and $S''(x)$ of $x$ in $X$ such that

$$v(y) \leq 1 \text{ for all } y \in S'(x) \text{ and } f(y) - g_x(y) \in H \text{ for all } y \in S''(x).$$

Put $S(x) = S'(x) \cap S''(x)$. Then open cover $\{S(x) : x \in K\}$ has a finite subcover $\{S(x_j) : 1 \leq j \leq m\}$ (say). Let $\mathcal{U}$ be an open refinement of $\{X\setminus K, S(x_j) : 1 \leq j \leq m\}$ such that any point of $X$ belongs to at most $n + 1$ members of $\mathcal{U}$. Choose $\{T_i : 1 \leq i \leq r\} \subseteq \mathcal{U}$ which covers $K$. Since $K \subseteq X \setminus z(\mathcal{M})$, we can apply Lemma 4.1.2 to obtain a collection $\{\varphi_i : 1 \leq i \leq r\} \subseteq \mathcal{M}$ such that $0 \leq \varphi_i \leq 1$, $\varphi_i = 0$ outside $T_i$, $\sum_{i=1}^{r} \varphi_i = 1$ on $K$, and $\sum_{i=1}^{r} \varphi_i \leq 1$ on $X$. Then $g = \sum_{i=1}^{r} \varphi_i g_{j_i} \in \mathcal{A}$ and, for any $y \in X$,

$$v(y)[g(y) - f(y)] = v(y)\sum_{i=1}^{r} \varphi_i(y)[g_{j_i}(y) - f(y)] + \left[\sum_{i=1}^{r} \varphi_i(y) - 1\right]v(y)f(y)$$

$$\in H + \ldots + H \text{ (at most } (n+1)\text{-terms}) + H \subseteq G.$$
(by considering the cases \( y \in K \) and \( y \notin K \)); that is, \( g - f \in N(v, G) \). Thus \( f \) belongs to the \( \omega_{V}\)-closure of \( \mathcal{A} \). \( \square \)

**Corollary 4.2.2.** Let \( X \), \( E \), \( V \) and \( \mathcal{A} \) be as in Theorem 4.2.1 and \( \mathcal{M} = \mathcal{C}_b(X) \). Then \( \mathcal{A} \) is \( \omega_{V}\)-dense in \( CV_0(X, E) \) iff for each \( x \in X \), \( \mathcal{A}(x) \) is dense in \( E \). In particular, \( CV_0(X) \otimes E \) is \( \omega_{V}\)-dense in \( CV_0(X, E) \).

**Theorem 4.2.3.** [Kh87] (a) \( CV_0(X, E) \) is \( \omega_{V}\)-closed in \( CV_b(X, E) \).

(b) If \( X \) is locally compact, then \( C_{oo}(X, E) \) is \( \omega_{V}\)-dense in \( CV_0(X, E) \).

**Proof.** (a) Let \( f \) belong to the \( \omega_{V}\)-closure of \( CV_0(X, E) \) in \( CV_b(X, E) \). To show that \( f \in CV_0(X, E) \), let \( v \in V \). Then, for any \( W \in \mathcal{W} \), the set \( F = \{ x \in X : v(x)f(x) \notin W \} \) is compact, as follows. Choose a balanced \( G \in \mathcal{W} \) with \( G + G \subseteq W \). There exists a function \( g \in CV_0(X, E) \) such that

\[
g - f \in N(v, G).
\]

Now the set \( K = \{ x \in X : v(x)g(x) \notin G \} \) is compact. If \( x \in X \setminus K \), then

\[
v(x)f(x) \in v(x)g(x) + G \subseteq G + G \subseteq W;
\]

and so \( x \in X \setminus F \). Therefore, \( F \subseteq K \) which implies that \( F \) is compact.

Thus \( f \in CV_0(X, E) \).

(b) Let \( f \in CV_0(X, E) \). To show that \( f \) belongs to the \( \omega_{V}\)-closure of \( C_{oo}(X, E) \), let \( v \in V \) and \( W \in \mathcal{W} \) be balanced. Now \( K = \{ x \in X : v(x)f(x) \notin W \} \) is compact. Since \( X \) is locally compact, there exists a \( \psi \in C_{oo}(X) \) such that \( 0 \leq \psi \leq 1 \) and \( \psi = 1 \) on \( K \). Then clearly \( g = \psi f \in C_{oo}(X, E) \). Let \( x \in X \). If \( x \in K \), then

\[
v(x)[f(x) - g(x)] = v(x)[f(x) - f(x)] = 0 \in W;
\]

If \( x \in X \setminus K \), then

\[
v(x)[f(x) - g(x)] = [\psi(x) - 1]v(x)f(x) \in [\psi(x) - 1]W \subseteq W;
\]

that is \( g - f \in N(v, W) \). Thus \( C_{oo}(X, E) \) is \( \omega_{V}\)-dense in \( CV_0(X, E) \). \( \square \)

**Theorem 4.2.4.** [Kh87] Let \( X \) be a completely regular Hausdorff space. Suppose \( E \) is a locally bounded TVS. Then \( \mathcal{C}_b(X, E) \cap CV_0(X, E) \) is \( \omega_{V}\)-dense in \( CV_0(X, E) \).

**Proof.** Let \( B \) be a bounded neighborhood of \( 0 \) in \( E \). There exists a closed shrinkable neighborhood \( S \) of \( 0 \) in \( E \) with \( S \subseteq B \). The Minkowski functional \( \rho = \rho_S \) of \( S \) is continuous and positively homogeneous, and \( S = \{ a \in E : \rho(a) \leq 1 \} \). Consequently, for any \( t > 0 \), the function
that is, \( x \).

\[
F
\]

\[
\therefore
\]

Thus bounded since \( v \).

Further, \( h_t(E) \subseteq tS \). \[ Let \ a \in E. \ If \ a \in tS, \ clearly \ h_t(a) = a \in tS; \ if \ a \notin tS, \ then \ \rho(a) > t > 0 \ and \ \frac{1}{\rho(a)}a \in S, \ and \ so \ h_t(a) = \frac{1}{\rho(a)}a \in tS. \]

Now, for any \( f \in CV_o(X, E) \), let \( F_t = h_t \circ f \). We claim that \( F_t \in C_b(X, E) \cap CV_o(X, E) \). Clearly \( F_t \) is continuous on \( X \); further, \( F_t \) is bounded since

\[
F_t(X) = h_t(f(X)) \subseteq h_t(E) \subseteq tS \subseteq tB
\]

and \( B \) is bounded. Hence \( F_t \in C_b(X, E) \). Now, let \( v \in V \). To show that \( vF_t \) vanishes at infinity, we need to show that, for any balanced \( G \subseteq W \), the set \( A = \{ x \in X : v(x)F_t(x) \notin G \} \) is compact. \[ Since \ f \in CV_o(X, E), \ the \ set \ B = \{ x \in X : v(x)f(x) \notin G \} \) is compact. Let \( x \in X \setminus B \), so that \( v(x)f(x) \in G \). Then, if \( f(x) \in tS \),

\[
v(x)F_t(x) = v(x)h_t(f(x)) = v(x)f(x) \in G
\]

and if \( f(x) \notin tS \), \( \rho(f(x)) > t \) and

\[
v(x)F_t(x) = v(x)h_t(f(x)) = \frac{t}{\rho(f(x))}v(x)f(x) \in \frac{t}{\rho(f(x))}G \subseteq G;
\]

that is, \( x \in X \setminus A \). Therefore \( A \subseteq B \), which implies that \( A \) is compact.]

Finally, let \( f \in CV_o(X, E) \), and let \( v \in V \) and \( W \in W \) be balanced. We show that, for a suitable \( t > 0 \), the function \( F_t = h_t \circ f \) satisfies \( F_t - f \in N(v, W) \). \[ Choose \ r \geq 1 \ such \ that \ B \subseteq rW \ and \ B \subseteq rS. \ The \ set \ K = \{ x \in X : v(x)f(x) \notin \frac{1}{r}S \} \) is compact and so we choose \( t \geq 1 \) with \( f(K) \subseteq \frac{1}{t}B \). Now, let \( x \in X \). If \( f(x) \subseteq tS \), then

\[
v(x)[F_t(x) - f(x)] = v(x)[h_t(f(x)) - f(x)] = v(x)[f(x) - f(x)] = 0 \in W.
\]

If \( f(x) \notin tS \), then \( \rho(f(x)) > t \) and also \( x \notin K \) (since \( f(K) \subseteq \frac{1}{t}B \subseteq tS \)) and so \( v(x)f(x) \in \frac{1}{t}S \); hence

\[
v(x)[F_t(x) - f(x)] = v(x)[\frac{t}{\rho(f(x))}f(x) - f(x)] = [\frac{t}{\rho(f(x))} - 1]v(x)f(x)
\]

\[
\in [\frac{t}{\rho(f(x))} - 1](\frac{1}{r})S \subseteq [\frac{t}{\rho(f(x))} - 1]W \subseteq W.
\]

Thus \( F_t - f \in N(v, W) \).] \[ □ \]
Theorem 4.2.5. [Kh87] Let $X$ be a completely regular space. Suppose $E$ is an admissible TVS and $V \subseteq S^+_0(X)$. Then $C_b(X) \otimes E$ is $\omega_V$-dense in $CV_o(X, E)$.

Proof. Let $f \in C_b(X, E)$, and let $v \in V$ and $W \in \mathcal{W}$. Choose an open balanced $G \in \mathcal{W}$ such that $G + G + G \subseteq W$. Choose $r \geq \max\{1, \|v\|\}$ with $f(X) \subseteq rG$, and let $K = \{x \in X : v(x) \geq \frac{1}{r}\}$. Then $f(K)$ is a compact subset of $E$ and so, by hypothesis, there exists a continuous map $\varphi : f(K) \rightarrow E$ with range contained in a finite dimensional subspace of $E$ such that $\varphi(f(x)) - f(x) \in \frac{1}{r}G$ for all $x \in K$. We can write $\varphi \circ f = \sum_{i=1}^{n}(\varphi_i \circ f) \otimes a_i$, where $\varphi_i \circ f \in C(K)$ and $a_i \in E$. By the Tietze extension Theorem, there exist $\psi_i (1 \leq \psi_i \leq 1)$ in $C_b(X)$ such that $\psi_i = \varphi_i \circ f$ on $K$. Let $h = \sum_{i=1}^{n} \psi_i \otimes a_i$. For any $x \in K$,

$$h(x) = (\varphi \circ f)(x) \in f(x) + \frac{1}{r}G \subseteq rG + rG;$$

hence $K \subseteq h^{-1}(rG + rG) = F$ (say). Clearly $F$ is open in $X$, and so there exists a $\psi \in C_b(X)$ with

$$0 \leq \psi \leq 1, \psi = 1 \text{ on } K \text{ and } \psi = 0 \text{ on } X \setminus F.$$ 

Let $g = \psi h$. Then $g \in C_b(X) \otimes E$ and $g = h = \varphi \circ f$ on $K$. Further, $g(X) \subseteq rG + rG$. [Let $x \in X$. If $x \in F$, then]

$$g(x) = \psi(x) h(x) = \psi(x)(rG + rG) \subseteq (rG + rG);$$

if $x \notin F$, then $g(x) = 0 \in rG + rG.$] Finally, $g - f \in N(v, W)$, as follows. Let $x \in X$. If $x \in K$, then

$$v(x)[g(x) - f(x)] = v(x)[\varphi(f(x)) - f(x)] \in v(x)\frac{1}{r}G \subseteq G \subseteq W;$$

if $x \notin K$, then $v(x) < \frac{1}{r}$ and so

$$v(x)[g(x) - f(x)] \in v(x)[rG + rG - rG] \subseteq v(x)rW \subseteq W. \quad \Box$$

Corollary 4.2.6. If $E$ is locally bounded and admissible and $V \subseteq S^+_0(X)$, then $C_b(X) \otimes E$ is $\omega_V$-dense in $CV_o(X, E)$.

We do not know whether, for any Nachbin family $V$ and $E$ admissible, $CV_o(X) \otimes E$ is $\omega_V$-dense in $CV_o(X, E)$. However, under some restrictions on $X$, this is true for $E$ any TVS.

Theorem 4.2.7. [Kh87] Let $X$ be a locally compact space of finite covering dimension. Then $C_{\omega}(X) \otimes E$ is $\omega_V$-dense in $CV_o(X, E)$.

Proof. By Theorem 4.2.3 (b), $C_{\omega}(X, E)$ is $\omega_V$-dense in $CV_o(X, E)$. Therefore, it suffices to show that $C_{\omega}(X) \otimes E$ is $\omega_V$-dense in $C_{\omega}(X, E)$. Let $f \in C_{\omega}(X, E)$, and let $v \in V$ and $W \in \mathcal{W}$ be balanced. There exists
a compact set $K \subseteq X$ such that $f(x) = 0$ for $x \notin K$. Choose $r \geq 1$
with $v(x) \leq r$ for all $x \in K$. Since $X$ is locally compact and $E$ is of
finite covering dimension, by Corollary 4.1.7, $C_o(X) \otimes E$ is $u$-dense in
$C_o(X, E)$; hence $C_{oo}(X) \otimes E$ is $u$-dense in $C_{oo}(X, E)$. Then there exists
a function $g \in C_{oo}(X) \otimes E$ with $g = 0$ outside $K$ and such that
$$g(x) - f(x) \in \frac{1}{r}W$$
for all $x \in X$.

Now, let $x \in X$. If $x \in K$, then
$$v(x)[g(x) - f(x)] \in \frac{v(x)}{r}W \subseteq W;$$
if $x \notin K$, then
$$v(x)[g(x) - f(x)] = v(x)[0 - 0] = 0 \in W.$$
Thus $g - f \in N(v, W)$. \qed
3. Weierstrass Polynomial Approximation

In this section, we present an infinite dimensional version of the Weierstrass polynomial approximation theorem, due to Bruno [Bru84].

Let $E$ and $F$ be real TVSs and $C(E, F)$ the space of all continuous functions from $E$ into $F$, and let

$$P(E, F) = \text{span}\{(\varphi(x))^n y : \varphi \in E^*, x \in X, y \in F, n \geq 1\},$$

the set of all continuous polynomials of finite type from $E$ into $F$.

**Definition.** A sequence $\{S_n\}$ of continuous linear operators from a TVS $E$ into itself is said to have the Grothendieck approximation property (GAP) if it satisfies:

(i) each $S_n$ has finite rank,

(ii) $\{S_n\}$ is equicontinuous,

(iii) $\{S_n\}$ converges uniformly on compact subsets to the identity on $E$.

For example, in a complete metric TVS with a Schauder basis $\{e_n\}$,

$$S_n : \sum_{i=1}^{\infty} a_i e_i \to a_n e_n$$

has the GAP.

**Example.** For $0 < p < 1$, the spaces $\ell_p(\mathbb{R})$, given by

$$\ell_p(\mathbb{R}) = \{a = \{a_i\} \subseteq \mathbb{R}: \sum_{i=1}^{\infty} |a_i|^p < \infty\},$$

with metric $d(a, b) = \sum_{i=1}^{\infty} |a_i - b_i|^p$, are complete metric vector spaces with Schauder bases and are not locally convex [Simo65].

**Theorem 4.3.1.** [Bru84] Let $E$ and $F$ be real TVS. Suppose $E$ has a sequence $\{S_n\}$ of projections with the GAP and $F$ has a sequence $\{T_n\}$ with the GAP. Then the space $P(E, F)$ of continuous polynomials from $E$ into $F$ of finite rank is $k$-dense in $C(E, F)$.

**Proof.** The proof is divided into three steps.

**Step I.** We show that, for any $f \in C(E, F)$ and $n \geq 1$, $T_n f S_n \xrightarrow{k} f$.

Let $K \subseteq E$ be a compact set and $W$ a neighborhood of 0 in $F$. Choose a balanced neighborhood $V$ of 0 in $F$ with $V + V \subseteq W$. Since $\{T_n\}$ is equicontinuous, there exists a neighborhood $U$ of 0 in $E$ such that $T_n(U) \subseteq V$ for all $n \geq 1$. 

Since $f$ is uniformly continuous on $K$ and $\{S_n\}$ converges uniformly to the identity of $E$ on $K$, it easily follows that $\{fS_n\}$ converges uniformly to $f$ on $K$. So there exists an integer $n_1$ such that
$$fS_n(x) - f(x) \in U \text{ for all } x \in K \text{ and } n \geq n_1.$$Since $\{T_n\}$ converges uniformly to the identity of $F$ on $f(K)$, there exists an integer $n_2$ such that
$$T_nf(x) - f(x) \in V \text{ for all } x \in K \text{ and } n \geq n_2.$$Hence, if $n \geq \max\{n_1, n_2\}$ and $x \in K$,
$$T_nfS_n(x) - f(x) = T_n(fS_n(x) - f(x)) + (T_nf(x) - f(x)) \in T_n(U) + V \subseteq V + V \subseteq W.$$

**Step II.** We show that, for any $f \in C(E, F)$ each $T_nfS_n$ belongs to the $k$-closure of $P(E, F)$. Put $f_n = T_nfS_n$. Let $K \subseteq E$ be a compact set and $V$ a balanced neighborhood of 0 in $F$. Since $E$ and $F$ are Hausdorff TVSs and $S_n$, $T_n$ are of finite rank, $S_n(E)$ and $T_n(F)$ are linearly homeomorphic to finite-dimensional Euclidean spaces. By the classical Stone-Weierstrass theorem [St48], $P(S_n(E), T_n(F))$ is $k$-dense in $C(S_n(E), T_n(F))$. Since $f'_n = f_n|S_n(E)$ belong to $C(S_n(E), T_n(F))$ and $S_n(K)$ is compact in $S_n(E)$, there exists a $p'_n \in P(S_n(E), T_n(F))$ such that
$$f'_n(x) - p'_n(x) \in V \text{ for all } x \in S_n(K).$$We define an extension $p_n$ of $p'_n(x)$ from $S_n(K)$ to $E$ by
$$p_n(x) = p'_nS_n(x), \quad x \in E.$$Clearly $p_n \in P(E, F)$ and for any $x \in K$,
$$f_n(x) - p_n(x) = f_nS_n(x) - p'_nS_n(x) \in V.$$

**Step III.** We now verify that $P(E, F)$ is $k$-dense in $C(E, F)$. Let $f \in C(E, F)$, and let $K \subseteq E$ be compact and $W$ a neighborhood of 0 in $F$. Choose a balanced neighborhood $V$ of 0 in $F$ with $V + V \subseteq W$. By step I, there exists an integer $n$ such that
$$f(x) - T_nfS_n(x) \in V \text{ for all } x \in K.$$By step II, there exists a $p_n \in P(E, F)$ such that
$$T_nfS_n(x) - p_n(x) \text{ for all } x \in K.$$Hence, for any $x \in K$,
$$f(x) - p_n(x) = [f(x) - T_nfS_n(x)] + [T_nfS_n(x) - p_n(x)] \in V + V \subseteq W. \quad \square$$
4. Maximal Submodules in \((C_b(X, E), \beta)\)

This section includes characterizations of maximal and closed \(C_b(X)\)-submodules of \((C_b(X, E), \beta)\) and \(CV_o(X, E)\), as given in [Kh84, Kh85b].

Throughout this section, \(X\) will denote a completely regular Hausdorff space and \(E\) a Hausdorff TVS with non-trivial topological dual \(E^*\).

**Definition.** A TVS \(F\) is said to have the separation property (or, equivalently, the Hahn-Banach extension property) if, for any closed vector subspace \(S\) of \(F\) and \(a \in F \setminus S\), there exists a maximal (proper) closed vector subspace \(M\) of \(F\) such that \(S \subseteq M\) and \(a \notin M\). \(F\) is said to have the weaker separation property if every proper closed vector subspace of \(F\) is contained in some maximal closed subspace of \(F\). Clearly, the separation property implies the weaker separation property. Every locally convex TVS has the separation property [Scha66], and it is shown by Gregory and Shapiro [GrSh70] that if \((F, F')\) is a dual pair of a vector space \(F\) with the weak topology on \(F\) not equal to the Mackey topology, then there is a non-locally convex linear topology between them having the separation property.

We shall require the following version of the Stone-Weierstrass theorem.

**Theorem 4.4.1.** [Kh84] For any \(C_b(X)\)-submodule \(A\) of \(C_b(X, E)\), consider the following conditions.

1. \(A\) is \(\beta\)-dense in \(C_b(X, E)\).
2. For each \(x \in X\), \(A(x)\) is dense in \(E\).
3. For each \(x \in X\) and maximal closed subspace \(M\) of \(E\), there exists an \(f \in A\) such that \(f(x) \notin M\).

Then (1) \(\Rightarrow\) (2) \(\Rightarrow\) (3). If \(X\) has finite covering dimension, then (2) \(\Rightarrow\) (1); if \(E\) has the weaker separation property, then (3) \(\Rightarrow\) (2).

**Proof.** (1) \(\Rightarrow\) (2) is evident since \(\rho \leq \beta\).

(2) \(\Rightarrow\) (1) is same as Theorem 4.1.3.

(2) \(\Rightarrow\) (3). Let \(x \in X\) and \(M\) a maximal closed subspace of \(E\), and let \(a \in E \setminus M\). Choose an open neighborhood \(G\) of \(a\) in \(E\) such that \(G \subseteq E \setminus M\). Since \(A(x)\) is dense in \(E\), there exists an \(f \in A\) with \(f(x) \in G\). Hence \(f(x) \notin M\).

(3) \(\Rightarrow\) (2). Suppose \(E\) has the weaker separation property. If \(\overline{A(x)} \neq E\) for some \(x \in X\), there exists a maximal closed subspace \(M\) of \(E\) such that \(\overline{A(x)} \subseteq M\). But, by (3), there exists a \(g \in A\) with \(g(x) \notin M\), and so \(\overline{A(x)} \subseteq M\). This gives us the desired contradiction. \(\square\)
Corollary 4.4.2. [Kh84] Suppose $X$ has finite covering dimension and $E$ has the separation property. Let $A$ be a $C_b(X)$-submodules of $C_b(X,E)$, and $f \in C_b(X,E)$. Then the following are equivalent:

1. $f$ belongs to the $\beta$-closure of $A$.
2. For each $x \in X$, $f(x) \in \overline{A(x)}$.
3. For each $x \in X$ and maximal closed subspace $M$ of $E$, $A(x) \subseteq M$ implies that $f(x) \in M$.

We now characterize the maximal $\beta$-closed $C_b(X)$-submodules of $C_b(X,E)$. We begin with the following.

Lemma 4.4.3. [Kh84] For any $x \in X$ and $M$ a maximal closed subspace of $E$,

$$S(x,M) = \{ f \in C_b(X,E) : f(x) \in M \}$$

is a maximal $\beta$-closed $C_b(X)$-submodule of $C_b(X,E)$.

Proof. First note that, since $M$ is a vector subspace, $S(x,M)$ is a $C_b(X)$-submodule of $C_b(X,E)$. There exists a non-zero $\varphi \in E^*$ such that $M = \varphi^{-1}(0)$ (§ A.2) Given any $x \in X$, define a map $T_x : (C_b(X,E), \beta) \to E$ by

$$T_x(f) = f(x), \ f \in C_b(X,E).$$

If $W$ is any neighborhood of 0 in $E$, then

$$N(\{x\},W) = \{ f \in C_b(X,E) : f(x) \in W \}$$

is a $\beta$-neighborhood of 0 in $C_b(X,E)$ such that

$$T_x(N(\{x\},W)) \subseteq W.$$ 

Hence $T_x$ is continuous. It is easy to see that

$$S(x,M) = (\varphi \circ T_x)^{-1}(0),$$

and so $S(x,M)$ is maximal and $\beta$-closed. \qed

Theorem 4.4.4. [Kh84] Suppose $X$ has finite covering dimension.

1. If $E$ has the weaker separation property, then every maximal $\beta$-closed $C_b(X)$-submodule of $C_b(X,E)$ is of the form $S(x,M)$, where $x \in X$ and $M$ is a maximal closed subspace of $E$.

2. If $E$ has the separation property, then every $\beta$-closed $C_b(X)$-submodule of $C_b(X,E)$ is the intersection of all maximal $\beta$-closed $C_b(X)$-submodules which contain it.

Proof. (1) Let $A$ be a maximal $\beta$-closed $C_b(X)$-submodule of $C_b(X,E)$, and let $f \in C_b(X,E)$ with $f \notin A$. Since $A$ is $\beta$-closed, by Theorem 4.4.1, there exists an $x_o \in X$ such that $f(x_o) \notin \overline{A(x_o)}$. By the weaker separation property of $E$, there exists a maximal closed subspace $M$ of $E$
such that \( \overline{A(x_o)} \subseteq M \). Clearly, \( A \subseteq S(x_o, M) \). Hence, by Lemma 5.5.3, \( A = S(x_o, M) \).

(2) Let \( A_1 \) be a \( \beta \)-closed \( C(X) \)-submodule of \( C(X, E) \), and let \( g \in C(X, E) \) with \( g \notin A_1 \). Then, as in the above proof and using Corollary 4.4.2, there exist an \( x_1 \in X \) and a maximal closed subspace \( M_1 \) of \( E \) such that
\[
A \subseteq S(x_1, M_1) \text{ and } g \notin S(x_1, M_1).
\]

Remarks. (1) If \( E \) is assumed to be locally convex, then Theorems 4.4.1 and 4.4.4 hold without restricting \( X \) to have finite covering dimension (see Remark 4.1.8).

(2) In general, a maximal \( C(X) \)-submodule of \( C(X, E) \) need not be \( \beta \)-closed. For instance if \( E \) admits a discontinuous linear functional \( \varphi \) and \( M = \varphi^{-1}(0) \), then, for any \( x \in X \), \( S(x, M) \) is maximal but not \( \beta \)-closed by Theorem 4.4.1 (since \( M \) is not closed in \( E \)).

Finally, we give a characterization of the maximal \( \omega_V \)-closed \( C(X) \)-submodules of \( CV_o(X, E) \).

**Theorem 4.4.5.** [Kh85b] Let \( X \) be of finite covering dimension, \( E \) a TVS, and \( V > 0 \) a Nachbin family on \( X \). Suppose that \( E \) has the separation property. Then:

(a) Every maximal \( \omega_V \)-closed \( C(X) \)-submodule of \( CV_o(X, E) \) is of the form
\[
S_V(x, M) = \{ f \in CV_o(X, E) : f(x) \in M \},
\]
where \( x \in X \) and \( M \) is a maximal closed subspace of \( E \).

(b) Every \( \omega_V \)-closed \( C(X) \)-submodule of \( CV_o(X, E) \) is the intersection of all maximal \( \omega_V \)-closed \( C(X) \)-submodules which contain it.

**Proof.** (a) Let \( A \) be a maximal \( \omega_V \)-closed \( C(X) \)-submodule of \( CV_o(X, E) \), and let \( f \in CV_o(X, E) \) with \( f \notin A \). By Theorem 4.2.1, there exists an \( x \in X \) such that \( f(x) \notin \overline{A(x)} \). By the separation property of \( E \), there exists a maximal closed subspace \( M \) of \( E \) such that
\[
\overline{A(x)} \subseteq M \text{ and } f(x) \notin M.
\]
It can be easily shown that \( S(x, M) \) is a maximal \( \omega_V \)-closed \( C(X) \)-submodule of \( CV_o(X, E) \), and thus \( A = S_V(x, M) \).

(b) The proof is similar to that of the above part and Theorem 4.4.4 (2). \( \square \)
5. The Approximation Property

Recall that a TVS $E$ is said to have the *approximation property* if the identity map on $E$ can be approximated uniformly on precompact sets by continuous linear maps of finite rank (i.e. with range contained in finite dimensional subspaces of $E$).

**Theorem 4.5.1.** [Kat81] Let $X$ be a completely regular space of finite covering dimension and $E$ a Hausdorff TVS having the approximation property. Then $(C_b(X, E), \beta)$ has the approximation property.

**Proof.** Suppose $X$ has finite covering dimension $n$. Let $A$ be a $\beta$-precompact subset of $C_b(X, E)$, and let $\varphi \in B_a(X)$ with $0 \leq \varphi \leq 1$ and $W \in \mathcal{W}$ a balanced set. We show that there exists a continuous linear mapping $T : (C_b(X, E), \beta) \rightarrow (C_b(X, E), \beta)$ of finite rank such that

$$Tf - f \in N(\varphi, W) \quad \text{for all } f \in A.$$  \hspace{1cm} (1)

Choose a balanced $W_1 \in \mathcal{W}$ with $W_1 + ... + W_1$ $(n + 2 \text{ terms}) \subseteq W$.

Also choose an open balanced $W_2 \in \mathcal{W}$ such that $W_2 \subseteq W_1 \cap S^{-1}(W_2) = W_2 \cap \{a \in E : Sa \in W_2\}$. \hspace{1cm} (3)

Since $A$, being $\beta$-bounded, is $u$-bounded, there exists an $r \geq 1$ such that $f(X) \subseteq rW_2$ for all $f \in A$. Choose a compact $K \subseteq X$ such that $\varphi(x) < 1/r$ for all $x \in X \setminus K$. Put $B = \cup \{f(K) : f \in A\}$. It is easy to see that $B$ is a precompact subset of $E$. Since $E$ has the approximation property, there exists a continuous linear mapping $S : E \rightarrow E$ of finite rank such that

$$sa - a \in (1/r)W_2 \quad \text{for all } a \in B.$$ \hspace{1cm} (2)

Choose an open balanced $W_3 \in \mathcal{W}$ such that

$$W_3 \subseteq W_2 \cap S^{-1}(W_2) = W_2 \cap \{a \in E : Sa \in W_2\}.$$ \hspace{1cm} (3)

Since $A$ is $\beta$-precompact, there exists a finite set $A_1 \subseteq A$ such that

$$A \subseteq A_1 + N(\varphi, W_3).$$ \hspace{1cm} (4)

For each $x \in X$, the set

$$\{y \in X : g(y) - g(y) \in W_3\}$$ \hspace{1cm} (5)

is an open neighborhood of $x$. Since $K$ is compact, there exist $x_1, ..., x_p \in K$ such that $K \subseteq \bigcup_{i=1}^{p} G(x_i)$. The collection $\{G(x_1), ..., G(x_p), X \setminus K\}$ forms an open cover of $X$. Since $X$ has finite covering dimension $n$ and $K$ is compact, there exists an open cover $\{H_1, ..., H_m\}$ of $K$ which is a refinement of the cover $\{G(x_1), ..., G(x_p)\}$ and such that each point of $K$ belongs to at most $n + 1$ of the $H_i$’s. For each $H_i$, choose an
5. THE APPROXIMATION PROPERTY

index $j_i$ such that $H_i \subseteq G(y_i)$, $y_i = x_{j_i}$. By Lemma 4.1.2, there exist $h_1, \ldots, h_m \in C_b(X)$ with

$$0 \leq h_i \leq 1, \ h_i = 0 \ on \ X \setminus H_i, \ \sum_{i=1}^{m} h_i(x) = 1 \ on \ K,$$

and $\sum_{i=1}^{m} h_i(x) \leq 1$ for each $x \in X$.

Define $T : C_b(X, E) \to C_b(X, E)$ by

$$T(f) = \sum_{i=1}^{m} h_i \otimes Sf(y_i), \ f \in C_b(X, E).$$

It is easy to verify that $T$ is a $\beta$-continuous linear mapping of finite rank.

To complete the proof, we need to verify (1). Let $f \in A$, and let $x \in X$.

By (4), there exists a $g \in A$ such that $\varphi(f - g)(X) \subseteq W_3$. (6)

Note that $I_x = \{i : x \in H_i\}$ contains at most $n + 1$ indices. Now

$$Tf(x) - f(x) = \sum_{i=1}^{m} h_i(x)Sf(y_i) - f(x).$$

For any $i \in I_x$, $x \in H_i \subseteq G(y_i)$ and so, using (2), (3) and (6)

$$\varphi(x)[Sf(y_i) - f(x)] = S[\varphi(x)(f(y_i) - g(y_i)) + \varphi(x)[Sg(y_i) - g(y_i)]
+ S[\varphi(x)(g(y_i) - g(x)) + \varphi(x)[g(x) - f(x)]
\subseteq S(W_3) + (\varphi(x)/r)W_2 + \varphi(x)W_3 + W_3
\subseteq W_2 + W_2 + W_2 + W_2 \subseteq W.$$  

**Case I.** Suppose $x \in \bigcup_{i=1}^{m} H_i$. If $x \in K$, then $\sum_{i=1}^{m} h_i(x) = 1$ and so

$$\varphi(x)[Tf(x) - f(x)] = \varphi(x) \left[ \sum_{i=1}^{m} h_i(x)Sf(y_i) - \sum_{i=1}^{m} h_i(x)f(x) \right]
= \sum_{i=1}^{m} h_i(x)\varphi(x)[Sf(y_i) - f(x)]
\subseteq \sum_{i=1}^{m} h_i(x)W_1 \subseteq W_1 + \ldots + W_1 \subseteq W.$$
If $x \in (\bigcup_{i=1}^{m} H_i) \setminus K$, then $\sum_{i=1}^{n} h_i(x) \leq 1$ and $\varphi(x)r < 1$, and so

\[
\varphi(x)[Tf(x) - f(x)] = \varphi(x) \left[ \sum_{i=1}^{m} h_i(x)Sf(y_i) - \sum_{i=1}^{m} h_i(x)f(x) \right]
+ \sum_{i=1}^{m} h_i(x)f(x) - f(x)
= \sum_{i \in I_x} h_i(x)\varphi(x)[S(f(y_i)) - f(x)]
+ \left[ \sum_{i \in I_x} h_i(x) - 1 \right] \varphi(x)f(x)
\in \sum_{i \in I_x} h_i(x)W_1 + \left[ \sum_{i \in I_x} h_i(x) - 1 \right] \varphi(x)rW_2
\subseteq \sum_{i=1}^{m} h_i(x)W_1 + W_2
\subseteq \{W_1 + \ldots + W_1 \text{ (n + 1)-terms} \} + W_2 \subseteq W.
\]

**Case II.** Suppose $x \in X \setminus \bigcup_{i=1}^{m} H_i$. Then each $h_i(x) = 0$ and $\varphi(x)r < 1$ and so

\[
\varphi(x)[Tf(x) - f(x)] = \varphi(x)[0 - f(x)] \in \varphi(x)rW_2 \subseteq W.
\]

Thus (1) holds. \qed

**Theorem 4.5.2.** [Kh95] Suppose every compact subset of $X$ has finite covering dimension and $E$ has the approximation property. Then $(C(X, E), k)$ has the approximation property.

**Proof.** Let $\mathcal{A}$ be a $k$-precompact subset of $C(X, E)$, and let $K$ be a compact subset of $X$ and $W \in \mathcal{W}$. Suppose $K$ has covering dimension $n$, and choose a balanced set $V \in \mathcal{W}$ such that

\[ V + V + \ldots + V((n + 1)\text{-terms}) \subseteq W. \]

Now choose an open balanced set $U \in \mathcal{W}$ such that $U + U + U + U \subseteq V$. Since $\mathcal{A}$ is precompact in $C(X, E)$, it is easy to verify that $D = \bigcup\{f(K) : f \in \mathcal{A}\}$ is a precompact subset of $E$. By hypothesis, there exists a continuous linear map $S : E \to E$ of finite rank such that

\[ Sa - a \in U \quad \text{for all } a \in D. \]
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Choose an open balanced set $U_1 \in \mathcal{W}$ such that
$$U_1 \subseteq U \cap \{a \in E : Sa \in U\}.$$  

Since $\mathcal{A}$ is precompact, there exists a finite subset $\mathcal{A}_1$ of $\mathcal{A}$ such that
$$\mathcal{A} \subseteq \mathcal{A}_1 + N(K, U_1).$$

For each $x \in K$, the set $G(x) = \bigcap_{g \in \mathcal{A}} \{y \in X : g(y) - g(x) \in U_1\}$ is an open neighborhood of $x$ in $X$. Then there exists a finite open cover $\mathcal{U} = \{G(x_1), ..., (G(x_p))\}$ (say) of $K$. By hypothesis, we can choose an open cover $\mathcal{V} = \{H_i, ..., H_m\}$ of $K$ which is a refinement of $\mathcal{U}$ and such that each point of $K$ belongs to at most $n + 1$ members of $\mathcal{V}$. For each $H_i$, choose an index $j_i$ such that $H_i \subseteq G(y_i)$, $y_i = x_{j_i}$. Put $H_{m+1} = X \setminus K$. Then there exist $\varphi_1, ..., \varphi_{m+1} \in C_b(X)$ with

$$0 \leq \varphi_i \leq 1, \varphi_i = 0 \text{ on } X \setminus H_i, \sum_{i=1}^{m+1} \varphi_i(x) = 1 \text{ for } x \in K,$$

and $\sum_{i=1}^{m+1} \varphi_i(x) \leq 1$ for $x \in X$. Define $T : C(X, E) \to C(X, E)$ by

$$Tf = \sum_{i=1}^{m+1} \varphi_i \otimes Sf(y_i), \ f \in C(X, E).$$

It is easy to verify that $T$ is a continuous linear map of finite rank. To complete the proof, we show that

$$Tf - f \in N(K, W) \text{ for all } f \in \mathcal{A}.$$ 

[Let $f \in \mathcal{A}$, and let $x \in K$. Choose $g \in \mathcal{A}_1$ such that $(g - f) \subseteq U_1$. Note that $I_x = \{i : x \in H_i\}$ contains at most $n + 1$ indices. Now

$$Tf(x) - f(x) = \sum_{i=1}^{m+1} \varphi_i(x)[Sf(y_i) - f(x)] = \sum_{i \in I_x} \varphi_i(x)[Sf(y_i) - f(x)].$$

For each $i \in I_x$,

$$Sf(y_i) - f(x) = S[f(y_i) - g(y_i)] + [Sg(y_i) - g(y_i)] + g(y_i) - g(x) + [g(x) - f(x)]$$

$$\subseteq SU_1 + U_1 + U_1 \subseteq V.$$ 

Hence

$$Tf(x) - f(x) \in \sum_{i \in I_x} \varphi_i(x)V \subseteq W.$$

□
Remark. We mention, as before, that if $E$ is a locally convex TVS, then with slight modification of proofs, Theorems 4.5.1 and 4.5.2 hold without the hypothesis of finite covering dimension.
6. Notes and Comments

Section 4.1. The Stone-Weierstrass theorem was first proved by M.H. Stone [St37] in 1937 for the space \( (C_b(X), u) \) as a generalization of the classical Weierstrass’s approximation theorem of 1885 (see also [St48] for its refinement). Several authors have later generalized it or given its new proof; see, e.g. Hewitt [Hew47], Kaplansky ([Kap51], §3), de Branges [Bra59], Bishop [Bis61], Jewitt [Je63], Machado [Mac77], Brosowski and Deutsch [BD81], and Ransford [Ran84].

The Stone-Weierstrass theorems for \( (C_b(X), \beta) \) and \( (C_b(X, E), \beta) \) were first established by Buck [Buc58] in 1958 for \( E \) finite dimensional. Its extension to \( (C_b(X, E), \beta) \), \( E \) a locally convex TVS, was later given by Todd [Tod65] and Wells [Wel65], independently. For \( E \) any TVS, a special case of the theorem (that \( C_b(X) \otimes E \) is \( u \)-dense in \( C_b(X, E) \)) was established by Shuchat [Shu72a] for \( (C_b(X, E), u) \), assuming \( X \) a compact space of finite covering dimension (see also Waelbroeck [Wae73], § 8.1)). More general forms of the theorem were later obtained by Khan [Kh79, Kh80, Kh95], Katsaras [Kat81], Abel [Ab87] and Prolla [Pro93a, Pro93b] for \( (C_b(X, E), \beta) \), \( (C_b(X, E), u) \) and \( (C(X, E), k) \), assuming the hypothesis of finite covering dimension on \( X \). The present section contains results of these latter authors.

Section 4.2. The ‘weighted approximation problem’, formulated by Nachbin [Nac65] and further studied by Summers [SumW71] for \( E \) a real or complex field, consists in finding a characterization of the \( \omega \)-closure of \( A \) in \( CV_o(X, E) \), where \( A \subseteq CV_o(X, E) \) an \( M \)-submodule with \( M \subseteq C(X) \) a \( C_b(X) \)-submodule. Prolla [Pro71a, Pro71b, Pro72] extended some results of above authors to the case of \( E \) a locally convex TVS. This section contains some results from [Kh85, Kh87] when \( E \) is not necessarily locally convex. In particular, these results give extension of ([Nac67], Propositions 1 and 2, p.64) and ([Nac67], Proposition 5, p.66) to \( E \) a locally bounded TVS.

Section 4.3. The main result is an infinite dimensional version of the Weierstrass polynomial approximation theorem, due to Bruno [Bru84], for \( P(E, F) \), the set of all continuous polynomials of finite type from a TVS \( E \) into a TVS \( F \). Earlier results in this direction were obtained by Prenter [Pre70] for \( E = F \) real separable Hilbert space and by Machado-Prolla [MP72] for \( E \) and \( F \) real locally convex spaces. The hypotheses on \( E \) and \( F \) involve the existence of sequences of projections having the Grothendieck approximation property (GAP) (cf. [Scha71], pp.108-115).
Section 4.4. C. Todd [Tod65] has used the Stone-Weierstrass theorem to characterize the maximal $\beta$-closed $C_b(X)$-submodules of $C_b(X, E)$ for $E$ a locally convex TVS. In this section, we present the extension of these characterizations to a non-locally convex setting, as given in [Kh84]. These require the additional assumptions on $E$ to have the separation property (or, equivalently, the Hahn-Banach extension property) or the weaker separation property considered by Gregory and Shapiro [GrSh70].

Section 4.5. The approximation property for locally convex spaces was defined by Grothendieck in 1955, who also raised “the approximation problem” (i.e. the problem whether every locally convex TVS $E$ has the approximation property). This problem remained unresolved until 1973 when Enflo [Enf73] gave an example of a separable Banach space without the approximation property. However, despite this development, there still exist some concrete Banach spaces about which it is not yet known whether they have the approximation property (see Singer ([Sin70], Vol.I-II).

As regards the function spaces, Grothendieck [Gro55] was the first to show that if $X$ is a compact Hausdorff space, then $(C(X), \|\cdot\|)$ has the approximation property. In the case of the space $(C_b(X), \beta)$, this result was proved by Collins and Dorroh [CD68] for $X$ locally compact and by Fremlin, Garling and Haydon [FGH72] for $X$ completely regular. Bierstedt [Bie75] obtained a similar result for the weighted space $(CV_b(X), \omega_Y)$. In the case of vector-valued functions, Fontenot [Fon74] has shown that if $X$ is completely regular and $E$ a normed space with the metric approximation property, then $(C_b(X, E), \beta)$ has the approximation property. Further extensions of this result have been obtained by Prolla [Pro77] for $E$ a locally convex TVS and by Katsaras [Kat81] for $E$ any TVS but assuming $X$ to have a finite covering dimension. This section contains results from [Kat81, Kh95].
CHAPTER 5

Maximal Ideal Spaces

In this chapter we give characterization of maximal closed ideals in the algebras \((C_b(X, A), \beta)\) and \((C(X, A), k)\) and also of the maximal ideal spaces of these algebras.
1. Maximal Ideals in \((C_b(X, A), \beta)\)

We first show that, if \(A\) is topological algebra with jointly continuous multiplication, then \((C_b(X, A), \beta)\) and \((C(X, A), k)\) become topological algebras with jointly continuous multiplication, as follows.

**Theorem 5.1.1.** Let \(X\) be a topological space and \(A\) a topological algebra with jointly continuous multiplication. Then \((C_b(X, A), \beta)\) and \((C(X, A), k)\) are topological algebras with jointly continuous multiplication.

**Proof.** Let \(W\) be a base of \(\tau\)-neighborhoods of 0 in \(A\) consisting of absorbing and balanced sets. Let \(f, g \in C_b(X, A)\), and let \(\{f_\alpha\}\) and \(\{g_\alpha\}\) be nets in \(C_b(X, A)\) such that \(f_\alpha \xrightarrow{\beta} f\) and \(g_\alpha \xrightarrow{\beta} g\). We need to show that \(f_\alpha g_\alpha \xrightarrow{\beta} fg\).

Let \(\varphi \in B_0(X), 0 \leq \varphi \leq 1,\) and \(W \in W\). Choose a \(V \subseteq W\) such that \(V + V + V \subseteq W\). Since \(A\) is a topological algebra with jointly continuous multiplication, there exists a \(U \in W\) such that \(U^2 \subseteq V\). Define \(\varphi'(x) = \sqrt{\varphi(x)}, x \in X\). Then \(\varphi' \in B_0(X)\). Choose \(r \geq 1\) such that

\[
(\varphi'f)(X) \subseteq rU, (\varphi'g)(X) \subseteq rU. \tag{1}
\]

There exists an index \(\alpha_o\) such that

\[
f_\alpha - f, g_\alpha - g \in N(\varphi', \frac{1}{r}U) \text{ for all } \alpha \geq \alpha_o. \tag{2}
\]

Now, for any \(x \in X\) and \(\alpha \geq \alpha_o\), using (1) and (2),

\[
\varphi(x)[f_\alpha(x)g_\alpha(x) - f(x)g(x)] = \varphi'(x)[f_\alpha(x) - f(x)]\cdot \varphi'(x)[g_\alpha(x) - g(x)] + \varphi'(x)f(x)\cdot \varphi'(x)[g_\alpha(x) - g(x)] + \varphi'(x)[f_\alpha(x) - f(x)]\cdot \varphi'(x)g(x)
\]

\[
\in U.U + rU \cdot \frac{1}{r}U + \frac{1}{r}U.uU \subseteq V + V + V \subseteq W;
\]

that is, \(f_\alpha g_\alpha - fg \in N(\varphi, W)\) for all \(\alpha \geq \alpha_o\). Hence the multiplication is jointly continuous in \((C_b(X, A), \beta)\).

Replacing both \(\varphi, \varphi'\) by \(\chi_K, K\) any compact set in \(X\), in the above proof, we conclude that \((C(X, A), k)\) is also topological algebras. \(\square\)

We shall require the following version of the Stone-Weierstrass Theorem from Chapter 4 for the characterization of \(\beta\)-closed and \(k\)-closed ideals.
Theorem 5.1.2. Let $X$ be a completely regular space of finite covering dimension and $E$ a TVS, and let $A$ be a $C_b(X)$-(resp. $C(X)$)-submodule of $C_b(X, E)$ (resp. $C(X, E)$). Then the following conditions are equivalent:

(1) $f$ belongs to the $\beta$-closure (resp. $k$-closure) of $A$.

(2) For each $x \in X$, $f(x) \in A(x)$.

In the remainder of this section, we shall assume that $A$ is a topological algebra with jointly continuous multiplication and identity $e$. Further, all ideals considered are the proper ideals.

Theorem 5.1.3. [Kh86] For any $x \in X$ and any closed left (resp. right, two-sided) ideal $M$ of $A$,

$$ S(x, M) = \{ f \in C_b(X, A) : f(x) \in M \} $$

is $\beta$-closed left (resp. right, two-sided) ideal in $C_b(X, A)$; if $M$ is maximal, so is $S(x, M)$.

Proof. The first part is straightforward. Let $M$ be a maximal closed left ideal in $A$ and $x \in X$, and let $J$ be a left ideal in $C_b(X, A)$ such that $S(x, M) \subseteq J$. Suppose there is a function $g \in J$ such that $g(x) \notin M$. Then $g(x)$ has a left inverse $c$ (say) in $E$. Since $J$ is a left ideal, $(1 \otimes c)g \in J$, where $1 \in C_b(X)$ is the function identically 1 on $X$. But,

$$ [(1 \otimes c)(1 \otimes g(x))](y) = cg(x) = (1 \otimes e)(y) $$

for all $y \in X$, and so the identity function $1 \otimes e \in J$. This contradiction shows that $S(x, M) = J$, as required. \qed

The following theorem gives us a characterization of $\beta$-closed (left, right) ideals in $C_b(X, A)$.

Theorem 5.1.4. [Kh86] Suppose $X$ has finite covering dimension. Then every $\beta$-closed left (resp. right) ideal $J$ in $C_b(X, A)$ is of the form

$$ J = \bigcap_{x \in X} S(x, J_x), $$

where, for each $x \in X$, $J_x$ represents a closed left (resp. right) ideal in $A$.

Proof. Let $J$ be a $\beta$-closed left ideal in $C_b(X, A)$. It is easily seen that, for each $x \in X$, $J_x = J(x)$ is a closed left ideal in $A$. Now, let $f \in J$ and $\varphi \in C_b(X)$. Then, for any $y \in X$,

$$ \varphi(y)f(y) = \varphi(y)[e f(y)] = (\varphi \otimes e)(y). $$

Since $(\varphi \otimes e)f \in J$, we have $\varphi f \in J$. Hence $J$ is a $C_b(X)$-submodule of $C_b(X, E)$ and so, by Theorem 6.1.2,

$$ J = \{ f \in C_b(X, E) : f(x) \in J(x) \text{ for all } x \in X \} = \bigcap_{x \in X} S(x, J_x). $$

We next consider the $\beta$-closed two-sided ideals.
Theorem 5.1.5. [Kh86] Suppose $X$ has finite covering dimension and $A$ is a simple topological algebra. Then any $\beta$-closed two-sided ideal $J$ in $C_b(X, A)$ is of the form

$$J_F = \{ f \in C_b(X, A) : f(x) = 0 \text{ for all } x \in F \},$$

where $F \subseteq X$ is a closed subset.

Proof. First note that, for any closed subset $F \subseteq X$, $J_F$ is a $\beta$-closed two-sided ideal in $C_b(X, A)$. Now, let $J$ be a $\beta$-closed two-sided ideal in $C_b(X, A)$. Then the set $F = \{ x \in X : f(x) = 0 \text{ for all } f \in J \}$ is closed in $X$, and clearly $J \subseteq J_F$. Let $g \in J_F$, and suppose that $g \notin J$. Since $J$ is $\beta$-closed, by Theorem 5.1.2, there exists a $y \in X$ such that $g(y) \notin J_y = J(y)$. Therefore, $J_y \neq A$ and so, by hypothesis, $J_y = \{ 0 \}$. Hence $y \in F$. But $g(y) \notin J_y$ implies that $g(y) \neq 0$, a contradiction. Thus $J = J_F$, as required.

Notice that, in the notation of Theorem 5.1.4, we can write the above $J$ as $J = \bigcap_{x \in F} S(x, \{ 0 \})$.

We now characterize the maximal $\beta$-closed ideals in $C_b(X, A)$.

Theorem 5.1.6. [Kh86] Suppose $X$ has finite covering dimension and $A$ is a $Q$-algebra. Then any maximal $\beta$-closed left (resp. right, two-sided) ideal in $C_b(X, A)$ is of the form $S(x, M)$, where $x \in X$ and $M$ is a maximal left (resp. right, two-sided) ideal in $A$.

Proof. Let $J$ be a maximal $\beta$-closed left ideal in $C_b(X, A)$, and let $f \in C_b(X, A)$ with $f \notin J$. By Theorem 5.1.2, there exists an $x \in X$ such that $f(x) \notin J_x = J(x)$. Since $A$ is a $Q$-algebra, there exists a maximal (closed) proper left ideal $M$ in $A$ such that $J_x \subseteq M$. Clearly, $J \subseteq S(x, M)$, and so, by the maximality of $J$ and Theorem 5.1.3, $J = S(x, M)$.

We next state, without proof, the analogues of Theorems 5.1.3-5.1.6 for the algebra $(C(X, A), k)$.

Lemma 5.1.7. For any $x \in X$ and any closed left (resp. right, two-sided) ideal $M$ of $A$,

$$S(x, M) = \{ f \in C(X, A) : f(x) \in M \}$$

is $k$-closed left (resp. right, two-sided) ideal in $C(X, A)$; if $M$ is maximal, so is $S(x, M)$.

The following theorem gives us a characterization of $k$-closed (left, right) ideals in $C(X, A)$.

Theorem 5.1.8. [Kh86] Suppose $X$ has finite covering dimension. Then every $k$-closed left (resp. right) ideal $I$ in $C(X, A)$ is of the form
1. MAXIMAL IDEALS IN \((C_b(X, A), \beta)\)

\[ I = \bigcap_{x \in X} S(x, I_x), \text{ where, for each } x \in X, I_x \text{ represents a closed left (resp. right) ideal in } A. \]

**Theorem 5.1.9.** [Kh86] Suppose \(X\) has finite covering dimension and \(A\) is a simple topological algebra. Then any \(k\)-closed two-sided ideal \(J\) in \(C(X, A)\) is of the form

\[ J = J_F = \{ f \in C(X, A) : f(x) = 0 \text{ for all } x \in F \}, \]

where \(F \subseteq X\) is a closed subset.

Notice that, in the notation of Theorem 5.1.8, we can write the above \(I\) as \(I = \bigcap_{x \in F} S(x, \{0\})\).

**Theorem 5.1.10.** [Kh86] Suppose \(X\) has finite covering dimension and \(A\) is a \(Q\)-algebra. Then any maximal \(k\)-closed left (resp. right, two-sided) ideal in \(C(X, A)\) is of the form \(S(x_o, M_o)\), where \(x_o \in X\) and \(M_o\) is a maximal left (resp. right, two-sided) ideal in \(A\).

**Remarks.**

1. In general, a maximal ideal in \(C_b(X, A)\) need not be \(\beta\)-closed. For instance, if \(A\) admits a discontinuous multiplicative linear functional \(\varphi\) and \(M = \varphi^{-1}(0)\), then, for any \(x \in X\), \(S(x, M)\) is a maximal ideal but not \(\beta\)-closed. Thus, if \(A\) admits no discontinuous multiplicative linear functional, then every maximal ideal in \(C_b(X, A)\) is \(\beta\)-closed. This remark also applies to \((C(X, A), k)\).

2. If the algebra \(A\) is locally convex, then Theorems 5.1.4 - 5.1.6 and 5.1.8 - 5.1.10 hold without restricting \(X\) to have finite covering dimension.
2. Maximal Ideal Space of \((C_b(X, A), \beta)\)

Let \(X\) be a completely regular space and \(A\) be a topological algebra with jointly continuous multiplication over \(\mathbb{C}\) and with non-trivial dual \(A^*\). Recall that \(\Delta(A)\) (resp. \(\Delta_c(A)\)) denotes the set of all non-zero multiplicative (multiplicative continuous) linear functionals on \(A\).

The set \(\Delta_c(A)\) equipped with the Gelfand topology, (i.e., the relative \(w' = w(A^*, A)\)-topology induced from \(A^*\)) is called the maximal ideal space (or the carrier space) of \(A\). We shall assume that \(\Delta_c(A)\) is non-empty. In the literature, \(\Delta_c(A)\) is also called the space of continuous homomorphisms of \(A\), or equivalently the space of closed maximal ideals with codimension 1 of \(A\).

For each \(x \in A\), let \(\hat{x} : \Delta_c(A) \to \mathbb{C}\) be the Gelfand function defined by \(\hat{x}(\varphi) = \varphi(x)\) for each \(\varphi \in \Delta_c(A)\) and let \(\hat{A} = \{\hat{x} : x \in A\}\). The mapping \(x \to \hat{x}\), for each \(x \in A\), will be called the Gelfand transform. It is clear that the Gelfand transform is an algebra homomorphism from \(A\) into \(C(\Delta_c(A))\).

It is well-known (see e.g. [Simm62]) that if \(X\) is a compact (resp. completely regular) Hausdorff space, then the maximal ideals in \(C_b(X, A)\) are in one-one correspondence with the points in \(X\) (resp. \(\beta X\)); i.e., \(\Delta_c(C_b(X)) \cong X\) (resp. \(\Delta_c(C(X)) \cong \beta X\)). For the study of the ideal structures of vector-valued algebras \((C_b(X, A), \beta)\) and \((C(X, A), k)\), the situation is not so simple. However, the space \(\Delta_c(C(X, A))\) has been studied in many papers under various kinds of topological assumptions on \(X, A\) and \(C(X, A)\). Further, results similar in spirit to \(\Delta_c(C(X, A)) \cong X \times \Delta_c(A)\) have also been obtained by various authors.

In this section, we present results concerning \(\Delta_c((C_b(X, A), \beta) \cong X \times \Delta_c(A)\). We follow the approach of Proulla [Pro81], but with no convexity assumptions is made about \(A\).

We begin with the following classical result which gives a characterization of \(\Delta_c(C(X), k)\).

**Theorem 5.2.1.** (i) The \(k\)-closed ideals of \(C(X)\) are of the form

\[
I_F = \{f \in C(X) : f|F = 0\},
\]

where \(F\) is a closed ideal in \(X\).

(ii) For any \(x \in X\), define \(\delta_x : C(X) \to \mathbb{K}\) by \(\delta_x(f) = f(x)\), \(f \in C(X)\). Then \(\delta_x \in \Delta_c(C(X))\) and the map \(G : x \to \delta_x\) is a homeomorphism. Thus \(X \cong \Delta_c(C(X), k)\).

**Proof.** ([Diet69], p. 203) If \(I\) is an ideal in \(C(X)\), let

\[
F = \cap\{z(f) : f \in I\} = \{x \in X : f(x) = 0 \text{ for all } f \in I\}.
\]
Then $F$ is closed in $X$ and $I$ is contained in the closed ideal $I_F$. To show that $I_F \subseteq I$, it is sufficient to show that $I$ is $k$-dense in $I_F$. \[\text{Let } h \in I_F, K \subseteq X \text{ a compact set and } \varepsilon > 0. \text{ Then } I|_K = \{f|K : f \in I\} \text{ is a subalgebra of } C(X). \text{ It is actually an ideal, since if } g \in C(K), g \text{ extends to a } g_1 \in C(X) \text{ and } g(f|K) = g_1 f|K \in I|_K. \text{ Using the Tietze extension theorem and the Urysohn lemma, as in ([Simm62], pp.329-330), } I|_K \text{ is norm dense in } \{f \in C(K) : f|F \cap K = 0\}, \text{ since } F \cap K = \cap\{z(f|K) : f \in I\}. \text{ Since } h|F \cap K = 0, \text{ there is a } g \in I \text{ with } p_K(g - h) \leq \varepsilon. \text{ Thus } h \text{ belongs to the } k\text{-closure of } I; \text{ that is, } k\text{-cl}(I) = I_F. \text{ So, if } I \text{ is } k\text{-closed, } I = I_F.

(ii) Clearly $\delta_x : C(X) \to \mathbb{K}$ is a non-zero continuous multiplicative linear functional. We first show that the map $G : x \to \delta_x$ is one-one. \[\text{Let } x \neq y \in X. \text{ Since } X \text{ is completely regular, there exist an } f \in C(X) \text{ such that } f(x) = 0 \text{ and } f(y) = 1. \text{ Then } G(x)(f) = 0 \text{ and } G(y)(f) = 1, \text{ and so } G(x) \neq G(y).\] We next show that $G$ is onto. \[\text{Let } \varphi \in \Delta_c(C(X)). \text{ Since } \ker \varphi \text{ is a closed maximal ideal of } C(X), \text{ by (i), there is a closed set } F \text{ such that } \ker \varphi = I_F. \text{ Since } \ker \varphi \text{ is maximal (and proper), } F \text{ reduces to a point } x \in X. \text{ So } \ker \varphi = \ker \delta_x \text{ and thus } \varphi = \delta_x ([Simm62], p. 321).\] So $G$ is a bijection.

Finally, $G$ is continuous and open, as follows. \[\text{For } f \in C(X), \text{ define } \hat{f} : \Delta_c(C(X)) \to \mathbb{K} \text{ by } \hat{f}(\varphi) = \varphi(f), \varphi \in \Delta_c(C(X)).\] The relative topology $\nu(C(X)^*, C(X))$ on $\Delta_c(C(X))$ is generated by \[\{\hat{f} : f \in C(X)\}, \text{ and so } \{\hat{f}^{-1}(V) : V \text{ open in } \mathbb{K}, \ f \in C(X)\}\] is a subbase for the topology on $\Delta_c(C(X))$. Since $X$ is completely regular, its topology equals the weak topology generated by $C(X)$, and so \[\{f^{-1}(V) : V \text{ open in } \mathbb{K}, f \in C(X)\}\] is a subbase for the topology on $X$ ([Simm62], p. 134). But \[\hat{G}(x) = \hat{f}(\delta_x) = \delta_x(f) = f(x), \text{ so } G^{-1}(\hat{f}^{-1}(V)) = \hat{f}G^{-1}(V) = f^{-1}(V); \text{ that is, } G \text{ is continuous and open.}\] Since $G$ is a bijection, it is a homeomorphism. \[\square\]

**Definition.** $\Delta_c(A)$ is called **locally equicontinuous** if each $h \in \Delta_c(A)$ has an equicontinuous neighborhood; this always happens in the case of $A$ a normed algebra, since in this case $\Delta_c(A)$ is a subset of the unit ball of $E^*$. 
Notation. If $A$ is a topological algebra, we can now define a mapping $G : X \times \Delta_\varepsilon(A) \to \Delta_\varepsilon(C(X, A))$ by

$$G(x, \varphi) = \varphi \circ \delta_x, (x, \varphi) \in X \times \Delta_\varepsilon(A),$$

where $\delta_x : C(X, A) \to A$ is the algebra homomorphism given by

$$\delta_x(f) = f(x), f \in C(X, A).$$

**Theorem 5.2.2.** Let $X$ be a locally compact space of finite covering dimension and $A$ a topological algebra with identity, and let $G$ be the mapping from $X \times \Delta_\varepsilon(A)$ into $\Delta_\varepsilon((C_b(X, A), \beta))$, as defined in (*). Then:

(a) $G$ is a bijection.

(b) The inverse mapping $G^{-1}$ is continuous.

(c) If $\Delta_\varepsilon(A)$ is locally equicontinuous, then $G$ itself is continuous.

Proof. Obviously, by (*), $G(x, \varphi) \in \Delta_\varepsilon(C_b(X, A))$ if $(x, \varphi) \in X \times \Delta_\varepsilon(A)$.

(a) To show that $G$ is one-one, let $(x, h) \neq (y, k)$ be given in $X \times \Delta_\varepsilon(A)$. If $x = y$, then $h \neq k$. Choose $u \in A$ such that $h(u) \neq k(u)$. Since $X$ is locally compact, $C_b(X, E)$ is essential in the sense that we can choose $f \in C_b(X, E)$ such that $f(x) = u$. Then

$$G(x, h)(f) = h(f(x)) = h(u) \neq k(u) = G(x, k)(f).$$

If $x \neq y$, choose $\varphi \in C_b(X)$ such that $\varphi(x) = 0$ and $\varphi(y) = 1$. Choose now $u \in A$ with $k(u) = 1$. Choose $f \in C_b(X, E)$, say $f = \varphi \otimes u$, such that $f(y) = u$. Then $g = \varphi f \in C_b(X, E)$ and we have

$$G(x, k)(g) = h(g(x)) = h(0) = 0, \text{ but } G(y, k)(g) = k(g(y)) = k(u) = 1.$$

To show that $G$ is onto, let $H \in \Delta_\varepsilon(C_b(X, A))$. Then $I = \ker H$ is a proper closed two-sided maximal ideal of $(C_b(X, A), \beta)$. By Theorem 5.2.1, there exists a unique $x \in X$ such that the ideal $I_x = \text{cl}(I(x))$ is proper. Choose $f_o \in C_b(X, A)$ such that $H(f_o) = 1$. Then $f_o$ is a modular identity for $I$.

For each $u \in A$, $uf \in C_b(X, A)$. Define $h = h_{f_o} : A \to \mathbb{K}$ by setting

$$h(u) = h_{f_o}(u) := H(u f_o), \quad u \in A.$$

Clearly, $h \in A^\ast$. Now, if $u, t \in A$, then $uf \in I$. Choose $g \in C(X, E)$ with $g(x) = u$. Then, since $H(f_o) = 1$,

$$h(ut) = H(uf_o)H(tf_o) = H(f_o H(uf_o)H(t f_o) = H(f_o H(h(t) = H(f_o h(t) = h(u)h(t));$$
hence $h$ is multiplicative. Let $J = \ker h$. We claim that $J \subseteq I(x)$. [Indeed, if $u \in J$, then $uf \in I$. Choose $g \in C_b(X, A)$ such that $g(x) = u$. Then $gf - g \in I$, and therefore $uf(x) - u \in I(x)$. Now $uf(x) = uf(x) \in I(x)$, so $u \in I(x).]$ Since $I_x$ is proper, it now follows that $h \neq 0$, i.e., $h \in \Delta_c(A)$. Let $L = \ker(h \circ \delta_x)$. We claim that $I \subseteq L$. [Indeed, let $g \in C_b(X, A)$ be such that $g \notin L$. Then $g(x) \notin J$. On the other hand, $J$ is a maximal ideal and $J \subseteq I(x) \subseteq I_x$. Since $I_x$ is proper, it follows that $J = I_x$. Hence $g(x) \notin I_x$. By Theorem 5.2.1, $g \notin I$.] Since $I$ is maximal and $L$ is closed and proper, $I = L$. This shows that $H$ and $h \circ \delta_x$ have the same kernel. Since both are multiplicative,

$H = h \circ \delta_x = G(x, h)$.

(c) Using the assumption that $\Delta_c(A)$ is locally equicontinuous, $G$

is continuous, as follows. Let $(x_0, h_0) \in X \times \Delta_c(A), \varepsilon > 0$ and $g \in C_b(X, E)$ be given. Choose an equicontinuous neighborhood $N$ of $h_0$

in $\Delta_c(A)$ such that

$N \subseteq \{h \in \Delta_c(A) : |(h - h_0)(g(x_0))| < \varepsilon/2\}$

Let $W$ be a neighborhood of $g(x)$ in $A$ such that

$|h(w - g(x_0))| < \varepsilon/2$ for all $w \in W, h \in N$.

Next, choose a neighborhood $U$ of $x_0$ in $X$ such that $g(x) \in W$ for all $x \in U$. Then $(x, h) \in U \times N$ implies $g(x) \in W$ and $h \in N$. Therefore

$|h(g(x) - g(x_0))| < \varepsilon/2$ and $|(h - h_0)(g(x_0))| < \varepsilon/2$.

It follows that

$|G(x, h) - G(x_0, h_0)(g)| = |h(g(x)) - h_0(g(x_0))|$

$\leq |(h - h_0)(g(x_0))| + |h(g(x) - g(x_0))| < \varepsilon$

for all $(x, h) \in U \times N$.

(b) To show that

$G^{-1}$

is continuous, let $H_\alpha \to H$ in $\Delta_c(C_b(X, A))$. Since $G$ is onto, there exist nets $\{x_\alpha\}$ in $X$ and $\{h_\alpha\}$ in $\Delta_c(A)$ such that $H_\alpha = G(x_\alpha, h_\alpha)$, and points $x$ in $X$ and $h$ in $\Delta_c(A)$ such that $H = G(x, h)$. Since $H \neq 0$, there is some $f \in C_b(X, A)$ such that $H(f) = 1$. Choose $\alpha_0$ such that $H_\alpha(f) \neq 0$ for all $\alpha \geq \alpha_0$. Take any $g \in C_b(X)$. Then $gf \in C_b(X, A)$ and, for all $\alpha \geq \alpha_0$,

$g(x_\alpha) = \frac{g(x_\alpha)h_\alpha(f(x_\alpha))}{h_\alpha(f(x_\alpha))} = \frac{H_\alpha(gf)}{H_\alpha(f)}$. 
Now
\[ g(x_\alpha) \to \frac{H(gf)}{H(f)} \]
\[ = H(gf) = h(g(x)f)x = g(x)h(f(x)) = g(x)H(f) = g(x). \]

Since \( X \) is a completely regular Hausdorff space and \( g \in C^b(X) \) was arbitrary, \( x_\alpha \to x \). Next, for any \( u \in A \), \( uf \in C^b(X,A) \) and so for \( \alpha \geq \alpha_0 \),
\[ h_\alpha(u) = \frac{h_\alpha(u)h_\alpha(f(x_\alpha))}{h_\alpha(f(x_\alpha))} = \frac{H_\alpha(uf)}{H_\alpha(f)}. \]

Hence
\[ h_\alpha(u) \to \frac{H(uf)}{H(f)} = H(uf) = h(uf(x)) = h(u)h(f(x)) = h(u)H(f) = h(u), \]
i.e., \( h_\alpha \to h \) in the relative weak topology of \( \Delta_c(A) \). Thus
\[ G^{-1}(H_\alpha) = G^{-1}[G(x_\alpha, h_\alpha)] = (x_\alpha, h_\alpha) \to (x, h) = G^{-1}(H), \]
and so \( G^{-1} \) is continuous. \( \square \)

**Corollary 5.2.3.** Let \( X \) be a completely regular space of finite covering dimension and \( A \) topological algebra with identity, and let \( G \) be the mapping from \( X \times \Delta_c(A) \) into \( \Delta_c((C^b(X,A), \beta), \) as defined in \( (*) \). If \( \Delta_c(A) \) is locally equicontinuous, then \( G \) is a homeomorphism.

**Theorem 5.2.4.** Let \( X \) be a completely regular space of finite covering dimension and \( A \) topological algebra with identity, and let \( G \) be the mapping from \( X \times \Delta_c(A) \) into \( \Delta_c((C(X,A), k), \) as defined in \( (*) \). If \( \Delta_c(A) \) is locally equicontinuous, then \( G \) is a homeomorphism.

**Theorem 5.2.5.** Let \( X \) be a locally compact space of finite covering dimension and \( A \) topological algebra with identity, and let \( G \) be the mapping from \( X \times \Delta_c(A) \) into \( \Delta_c((C_0(X,A), u), \) as defined in \( (*) \). If \( \Delta_c(A) \) is locally equicontinuous, then \( G \) is a homeomorphism.

**Note.** If the algebra \( A \) is locally convex, then 5.2.2-5.2.5 hold without restricting \( X \) to have finite covering dimension.
3. NOTES AND COMMENTS

Section 5.1. The classical results on characterization of maximal and closed ideals in \((C(X, A), k)\) and \((C_b(X, A), u)\) were obtained by Kaplansky [Kap47] in the case of \(A\) a topological ring. These results are further studied in ([Nai72], Ch.V, §26.2) for \(A\) a Banach algebra and in ([Pro77], Ch.1, §6) for \(A\) a locally convex algebra.

In 1958, Buck [Buc58] showed that there is a one-one correspondence between the maximal \(\beta\)-closed ideals in \(C_b(X)\) and the points of \(X\). This section contains characterizations of the maximal and \(\beta\)-closed (resp. \(k\)-closed) ideals in \(C_b(X, A)\) (resp. \(C(X, A)\)) where \(X\) is a completely regular Hausdorff space of finite covering dimension and \(A\) a Hausdorff topological algebra (not necessarily locally convex) with jointly continuous multiplication, as given in [Kh86].

Section 5.2. Concerning the results similar in spirit to \(\Delta_c(C(X, A)) \cong X \times \Delta_c(A)\), Hausner ([Hau57], p. 248) showed that if \(X\) is compact and \(A\) a commutative Banach algebra, then

\[
\Delta_c((C(X, A), u)) \cong X \times \Delta_c(A).
\]

This was an improvement of an earlier result of Yood [Y51], Theorem 3.1, Lemma 5.1, where \(A\) is a commutative \(B^*\) algebra with identity (or \(A\) a commutative Banach algebra with identity and assuming that every maximal ideal in \(C(X, E)\) was of the form \(S(x, M) = \{f \in C(X, E) : f(x) \in M\}\), with \(x \in X\) and \(M\) a maximal ideal in \(A\)).

Using tensor products, Mallios ([Mal66], Theorem 5.1), showing

\[
\Delta_c((C_0(X, A), u)) \cong X \times \Delta_c(A),
\]

extended the Hausner-Yood theorem to the case of \(X\) a locally compact space and \(A\) a complete locally \(m\)-convex algebra \(Q\)-algebra for which \(\Delta_c(A)\) is locally equicontinuous; and (again using tensor products) Dietrich ([Diet69], Theorem 4) showed that \(\Delta_c((C(X, A, k)) \cong X \times \Delta_c(A)\) if \(X\) is a completely regular \(k\)-space and \(A\) is a complete locally convex algebra with \(\Delta_c(A)\) locally equicontinuous. Prolla [Pro81] obtained a similar result for the weighted function algebra \((CV_\omega(X, A), \omega_V)\), without using tensor product techniques. Hery [Her76] extended some results of the above authors to the case of \(A\) a certain commutative topological algebras, not necessarily locally convex, with identity (see also Arhippainen [Arhi92, Arhi96]. This section contains results about \(\Delta_c((C_b(X, A), B) \cong X \times \Delta_c(A)\), following the approach of Prolla [Pro81] without the convexity assumptions on \(A\).
Some generalizations of Hery’s results to the case when $X$ is realcompact and $A$ is a metrizable algebra or $X$ is a completely regular Hausdorff space and $A$ is a realcompact metrizable algebra are considered in MR (99k:46089) and MR (2002b:46084) and in the case of topological module-algebras in MR (87c:46056) (here are given conditions when all ideals (not necessarily two-sided) are fixed).

Moreover, descriptions of $\Delta_c(C_b(X,A),\beta)$, $\Delta_c(C_o(X,A),u)$, $\Delta_c(CV_b(X,A),\beta)$ and $\Delta_c(Crc(X,A),u)$ are given in [Ab04] as well and, when $A$ is locally convex, in MR (58#30268) and (for closed (maximal, but not maximal closed) one-sided ideals) in MR (84d:46084).

In addition, the description of all closed maximal (not maximal closed) modular left (right and two-sided) ideals in subalgebras of $C(X;A;S)$ (the algebra of $A$-valued continuous functions $f$ on $X$ for which $f(S)$ is relatively compact in $A$ for each $S \in \mathcal{S}$, $\mathcal{S}$ is a cover of $X$ which is closed with respect to finite unions) is given in MR (2005c:46064) and in MR (2004a:46045) (the description $\Delta_c(A)$ of $A$, where $A$ is a subalgebra of $C(X;A;\mathcal{S})$, is included) by Mart Abel in case when $X$ is a completely regular Hausdorff space, $\mathcal{S}$ is a compact cover of $X$ and $A$ is a locally $m$-pseudoconvex Hausdorff algebra, a locally pseudoconvex Waelbroeck Hausdorff algebra or an exponentially galbed Hausdorff algebra with bounded elements. Similar results for real topological algebras are given by Olga Panova in MR 2326371 and in [O. Panova, Real Gelfand-Mazur algebras. Dissertation, University of Tartu, Tartu, 2006, Dissertationes Mathematicae Universitatis Tartuensis, 48. Tartu University Press, Tartu, 2006, 84 pp.].

Most recently, Oubbi [Oub07] has given several interesting results related to the above ones.
CHAPTER 6

Separability and Trans-separability

In this chapter, we consider various characterizations of separability and trans-separability of the function spaces \((C_b(X, E), \beta), (C(X, E), k), (C_b(X, E), \sigma),\) and \((C_b(X, E), u)\).
1. Separability of Function Spaces

Recall that $X$ is called submetrizable if it can be mapped by a one-to-one continuous function onto some metric space $Z$ (or, equivalently, if its topology is stronger than a metrizable topology on $X$). If, in addition, $Z$ is separable, $X$ is called separably submetrizable. A submetrizable space is Hausdorff but not necessarily regular. A topological space $X$ is separably submetrizable iff there exists a sequence $\{g_n\} \subseteq C_b(X)$ which separates points of $X$. The space $\ell_\infty(N)$ is submetrizable under its weak topology $w(\ell_\infty(N), M(N))$, where $M(N)$ is the dual of $C(N)$. The space $M[0,1]$ is a non-separable Banach space, although the weak topology of $M[0,1]$ is submetrizable. A TVS is submetrizable iff its origin is a $G_\delta$.

We state the following three theorems without proof for reference purpose.

**Theorem 6.1.1.** [KK40, SZ57] (i) $(C_b(X), u)$ is separable iff $X$ is compact metric space.

(ii) If $X$ is locally compact, then $(C_o(X), u)$ is separable iff $X$ is $\sigma$-compact metric space.

**Theorem 6.1.2.** [War58, GuSc72, SumW72] The following statements are equivalent:

(a) $(C_b(X), \beta)$ is separable.

(b) $(C(X), k)$ is separable.

(c) $X$ is separably submetrizable.

**Theorem 6.1.3.** [GuSc72] The following statements are equivalent:

(a) $(C_b(X), t)$ is separable, where $t$ is either $\sigma_o, \sigma$ or $u$.

(b) $X$ is a compact metric space.

We now present the extension of above results to the case of $E$-valued functions. We first obtain an extension of Theorem 7.1.2.

**Theorem 6.1.4.** [Kat81, Kh86b] The following are equivalent:

(a) $(C_b(X) \otimes E, \beta)$ is separable.

(b) $(C(X) \otimes E, k)$ is separable.

(c) $X$ is separably submetrizable and $E$ is separable.

Proof. (a) $\Rightarrow$ (b) If $(C_b(X) \otimes E, \beta)$ is separable, then $(C_b(X) \otimes E, k)$ is also separable. Since $C_b(X)$ is $k$-dense in $C(X)$, it follows that $C_b(X) \otimes E$ is $k$-dense in $C(X) \otimes E$. Thus $(C(X) \otimes E, k)$ separable.

(b) $\Rightarrow$ (c) We first note the fact that both $(C(X), k)$ and $E$ are isomorphic to subspaces of $(C(X) \otimes E, k)$ via the maps $g \to g \otimes a$ ($0 \neq a \in E$ fixed) and $b \to 1_X \otimes b$, respectively. Hence, by (b), $(C(X), k)$ is separable and so, by Theorem 6.1.2, $X$ is separably submetrizable. Further, $E$ is also separable.
(c) ⇒ (a) Suppose $X$ is separably submetrizable. Then, by Theorem 6.1.2, $(C_b(X), \beta)$ is separable. Let $\{\varphi_m\}$ and $\{a_n\}$ be countable dense subsets of $(C_b(X), \beta)$ and $E$, respectively. Let $\mathcal{A}$ be the countable subspace generated by $\{\varphi_m \otimes a_n : m, n = 1, 2, \ldots\}$ over rationals. We show that $\mathcal{A}$ is $\beta$-dense in $C_b(X) \otimes E$. Let $g \in C_b(X) \otimes E$, $\varphi \in B_o(X)$ with $0 \leq \varphi \leq 1$, and $W \in W$. We can write $f = \sum_{i=1}^p \theta_i \otimes b_i$ ($\theta_i \in C_b(X)$, $b_i \in E$). Let $V \in W$ be balanced with $V + V + \ldots + V (2p\text{-terms}) \subseteq W$. Choose $r \geq 1$ such that each $b_i \in rV$. Let $s = \max\{|\theta_i| : i \leq i \leq p\}$. For each $i = 1, \ldots, p$, there exist $\varphi_{m_i} \in \{\varphi_m\}$ and $a_{n_i} \in \{a_n\}$ such that $||\varphi(\varphi_m_i - \theta_i)|| < 1/r(s + 1)$ and $a_{n_i} - b_i \in (1/r(s + 1))V$. Note that $||\varphi(\varphi_m_i)|| < s + 1$. Let $g = \sum_{i=1}^n \varphi_{m_i} \otimes a_{n_i}$. Then $g \in \mathcal{A}$ and, for any $x \in X$,

$$
\varphi(x)[g(x) - f(x)] = \sum_{i=1}^p \varphi(x)\varphi_{m_i}(x)[a_{n_i} - b_i]
+ \sum_{i=1}^p \varphi(x)[\varphi_{m_i}(x) - \theta_i(x)]b_i
\leq \frac{1}{r}[V + \ldots + V(2p\text{-terms})] + \frac{1}{s + 1}[V + \ldots V(p\text{-terms})]
\subseteq W.
$$

Hence $g - f \in N(\varphi, W)$, as required. \hfill \square

**Corollary 6.1.5.** Let $X$ be a locally compact $\sigma$-compact space. Then $(C_b(X) \otimes E, \beta)$ is separable iff $X$ is metrizable and $E$ is separable.

**Corollary 6.1.6.** Suppose either $X$ has finite covering dimension or $E$ is locally convex, or $E$ is complete metrizable with a basis. Then the following are equivalent:

(a) $(C_b(X, E), \beta)$ is separable.

(b) $(C(X, E), k)$ is separable.

(c) $X$ is separably submetrizable and $E$ is separable.

In fact, each of the above restrictions on $X$ or $E$ implies that $C_b(X) \otimes E$ is $\beta$-dense in $C_b(X, E)$ and that $C(X) \otimes E$ is $k$-dense in $C(X, E)$ (see Chapter 5). It is not known whether or not these ‘density’ results hold for $E$ a locally bounded space. However, we can prove

**Theorem 6.1.7.** [Kh86b] Let $X$ be any Hausdorff space and $E$ any locally bounded TVS. Then $(C_b(X, E), \beta)$ is separable iff $(C(X, E), k)$ is so.

Proof. Let $\{f_n\}$ be a countable $k$-dense subset of $C(X, E)$. Let $V$ be a balanced bounded neighborhood of 0 in $E$, and let $S$ be a closed
are isomorphic to subspaces of \((\mathbb{C} = 1^m \neq 0)\). So we only need to show that

\[ h_m : E \to E \]

is continuous. Further, \( h_m(E) \subseteq mS \subseteq mV \), which shows that the functions \( h_m \circ f_n \in C_b(X, E) \). We show that \( \{ h_m \circ f_n : m, n = 1, 2, \ldots \} \) is \( \beta \)-dense in \( C_b(X, E) \). Let \( f \in C_b(X, E), \varphi \in B_\beta(X) \) with \( 0 \leq \varphi \leq 1 \), and \( W \in W \). Choose \( r \geq 1 \) such that \( V + V \subseteq rSV + V \subseteq rW \). Choose an integer \( M \geq 1 \) with \( f(X) \subseteq (M/r)V \). Let \( K \) be a compact subset of \( X \) such that \( \varphi(x) < 1/rM \) for \( x \in X \setminus K \). There exists an integer \( N \) such that \( (f_N - f)(K) \subseteq (1/r)V \). Let \( y \in X \). If \( y \in K \), then \( f_N(y) \in MS \) and so

\[ \varphi(y)(h_N \circ f_N(y) - f(y)) = \varphi(y)(f_N(y) - f(y)) \in W. \]

If \( y \in X \setminus K \), then

\[
\varphi(y)[h_M \circ f_N(y) - f(y)] \subseteq \varphi(y)\left[MS - \frac{M}{r}V\right] \\
\subseteq \frac{M}{rM}\left[V - \frac{1}{r}V\right] \subseteq V + V \subseteq W.
\]

Thus \( h_M \circ f_N - f \in N(\varphi, W) \).

Conversely, suppose \((C_b(X, E), \beta)\) is separable. Then \((C_b(X, E), k)\) is also separable. So we only need to show that \( C_b(X, E) \) is \( k \)-dense in \( C(X, E) \) (cf. Theorem 1.1.11). Let \( f \in C(X, E) \) and \( K \) a compact subset of \( X \). If \( V \) is a bounded neighborhood of \( 0 \) in \( E \), let \( S \) be a closed shrinkable neighborhood of \( 0 \) with \( S \subseteq V \). Choose \( r \geq 1 \) with \( f(K) \subseteq rS \). Then, as in the above proof, we get a function \( h_r \circ f \in C_b(X, E) \) such that \( h_r \circ f(x) - f(x) = 0 \) for all \( x \in K \).

**Theorem 6.1.8.** [KR91] The following statements are equivalent:

(a) \((C_b(X) \otimes E, t)\) is separable, where \( t \) is either \( \sigma_o, \sigma \), or \( u \).

(b) \( X \) is a compact metric space and \( E \) is separable.

**Proof.** \((a) \Rightarrow (b)\) We first note the fact that both \((C_b(X), t)\) and \( E \) are isomorphic to subspaces of \((C_b(X) \otimes E, t)\) via the maps \( g \to g \otimes a \) \((0 \neq a \in E \text{ fixed})\) and \( b \to 1_X \otimes b \), respectively. Hence, by \((a)\), \((C_b(X), t)\) is separable and so, by Theorem 6.1.3, \( X \) is a compact metric space. Further, \( E \) is also separable.

\((b) \Rightarrow (a)\) By Theorem 6.1.1 \((i)\), \((C_b(X), u)\) is separable. Let \( \{g_m : m = 1, 2, \ldots\} \) and \( \{g_n : n = 1, 2, \ldots\} \) be countable dense subsets of \((C_b(X), u)\) and \( E \), respectively and let \( A \) consist of finite sums of the
elements \( \{g_m \otimes a_n : m, n = 1, 2, \ldots\} \). Let \( f \in C_b(X) \otimes E \) and \( W \in \mathcal{W} \) any neighborhood of 0 in \( E \). The function \( f \) may be written as 
\[
 f = \sum_{i=1}^{p} f_i \otimes b_i, \quad \text{where} \quad f_i \in C_b(X) \text{ and } b_i \in E. 
\]
Choose a \( V \in \mathcal{W} \) such that \( V + V + \ldots + V \) (2p-terms) \( \subseteq W \) and choose \( r \geq 1 \) such that \( b_i \in rV \) \((1 \leq i \leq p)\). Let \( s = \max \{\|f_i\| : 1 \leq i \leq p\} \). Then, for each \( 1 \leq i \leq p \), there exist \( g_{m_i} \in \{g_m\} \) and \( a_{n_i} \in \{a_n\} \) such that 
\[
 \|f_i - g_{m_i}\| < \frac{1}{r(s+1)} \quad \text{and} \quad a_{n_i} - b_i \in \frac{1}{r(s+1)} V. 
\]
Let \( g = \sum_{i=1}^{p} g_{m_i} \otimes a_{n_i} \). Then \( g \in A \) and, for any \( x \in X \),
\[
g(x) - f(x) = \sum_{i=1}^{p} g_{m_i}(x)[a_{n_i} - b_i] + \sum_{i=1}^{p} g_{m_i}(x) - f(x)]b_i \\
\quad \leq \frac{1}{r} (V + \ldots + V(\text{p-terms})) + \frac{1}{r(s+1)} (V + \ldots + V(\text{p-terms})) \\
\quad \subseteq W.
\]
Thus \( g - f \in N(X, W) \), and so \( A \) is \( u \)-dense in \( C_b(X) \otimes E \). \qed

Corollary 6.1.9. [KR91] Suppose \((C_b(X, E), u)\) is metrizable and \(C_b(X) \otimes E\) is \( u \)-dense in \( C_b(X, E) \). Then the following statements are equivalent:
(a) \((C_b(X, E), t)\) is separable, where \( t \) is either \( \sigma_o \), \( \sigma \) or \( u \).
(b) \(X\) is a compact metric space and \( E\) is separable.

Theorem 6.1.10. \((C_b(X, E), u)\) is separable iff \((C_b(X, E), \sigma_o)\) is separable.

Proof. If \((C_b(X, E), u)\) is separable, then clearly \((C_b(X, E), \sigma_o)\) is separable. Suppose that \((C_b(X, E), \sigma_o)\) is separable but that \((C_b(X, E), u)\) is not. Then, for any sequence \( \{f_n : n = 1, 2, \ldots\} \) in \( C_b(X, E) \), there exist an \( f \in C_b(X, E) \), \( W \in \mathcal{W} \), and a sequence \( \{x_n\} \) in \( X \) such that \( f_n(x_n) - f(x_n) \notin W \) for all \( n \). Thus, if \( A = \{x_n\} \), then \( f_n - f \notin N(\chi_{A}, W) \). This contradicts the separability of \((C_b(X, E), \sigma_o)\). \qed

Next, using Theorem 6.1.1(ii) instead of Theorem 6.1.3, we can easily obtain:

Theorem 6.1.11. Let \( X \) be locally compact. Then \((C_o(X) \otimes E, u)\) is separable iff \( X \) is a \( \sigma \)-compact metric space and \( E \) is separable.

Corollary 6.1.12. [KR91] Suppose that either \( X \) has finite covering dimension or \( E \) locally convex, or \( E \) has the approximation property. Then
(i) \((C_b(X, E), t)\) is separable iff \( X \) is compact metric space and \( E \) is separable, where \( t \) is either \( \sigma \), \( \sigma_o \) or \( u \).
(ii) If $X$ is locally compact, $(C_b(X,E), u)$ is separable iff $X$ is $\sigma$-compact metric space and $E$ is separable.

Proof. Each of the above restrictions on $X$ or $E$ implies that, in case (i), $C_b(X) \otimes E$ is $u$-dense in $C_b(X, E)$ and that, in case (ii), $C_o(X) \otimes E$ is $u$-dense in $C_b(X, E)$. The result now follows by applying Theorems 6.1.8 and 6.1.11. \hfill \Box

Finally we give some examples concerning the above results.

Examples 6.1.13. For $0 < p < 1$, let $\ell_p = \ell_p(\mathbb{N})$ denote the usual space of scalar sequences, and let $h_p = h_p(D)$ denote the Hardy space of certain harmonic functions on the unit disc $D$ of the complex plane (see [Shap85]). $\ell_p$ and $h_p$ are not locally convex but are locally bounded (hence metrizable) and their duals separate points; $\ell_p$ is separable while $h_p$ is not since it contains a copy of $\ell_\infty$ ([Shap85], Theorem 3.5). Let $\omega$ be the first uncountable ordinal and $\alpha < \omega$ a limit ordinal. Let $Y = [0, \omega]$, $Y_o = [0, \omega)$, $Z = [0, \alpha]$, and $Z_o = [0, \alpha)$, each endowed with the order topology ([Will70], [StSe78]). Then

1. $(C_b(X) \otimes \ell_p)$ is (a) $\beta$-dense in $C_b(X, \ell_p)$ (b) $k$-dense in $C(X, \ell_p)$, and (c) $u$-dense (hence also $\sigma$, $\sigma$-$\sigma$-dense) in $C_{rc}(X, \ell_p)$ for any completely regular Hausdorff space $X$ since $\ell_p$ is admissible (see Chapter 4).

2. $(C_b(Z, \ell_p), u)$ is separable since $Z$ is compact and metrizable, but $(C_b(Y, \ell_p), u)$ and $(C_b(Y_o, \ell_p), u)$ are not separable since $Y$ and $Y_o$ are not metrizable.

3. $(C_o(Z_o, \ell_p), u)$ is separable since $Z_o$ is $\sigma$-compact and metrizable; $(C_o(Y_o, \ell_p), u)$ is not separable since $Y_o$ is not $\sigma$-compact and also not metrizable.

4. $(C_b(Z_o, E), \beta)$ and $(C(Z_o, E), k)$ are separable for $E = \ell_p$ but not for $E = h_p$, since $Z_o$ is separable and metrizable, $\ell_p$ is separable, and $h_p$ is not separable.
2. Trans-separability of Function Spaces

In this section we present some results on trans-separability, as given in [Kh99, Kh04, Kh08b].

Recall that a locally convex TVS $L$ is called seminorm-separable if, for each continuous seminorm $p$ on $L$, $(L, p)$ is separable. The following two classical results are stated for reference purpose.

**Theorem 6.2.1.** [GuSc72] The following statements are equivalent:
(a) $(C_b(X), \beta)$ is seminorm-separable.
(b) $(C(X), \kappa)$ is seminorm-separable.
(c) Every compact subset of $X$ is metrizable.

**Theorem 6.2.2.** [GuSc72] The following statements are equivalent:
(a) $(C_b(X), \sigma)$ is seminorm-separable.
(b) $(C_b(X), \sigma_0)$ is seminorm-separable.
(c) The closure in $\beta X$ of each $\sigma$-compact subset of $X$ is metrizable in $\beta X$.

**Definition.** A uniform space $L$ is called trans-separable if every uniform cover of $L$ admits a countable subcover. In particular, a TVS $L$ is trans-separable if, for each neighborhood $W$ of 0 in $L$, the open cover \{${a + W : a \in L}$\} of $L$ admits a countable subcover.

For TVSs, a generalized notion of separability, namely, the neighborhood-separability may be defined, as follows.

**Definition.** Let $L$ be a TVS, and let $V$ be a neighborhood of 0 in $L$. A subset $H$ of $L$ is said to be $V$-dense in $L$ if, for any $z \in L$ and $\delta > 0$, there exists an element $y \in H$ such that $y - z \in \delta V$. $L$ is called neighborhood-separable if, for each neighborhood $V$ of 0, there exists a countable $V$-dense subset of $L$.

Another notion of generalized separability may also be considered, as follows.

**Definition.** Let $(L, \tau)$ be a TVS whose topology is generated by a family $Q(\tau)$ of continuous F-seminorms. Then $(L, \tau)$ is called F-seminorm-separable if $(L, q)$ is separable for each $q \in Q(\tau)$. Clearly, separability implies F-seminorm-separability; the converse holds in metrizable spaces.

The following result establishes the equivalence of all the above notions of generalized separability.

**Theorem 6.2.3.** [Kh04, Kh08b] For a TVS $(L, \tau)$, the following are equivalent:
(1) \((L, \tau)\) is trans-separable.
(2) \((L, \tau)\) is neighborhood-separable.
(3) \((L, \tau)\) is F-seminorm-separable.

Proof. (1) \(\Rightarrow\) (2) Suppose \(L\) is trans-separable, and let \(V\) be a neighborhood of 0. For each \(n \geq 1\), \(U_n = \{x + \frac{1}{n}V : x \in L\}\) is a uniform cover of \(L\), and so it has a countable subcover \(U^*_n = \{x^{(n)}_k + \frac{1}{n}V : k \in \mathbb{N}\}\). Let \(D = \bigcup_{n=1}^{\infty} \{x^{(n)}_k : k \in \mathbb{N}\}\). To show that \(D\) is \(V\)-dense in \(L\), let \(y \in L\) and \(\delta > 0\). Choose \(N \geq 1\) such that \(\frac{1}{N} < \delta\). Since \(U^*_N\) is a cover of \(L\), \(y \in x^{(N)}_K + \frac{1}{N}V\) for some \(K \in \mathbb{N}\). Then \(y - x^{(N)}_K \in \delta V\). Hence \(L\) is neighborhood-separable.

(2) \(\Rightarrow\) (1) Suppose \(L\) is neighborhood-separable, and let \(U\) be a neighborhood of 0. Choose a balanced neighborhood \(V\) of 0 with \(V + V \subseteq U\). Let \(\{z_n\}\) be a countable \(V\)-dense subset of \(L\). Since \(L = \bigcup_{x \in L} (x + V)\), to each \(z_n \in L\), there exists some \(x_n \in L\) such that \(z_n - x_n \in V\). Let \(y \in L\). Choose \(z_k\) such that \(y - z_k \in V\). Then
\[
y - x_k = (y - z_k) + (z_k - x_k) \in V + V \subseteq U.
\]
Hence \(L = \bigcup_{n \geq 1} (x_n + U)\), and so \(L\) is trans-separable.

(1) \(\Rightarrow\) (3) Suppose \(L\) is trans-separable, and let \(q \in Q(\tau)\). For each \(n \geq 1\), let \(V_n = \{x \in L : q(x) < 1/n\}\), a balanced neighborhood of 0 in \(L\). Then, for each \(n \geq 1\), \(U_n = \{x + V_n : x \in L\}\) is a uniform cover of \(L\), and so it has a countable subcover \(U^*_n = \{x^{(n)}_k + V_n : k \in \mathbb{N}\}\). Let \(D = \bigcup_{n=1}^{\infty} \{x^{(n)}_k : k \in \mathbb{N}\}\). To show that \(D\) is dense in \((L, \tau)\), let \(y \in L\) and \(\varepsilon > 0\). Choose \(N \geq 1\) such that \(1/N < \varepsilon\). Since \(U^*_N\) is a cover of \(L\), \(y \in x^{(N)}_K + V_N\) for some \(K \in \mathbb{N}\). Then \(q(y - x^{(N)}_K) < 1/N < \varepsilon\). Hence \((L, \tau)\) is separable.

(3) \(\Rightarrow\) (1) Suppose \((L, q)\) is separable for each \(q \in Q(\tau)\). Let \(\{x + U : x \in L\}\) be any uniform cover of \(L\), where \(U\) is neighborhood of 0 in \(L\). Choose a balanced neighborhood \(V\) of 0 in \(L\) with \(V + V \subseteq U\). Choose \(q \in Q(\tau)\) such that \(W = \{x \in L : q(x) < 1\} \subseteq V\). Let \(\{z_n\}\) be a countable dense subset in \((L, q)\). Since \(L = \bigcup_{x \in L} (x + W)\), to each \(z_n \in L\), there exists some \(x_n \in L\) such that \(z_n - x_n \in W\). Let \(y \in L\). Choose \(z_k\) such that \(q(y - z_k) < 1\). Then
\[
y - x_k = (y - z_k) + (z_k - x_k) \in W + W \subseteq U,
\]
and so \(L = \bigcup_{n \geq 1} (x_n + U)\). \(\square\)

**Theorem 6.2.4.** [K108b] Let \(E\) be any non-trivial TVS. Then the following statements are equivalent:

(a) \((C_b(X) \otimes E, \beta)\) is trans-separable.
(b) \((C(X) \otimes E, k)\) is trans-separable.

(c) Every compact subset of \(X\) is metrizable and \(E\) is trans-separable.

Proof. (a) \(\Rightarrow\) (b) This follows from the fact that \(k \leq \beta\) on \(C_0(X) \otimes E\) and that \(C_0(X) \otimes E\) is \(k\)-dense in \(C(X) \otimes E\).

(b) \(\Rightarrow\) (c) This follows from Theorem 6.2.1 and the fact that both \((C(X), k)\) and \(E\) are isomorphic to subspaces of \((C(X) \otimes E, k)\) via the maps \(g \to g \otimes a\) (\(0 \neq a \in E\) fixed) and \(a \to 1_X \otimes a\), respectively.

(c) \(\Rightarrow\) (a) By Theorem 6.2.1, \((C_0(X), \beta)\) is trans-separable. Fix a \(\varphi \in B_\alpha(X), 0 \leq \varphi \leq 1\) and a balanced \(W \in \mathcal{W}\). We need to show that there is a countable set \(H \subseteq C_0(X) \otimes E\) such that \(C_0(X) \otimes E = H + N(\varphi, W)\).

For every pair \(m, n \in \mathbb{N}\) choose a balanced \(U_{m,n} \in \mathcal{W}\) so that, denoting \(V_{m,n} = U_{m,n} + mU_{m,n} + U_{m,n}\), one has

\[
V_{m,n} + \cdots + V_{m,n} \text{ (n-summands)} \subseteq W.
\]

Also, choose a countable set \(D_{m,n}\) in \(E\) so that \(E = D_{m,n} + U_{m,n}\). Let \(D\) be the union of all these sets \(D_{m,n}\) \((m, n \in \mathbb{N})\).

Next, for each \(k \in \mathbb{N}\) denote \(B_k = \{f \in C_0(X) : \|f\|_\varphi \leq 1/k\}\) and choose a countable set \(G_k\) in \(C_0(X)\) so that \(C_0(X) = G_k + B_k\). Let \(G\) be the union of all these sets \(G_k\) \((k \in \mathbb{N})\).

We are going to show that the countable set \(H = H_{\varphi, W}\) of all functions in \(C_0(X) \otimes E\) of the form \(h = \sum_{i=1}^r g_i \otimes d_i\), where \(g_i \in G\) and \(d_i \in D\) \((i = 1, \ldots, r, r \in \mathbb{N})\), is as required.

Take any \(f \in C_0(X) \otimes E\). Then \(f = \sum_{i=1}^n f_i \otimes a_i\) for some \(f_1, \ldots, f_n \in C_0(X)\) and \(a_1, \ldots, a_n \in E\). Let \(m \in \mathbb{N}\) be such that \(\|f_i\|_\varphi \leq m\) for \(i = 1, \ldots, n\), and next choose \(k \in \mathbb{N}\) so that \(k^{-1}a_i \in U_{m,n}\) for \(i = 1, \ldots, n\).

By the definitions of \(D_{m,n}\) and \(G_k\), there are \(d_1, \ldots, d_n \in D_{m,n}\) and \(g_1, \ldots, g_k \in G_k\) such that

\[
a_i - d_i \in U_{m,n} \quad \text{and} \quad \|f_i - g_i\|_\varphi \leq 1/k \quad \text{for} \quad i = 1, \ldots, n.
\]

Now, for \(i = 1, \ldots, n\) and \(x \in A\),

\[
\varphi(x)[f_i(x)a_i - g_i(x)d_i] = \varphi(x)[f_i(x) - g_i(x)]a_i + \varphi(x)f_i(x)(a_i - d_i) + \varphi(x)[g_i(x) - f_i(x)](a_i - d_i);
\]

hence (using the fact that \(U_{m,n}\) is balanced)

\[
\varphi(x)[f_i(x)a_i - g_i(x)d_i] \in U_{m,n} + mU_{m,n} + U_{m,n} = V_{m,n}.
\]

In consequence, setting \(h = \sum_{i=1}^n g_i \otimes d_i\) we have \(h \in H\) and for every \(x \in A\),

\[
\varphi(x)[f(x) - h(x)] = \sum_{i=1}^n \varphi(x)[f_i(x)a_i - g_i(x)d_i] \in W
\]
so that \( f - h \in N(\varphi, W) \).

**Remark.** A somewhat more transparent variant of the above proof that (c) implies (a) can be based on Theorem 6.2.3. Thus, one has to show that for any \( \varphi \in B_o(X), 0 \leq \varphi \leq 1 \), and any continuous \( F \)-seminorm \( q \) on \( E \), the space \((C_0(X) \otimes E, p_\varphi)\) is separable, where

\[
p_\varphi(f) = \sup_{x \in X} q(\varphi(x)f(x)), \quad f \in C_0(X, E)
\]

Now, let \( G \) be a countable subset dense in \((C_b(X), \|\cdot\|_\varphi)\), and \( D \) a countable set dense in \((E, q)\). Take any \( f = \sum_{i=1}^n f_i \otimes a_i \) in \( C_b(X) \otimes E \), and choose \( m \in \mathbb{N} \) so that \( \|f_i\|_\varphi \leq m \) for each \( i \). Given \( \varepsilon > 0 \), let \( g = \sum_{i=1}^n g_i \otimes d_i \), where \( g_i \in G \) and \( d_i \in D \). Assume that \( \|f_i - g_i\|_\varphi \leq \delta \) for all \( i \) and some as yet unspecified \( 0 < \delta < 1 \). Then, making use of (*), it is easily seen that

\[
p_\varphi(f - g) \leq \sum_{i=1}^n [q(\|f_i - g_i\|_\varphi a_i) + q(\|f_i\|_\varphi (a_i - d_i)) + q(\|f_i - g_i\|_\varphi (a_i - d_i))]
\]

and this can be made smaller than \( \varepsilon \) by taking \( \delta \) sufficiently small and choosing the \( d_i \)'s in \( D \) sufficiently close to the \( a_i \)'s. It follows that the countable set of all \( g \)'s of the above form is dense in \((C_b(X) \otimes E, p_\varphi)\).

Next, we obtain:

**Theorem 6.2.5.** [Kh04] Let \( E \) be a non-trivial TVS. Then:

(a) \((C_b(X) \otimes E, u)\) is trans-separable iff \( X \) is a compact metric space and \( E \) is trans-separable.

(b) Suppose \( X \) is locally compact. Then \((C_o(X) \otimes E, u)\) is trans-separable iff \( X \) is a \( \sigma \)-compact metric space and \( E \) is trans-separable.

Proof. (a) In this case, \((C_b(X) \otimes E, u)\) is trans-separable iff it is separable iff \( X \) is a compact metric space. The proof now follows just as in above theorem.

(b) If \( X \) is locally compact, then \((C_o(X), u)\) is trans-separable iff it is separable iff \( X \) is a \( \sigma \)-compact metric space. Again the proof follows just as in above theorem.

Again we remark that, if \( C_b(X) \otimes E \) (resp. \( C_o(X) \otimes E \)) is \( u \)-dense in \( C_b(X, E) \) (resp. \( C_o(X, E) \)), the above theorem remains valid with \( C_b(X) \otimes E \) (resp. \( C_o(X) \otimes E \)) replaced by \( C_b(X, E) \) (resp. \( C_o(X, E) \)).
Remark. If $X$ has finite covering dimension or $E$ is locally convex, or $E$ has the approximation property or $E$ is complete metrizable with a basis, then $C_0(X) \otimes E$ is $\beta$-dense in $C_0(X,E)$ and that $C(X) \otimes E$ is $k$-dense in $C(X,E)$ (see Chapter 4). Hence, under these assumptions, the above two theorems hold with $C_0(X) \otimes E$, $C(X) \otimes E$ and $C_0(X) \otimes E$ replaced by $C_0(X), C(X,E)$ and $C_0(X,E)$, respectively. It is not known whether or not these ‘density’ results hold for a locally bounded space. However, we can prove

Theorem 6.2.6. [Kh08b] Let $X$ be any Hausdorff space and $E$ any locally bounded space. Then $(C_0(X,E), \beta)$ is trans-separable iff $(C(X,E),k)$ is so.

Proof Suppose $(C(X,E),k)$ is trans-separable. Let $\varphi \in B_o(X)$ with $0 \leq \varphi \leq 1$, and let $W \in \mathcal{W}$. Let $V$ be a balanced bounded neighborhood of 0 in $E$, and let $S$ be a closed shrinkable neighborhood of 0 with $S \subseteq V$. The Minkowski functional $\rho = \rho_S$ of $S$ is continuous and positive homogeneous and, consequently, for each $r > 0$, the function $h_r: E \to E$ defined by

$$h_r(a) = \begin{cases} \frac{a}{\rho(a)} & \text{if } a \in rS \\ a & \text{if } a \in E \setminus rS \end{cases}$$

is continuous. Further, $h_r(E) \subseteq rS \subseteq rV$, which shows that, for each $f \in C(X,E)$, the function $h_r \circ f \in C_0(X,E)$. Choose $t \geq 1$ such that $V + V \subseteq tS$ and $V + V \subseteq tW$. For each $m = 1, 2, \ldots$, there exists a compact set $K_m \subseteq X$ such that $\varphi(x) < 1/tm^2$ for $x \in X \setminus K_m$. Corresponding to each $K_m$, choose $\{f_{mn} : n = 1, 2, \ldots\}$ as a $N(K_m,V)$-dense of $C(X,E)$. We show that $\{h_m \circ f_{mn} : m, n = 1, 2, \ldots\}$ is $\beta$-dense in $C_0(X,E)$. Let $f \in C_0(X,E)$ and $0 \leq \delta \leq 1$. Choose integers $M \geq 1/\delta$ and $N \geq 1$ such that $f(X) \subseteq (M\delta/t)V$ and $(f_{MN} - f)(K_M) \subseteq (\delta/t)V$. Let $y \in X$. If

$$f_{MN}(y) \in f(y) + (\delta/t)V \subseteq (M\delta/t)V + (M\delta/t)V \subseteq MS,$$

and so

$$\varphi(y)[h_M \circ f_{MN}(y) - f(y)] = \varphi(y)[f_{MN}(y) - f(y)] \in \varphi(y)(\delta/t)V \subseteq \delta W.$$ 

If $y \in X \setminus K_M$, then, since $h_M(f_{MN}(y)) \in h_M(E) \subseteq MS$,

$$\varphi(y)[h_M \circ f_{MN}(y) - f(y)] \in \varphi(y)[MS - \frac{M\delta}{t}V] \subseteq \frac{M}{tM^2}[V + \frac{\delta}{t}V] \subseteq \frac{\delta}{t}[V + V] \subseteq \delta W.$$ 

Thus $h_M \circ f_{MN} - f \in \delta N(\varphi,W)$. 

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The converse follows from the fact that $C_b(X, E)$ is dense in $(C(X, E), k)$. Indeed, let $f \in C(X, E)$, $K$ a compact subset of $X$ and $W \in \mathcal{W}$. Let $S$ and $V$ be as above with $S \subseteq V$. Choose $r \geq 1$ with $f(K) \subseteq rS$. Then, as in the above proof, we get a function $h_r \circ f \in C_b(X, E)$ such that

$$h_r \circ f(x) - f(x) = f(x) - f(x) = 0 \in W$$

for all $x \in K$. □

Similar characterizations of trans-separability for the $\sigma$ and $\sigma_o$ topologies are stated without proof since they can be obtained by applying Theorem 6.2.2 in place of Theorem 6.2.1, with slight modifications.

Theorem 6.2.7. [Kh04] Let $E$ be any non-trivial TVS. Then the following statements are equivalent:

(a) $(C_b(X) \otimes E, \sigma)$ is trans-separable.
(b) $(C_b(X) \otimes E, \sigma_o)$ is trans-separable.
(c) The closure in $\beta X$ of each $\sigma$-compact subset of $X$ is metrizable in $\beta X$ and $E$ is trans-separable.

Note If $C_b(X) \otimes E$ is $u$-dense in $C_{rc}(X, E)$, then clearly Theorem 7.2.7 holds with $C_b(X) \otimes E$ replaced by $C_{rc}(X, E)$. We do not know of any general assumption on $X$ other than its compactness ensuring that $C_b(X) \otimes E$ is $\sigma$- or $\sigma_o$-dense in $C_b(X, E)$. However, without this density assumption, the following holds.

Theorem 6.2.8. [Kh04] $(C_b(X, E), \sigma)$ is trans-separable iff $(C_b(X, E), \sigma_o)$ is so.

Proof. $(\Rightarrow)$ This follows from the fact that $\sigma_o \leq \sigma$.

$(\Leftarrow)$ Suppose $(C_b(X, E), \sigma)$ is not separable. We show that $(C_b(X, E), \sigma_o)$ is also not trans-separable, i.e., if $H = \{f_n\}$ is any countable set in $C_b(X, E)$, then there exist a countable set $B = \{x_n\} \subseteq X$ and a $W \in \mathcal{W}$ such that $C_b(X, E) \neq H + N(B, W)$. Since $(C_b(X, E), \sigma)$ is not separable, there exists a $\sigma$-compact set $A \subseteq X$ and a $W \in \mathcal{W}$ such that $C_b(X, E) \neq H + N(B, W)$. Then there exists an $f \in C_b(X, E)$ such that $f - f_n \notin N(A, W)$ for all $n \geq 1$. So, for each $n \geq 1$, there exists an $x_n \in A$ such that

$$f(x_n) - f_n(x_n) \notin W.$$ 

Then, if $B = \{x_n\}$, $f - f_n \notin N(B, W)$ for all $n \geq 1$. Hence $C_b(X, E) \neq H + N(B, W)$, and so $(C_b(X, E), \sigma_o)$ is not trans-separable, as desired.

□

Next, using Theorem 6.1.1 and the method of proof of Theorem 6.2.5, we obtain:

Theorem 6.2.9. [Kh08b] Let $E$ be a TVS. Then:
(a) \((C_b(X) \otimes E, u)\) is trans-separable iff \(X\) is a compact metric space and \(E\) is trans-separable.

(b) Suppose \(X\) is locally compact. Then \((C_o(X) \otimes E, u)\) is trans-separable iff \(X\) is a \(\sigma\)-compact metric space and \(E\) is trans-separable.

Again we remark that, if \(C_b(X) \otimes E\) (resp. \(C_o(X) \otimes E\)) is \(u\)-dense in \(C_b(X, E)\) (resp. \(C_o(X, E)\)), Theorem 6.2.9 remains valid with \(C_b(X) \otimes E\) (resp. \(C_o(X) \otimes E\)) replaced by \(C_b(X, E)\) (resp. \(C_o(X, E)\)).

Finally, we give some examples concerning the above results and remarks.

**Examples 6.2.10.** Using the notations of Example 6.1.13, we have

1. \((C_b(Y_o, E), \beta)\) and \((C(Y_o, E), k)\) are trans-separable for \(E = \ell_p\) but not for \(E = h_p\). Here every compact subset of \(Y_o\) is metrizable although \(Y_o\) is not.

2. \((C_b(Z, \ell_p), u)\) and \((C_o(Z_o, \ell_p), u)\) are (trans-) separable (since \(Z\) is compact and metrizable and \(Z_o\) is \(\sigma\)-compact and metrizable) but \((C_b(Y, \ell_p), u)\) and \((C_o(Y_o, \ell_p), u)\) are not (since \(Y\) is compact but not metrizable and \(Y_o\) is not \(\sigma\)-compact and also not metrizable).

3. \((C_b(Y) \otimes \ell_p, u)\) and \((C_b(Z) \otimes h_p, u)\) are not trans-separable since \(Y\) is not metrizable and \(h_p\) is not trans-separable.

4. If \(\omega_o\) the first countable infinite ordinal and

\[ T = [0, \omega] \times [0, \omega_o] \setminus \{ (\omega, \omega_o) \}, \]

the deleted Tychonoff plank, with the product topology [StSe78; Will70], then \((C_b(T) \otimes \ell_p, \sigma)\) is not trans-separable since, if \(A = \bigcup_{n=1}^{\infty} [0, \omega] \times \{ n \}\), \(A\) is \(\sigma\)-compact and \(\bar{A^\beta T} = \beta T = [0, \omega] \times [0, \omega_o]\) which is not metrizable: in fact the point \((\omega, \omega_o)\) is not even a \(G_\delta\) in \(T\) (see [GuSc72], p. 259).
Section 6.1. The fundamental result on the characterization of separability of \((C_b(X), u)\) was obtained by M. Krein and S. Krein [KK40] in 1940. Later, similar results were obtained by Warner [War58] for \((C(X), k)\) and by Gulick and Schmets [GuSc72] and, independently, by Summers [SumW72] for \((C_b(X), \beta)\). The notions of submetrizable and separably submetrizable spaces are given in [War58, SumW72]. In the case of vector-valued functions, Todd [Tod65] and Choo [Cho79] have considered the separability of \((C_b(X, E), \beta)\) for \(E\) a locally convex TVS and by Katsaras [Kat83], Khan [Kh85, Kh86] and Khan-Rowlands [KR91] for \(E\) a TVS.

Section 6.2. As stated in [RobN91], Trans-separability appears under many different names in the literature. While workers in the field of uniform spaces tend to speak simply of separable space (see [Hag74, Isa64]), those who study locally convex spaces favour the phrase "seminorm separable". Pfister [Pf76] investigated TVSs which were of "countable type", whereas Drewnowski [Dre75] coined the word "trans-separable" while working with topological abelian groups and this also seems to be suitable in the case of TVSs. This notion has proved to be useful in the work [Pf76, CO87, RobN91] while studying the metrizability of precompact sets in locally convex spaces.

Characterizations of seminorm separability for \((C_b(X), u), (C(X), k)\) and \((C_b(X), \beta)\) were first given by Gulick and Schmets [GuSc72]. This section contains results from [Kh99, Kh04, Kh08b] which generalize the results of [GuSc72] to vector-valued functions by considering the notion of neighborhood separability, or equivalently, trans-separability in the non-locally convex setting.
CHAPTER 7

Weak Approximation in Function Spaces

In this chapter we are mainly concerned with the weak approximation type results for $C_b(X,E)$ and $CV_o(X,E)$. Throughout this chapter we shall use the notations of Chapters 1 and 4 concerning the strict topology $\beta$ and the weighted topology $\omega_V$. As seen in Chapter 4, $C_b(X) \otimes E$ is $\beta$-dense in $C_b(X,E)$ and $CV_o(X) \otimes E$ is $\omega_V$-dense in $CV_o(X,E)$ for a locally convex $E$, and also for some concrete classes of not necessarily locally convex spaces $E$ (such as admissible spaces and spaces having the approximation property); the general case of density problem remains open.

The main results in this section include a “convexified” version of the approximation problem. It states that, if $X$ is locally compact, then $CV_o(X) \otimes E$ is always weakly dense in $CV_o(X,E)$. This implies that $CV_o(X) \otimes E$ is dense in $CV_o(X,E)$ equipped with the locally convex topology $\omega_V$ associated to $\omega_V$ (i.e., the strongest locally convex topology on $CV_o(X,E)$ which is weaker than $\omega_V$), where $E^c$ is the locally convex TVS associated to $E$. These results are used later in Chapter 8 to obtain a representation of $\beta$- and $\omega_V$-continuous linear functionals on $C_b(X,E)$ and $CV_o(X,E)$, respectively.
1. Weak Approximation in \( (C_b(X, E), \beta) \)

In this section, we shall present some results regarding the weak approximation in the space \( (C_b(X, E), \beta) \), as given in [Naw85]. We shall use here the measure theoretic terminology of section A.7. Let \( k(X), z(X) \) and \( cz(X) \) denote the families of all compact, zero and cozero subsets of \( X \), respectively, let \( \mathcal{B}a(X) \) denote the \( \sigma \)-algebra of subsets of \( X \) generated by \( z(X) \). The family of all positive, tight Baire measures on \( X \) will be denoted by \( \mathcal{M}_t^+(\mathcal{B}a(X)) \). Let \( C(X, I) \) denote the subset of \( C_b(X) \) of all functions having values in \( I = [0, 1] \). If \( f \) is a function on \( X \) into \( E \) or \( \mathbb{R} \) and \( U \in cz(X) \), then \( f \prec U \) means \( \text{supp}(f) \subseteq Z \) for some \( Z \in z(X) \) with \( Z \subseteq U \). We recall that if \( A \in k(X) \) or \( A \in z(X) \) and \( B \in cz(X) \) with \( A \subseteq B \), then:

(a) there are sets \( F \in k(X) \), \( Z \in z(X) \) such that \( A \subseteq F \subseteq Z \subseteq B \);  
(b) there is a function \( \theta \in C(X, I) \) such that \( \theta \prec B \) and \( \theta(A) = 1 \).

If \( E \) is a TVS, then its topology may be generated by some family of \( F \)-seminorms (§ A.2). This implies that if \( W \) is a base of balanced neighborhoods of 0 in \( E \), then \( W \cap \mathcal{B}a(E) \) is also a base of 0 consisting of balanced and zero or cozero sets.

**Definition.** We define the \( m_t \)-topology on \( C_b(X, E) \) as the linear topology which has a base at 0 the family of all sets of the form

\[
G = G(\mu, W, \varepsilon) = \{ f \in C_b(X, E) : \mu(\{ x : f(x) \notin W \}) \leq \varepsilon \}, \quad (*)
\]

where \( \mu \in \mathcal{M}_t^+(\mathcal{B}a(X)), W \in W \cap \mathcal{B}a(E) \), and \( \varepsilon > 0 \). Let \( \mathcal{J} \) be the family of all linear topologies \( \tau \) on \( C_b(X, E) \) satisfying \( \tau|_H \leq m_t|_H \) for any \( u \)-bounded subset \( H \) of \( C_b(X, E) \). We define the \( \gamma \)-topology as \( \sup \mathcal{J} \).

**Lemma 7.1.1.** [Naw85] If \( T \) is a linear functional on \( C_b(X, E) \), then the following statements are equivalent:

(a) \( T \in (C_b(X, E), \gamma)^* \).

(b) \( T(f_n) \to 0 \) for every net \( \{ f_n \} \subseteq C_b(X, E) \) which is \( u \)-bounded and \( m_t \)-convergent to 0.

**Proof.** The implication (a) \( \Rightarrow \) (b) is obvious. (b) implies that the weak topology \( \sigma(T) \) induced on \( C_b(X, E) \) by \( T \) belongs to \( \mathcal{J} \). Thus \( \sigma(T) \leq \gamma \), and so \( T \) is \( \gamma \)-continuous. \( \square \)

**Lemma 7.1.2.** [Naw85] The space \( C_b(X) \otimes E \) is \( \gamma \)-dense in \( C_b(X, E) \).

**Proof.** Fix \( f \in C_b(X, E) \). Let \( S_f \subseteq C_b(X) \otimes E \) be the family of all functions of the form \( \sum_{i=1}^n \theta_i \otimes f(x_i) \), where \( \theta_i \in C(X, I), x_i \in X, \theta_i(x_i) = 1 \) and \( \text{supp}(\theta_i) \cap \text{supp}(\theta_j) = \emptyset \) if \( i \neq j \). It is easy to see that \( S_f \) is \( u \)-bounded. Let \( G \) be a \( m_t \)-neighborhood of 0 of the form \( G(\mu, W, \varepsilon) \) given by (*)). We can assume that \( W \) is balanced.
Choose a balanced $W_1 \in \mathcal{W}$ such that $W_1 + W_1 \subseteq W$ and $W_1 \in cz(E)$. By the tightness of $\mu$, we can find $K \in k(X)$ such that

$$\mu(B) \leq \varepsilon/2 \text{ for every } B \in \mathcal{B}(X), B \cap K = \emptyset.$$ 

The set $f(K)$ is compact, so $f(K) \subseteq F + W_1$ for some finite subset $F$ of $E$. It follows that there are sets $B_1, \ldots, B_p \in \mathcal{B}(X)$ such that $K \subseteq \bigcup_{i=1}^p B_i$, $B_i \cap B_j = \emptyset$ if $i \neq j$ and

$$f(x) - f(y) \in W \text{ for any } x, y \in B_i, i = 1, \ldots, p.$$ 

Let $Z_i \in z(X), U_i \in cz(X)$ be such that

$$Z_i \subseteq B_i \subseteq U_i \text{ and } \mu(U_i \setminus Z_i) \leq \varepsilon/(2p).$$ 

We can find sets $O_i \in cz(X)$ such that $Z_i \subseteq O_i \subseteq U_i$ and $O_i \cap O_j = \emptyset$ if $i \neq j, i = 1, \ldots, p$. There exist functions $\theta_i \in C(X,I)$ such that

$$\theta_i(Z_i) = 1, \supp(\theta_i) = O_i, i = 1, \ldots, p.$$ 

Choose $x_i \in Z_i, i = 1, \ldots, p$. Then the function $h = \sum_{i=1}^p \theta_i \otimes f(x_i)$ belongs to $S_f$ and

$$\mu(\{x : (f - h)(x) \notin W\}) \leq \sum_{i=1}^p \mu(U_i \setminus Z_i) + \mu(X \setminus \bigcup_{i=1}^p U_i) \leq \frac{p\varepsilon}{2p} + \varepsilon/2 = \varepsilon.$$ 

Therefore $f - g \in G(\mu, W, \varepsilon)$, so $C_b(X) \otimes E$ is $\gamma$-dense in $C_b(X,E)$. \hfill \Box

**Theorem 7.1.3.** [Naw85] *Every $\beta$-continuous linear functional on $C_b(X,E)$ is $\gamma$-continuous.*

**Proof.** Let $T \in (C_b(X,E), \beta)^*$. There exist $\varphi \in B_c(X)$ and $W_1 \in \mathcal{W}$ such that

$$|T(f)| \leq 1 \text{ for all } f \in N(\varphi, W_1). \quad (1)$$

We may assume that $W_1$ is balanced and belongs to $cz(E)$. Since $\beta \leq u$, we can find a balanced $W_2 \in \mathcal{W}$ such that

$$N(1, W_2) \subseteq N(\varphi, W_1), \quad W_2 \subseteq W_1, W_2 \in z(E). \quad (2)$$

We observe that:

- For every $\varepsilon > 0$, there is a $K_\varepsilon \in k(X)$ such that

$$|T(f)| \leq \varepsilon \text{ for all } f \in N(1, W_2) \text{ with } f(K_\varepsilon) = \{0\}. \quad (3)$$

[Indeed, let $K_\varepsilon$ be a compact subset of $X$ such that $\{x : |\varphi(x)| \geq \varepsilon\} \subseteq K_\varepsilon$. For any $f \in N(1, W_2), f(X) \subseteq W_1$ by (2). If, moreover, $f(K_\varepsilon) = \{0\}$, then $\varphi f(X) \subseteq \varepsilon W_1$. Therefore by (1), $|T(f)| \leq \varepsilon$.]
We define $F : cz(X) \to \mathbb{R}$ by

$$F(U) = \sup\{|T(f)| : f \in N(1, W_2), f \prec U\}$$

for any $U \in cz(X)$. Obviously $F$ is positive, finite and if $U_1, U_2 \in cz(X), U_1 \subseteq U_2$, then $F(U_1) \leq F(U_2)$. We will show that

$$F(U_1 \cup U_2) \leq F(U_1) + F(U_2) \text{ for any } U_1, U_2 \in cz(X). \tag{4}$$

Let $U_1, U_2 \in cz(X), f \in N(1, W_2)$ and $f \prec U_1 \cup U_2$. There is a $Z \in z(X)$ such that $supp(f) \subseteq Z \subseteq U_1 \cup U_2$. Fix $\varepsilon > 0$. Let $K_\varepsilon$ be as in (3). Put $K = Z \cap K_\varepsilon, H_1 = K \cap U_1, H_2 = K \setminus H_1$. For any $x \in H_1$ there are sets $U_x \in cz(X), Z_x \in z(X)$ such that

$$x \in U_x \subseteq Z_x \subseteq U_i, i = 1, 2.$$ 

The family $\{U_x : x \in K\}$ is an open cover of $K$. By the compactness of $K$ we can find a finite subset $S$ of $K$ such that $K \subseteq \cup\{U_x : x \in S\}$. Let $Z'_i = \cup\{Z_x : x \in S \cap H_i\}, i = 1, 2$. Then $Z'_i \in z(X), Z'_i \subseteq U_i, i = 1, 2$. There are sets $Z_i \in z(X)$ and $O_i \in cz(X)$ such that $Z'_i \subseteq O_i \subseteq Z_i \subseteq U_i, i = 1, 2$. Let $\varphi_0, \varphi_1, \varphi_2 \in C(X, I)$ be functions with supports $U_1 \cup U_2 \setminus (Z'_1 \cup Z'_2), O_1, O_2$ respectively. We define functions

$$\varphi_i = \varphi_i(\varphi_0 + \varphi_1 + \varphi_2)^{-1}, i = 0, 1, 2.$$ 

Let $f_i = \varphi_i f, i = 0, 1, 2$. Then

$$f_1 \prec U_1, f_2 \prec U_2 \text{ and } f_o(K_\varepsilon) = \{0\}.$$ 

Moreover, $f_i \in N(1, W_2), i = 1, 2$, and $f = f_o + f_1 + f_2$, so that

$$|T(f)| \leq |T(f_o)| + |T(f_1)| + |T(f_2)| \leq \varepsilon + F(U_1) + F(U_2).$$ 

This implies that $F(U_1 \cap U_2) \leq F(U_1) + F(U_2)$.]

Suppose additionally that $U_1 \cap U_2 = \emptyset$. Fix $\varepsilon > 0$. Let $f_1, f_2 \in N(1, W_2)$ be such that $Tf_i \geq F(U_i) + F(U_2) - \varepsilon$. Thus

$$F(U_1 \cup U_2) = F(U_1) + F(U_2) \text{ for any } U_1, U_2 \in cz(X), U_1 \cap U_2 = \emptyset. \tag{5}$$

From (3) it immediately follows that, for every $\varepsilon > 0$, there exists a $K_\varepsilon \in k(X)$ such that

$$F(U) \leq \varepsilon i f U \in cz(X) \text{ and } U \cap K_\varepsilon = \emptyset. \tag{6}$$

Moreover, for every $\varepsilon > 0$ and $U \in cz(X)$, there is a $Z \in z(X), Z \subseteq U$, such that

$$F(U \setminus Z) \leq \varepsilon. \tag{7}$$
[Indeed, if this statement fails to be true for some \( \varepsilon > 0 \), then by induction we can find a sequence \( \{f_n\} \subseteq N(1, W_2) \) such that
\[
supp(f_i) \cap supp(f_j) = \emptyset \text{ for } i \neq j \text{ and } T(f_i) > \varepsilon, i, j = 1, 2, \ldots
\]
But \( g_n = f_1 + \ldots + f_n \) belongs to \( N(1, W_2) \) for every \( n \in \mathbb{N} \) and \( T(g_n) > n\varepsilon \). This contradicts (1).

We define \( \mu(B) = \inf\{F(U) : U \in cz(X), U \supseteq B\} \) for \( B \subseteq X \). It is easy to see that the family \( B \) of all subsets \( B \) of \( X \) such that for any given \( \varepsilon > 0 \) there are \( Z \in z(X), U \in cz(X), Z \subseteq B \subseteq U \) satisfying \( \mu(U \setminus Z) \leq \varepsilon \) is an algebra. By (7), \( cz(X) \subseteq B \) and so \( Ba(X) \subseteq B \). The function \( F \) restricted to \( Ba(X) \) is a positive Baire measure on \( X \). From (6), it immediately follows that \( \mu \) is tight.

We will now show that \( T \) is \( \gamma \)-continuous. [Let \( \{f_\alpha\}_{\alpha \in A} \) be a \( u \)-bounded net in \( C_b(X, E) \) which is \( m_1 \)-convergent to 0. There is a \( \delta > 1 \) such that \( \{f_\alpha\} \subseteq \delta N(1, W_2) \). Fix \( \varepsilon > 0 \). Let
\[
Z_\alpha = \{x \in X : f_\alpha(x) \notin \varepsilon W_1\},
U_\alpha = \{x \in X : f_\alpha(x) \notin \varepsilon W_2\}.
\]
Then \( Z_\alpha \in z(X), U_\alpha \in cz(X) \) and \( Z_\alpha \subseteq U_\alpha \). We can find functions \( \theta_\alpha \in C(X, I) \) such that
\[
\theta_\alpha \prec U_\alpha \text{ and } \theta_\alpha(Z_\alpha) = 1, \alpha \in A.
\]
Let \( h_\alpha = \theta_\alpha f_\alpha \) and \( k_\alpha = (1 - \theta_\alpha)f_\alpha \). Then \( k_\alpha \prec U_\alpha \) and \( \delta^{-1}h_\alpha \in N(1, W_2) \), so that \( |T(h_\alpha)| \leq \delta \mu(U_\alpha) \). Moreover, \( k_\alpha(X) \subseteq \varepsilon W_1 \), and so \( k_\alpha \in s\varepsilon N(1, W_1) \) where \( s = \sup \{|\varphi(x)| : x \in X\} \). Thus
\[
|T(f_\alpha)| \leq |T(h_\alpha)| + |T(k_\alpha)| \leq \delta \mu(U_\alpha) + s\varepsilon.
\]
This implies that \( \lim_\alpha T(f_\alpha) = 0 \) and so, by Lemma 7.1.2, \( T \) is \( \gamma \)-continuous.]

**Theorem 7.1.4.** The space \( C_b(X) \otimes E \) is \( \sigma(Y, Y^*) \)-dense in \( C_b(X, E) \), where \( Y = (C_b(X, E), \beta) \).

**Proof.** By Theorem 7.1.3, \( \sigma(Y, Y^*) \leq \gamma \). Now, the statement immediately follows from Lemma 7.1.2. \( \square \)
2. Weak Approximation in $CV_o(X, E)$

We now present a “convexified” version of weighted approximation, as given in [Naw89].

**Notation.** Let $SM(X)$ be the family of all set functions $\mu : cz(X) \rightarrow [0, \infty]$, usually called the submeasures, satisfying:

(S1) $\mu(\emptyset) = 0$,
(S2) $\mu(U_1) \leq \mu(U_2)$ if $U_1 \subseteq U_2$,
(S3) $\mu(U_1 \cup U_2) \leq \mu(U_1) + \mu(U_2)$,
(S4) for every $U \in cz(X)$ and an $\varepsilon > 0$ there is a $Z \in z(X)$ such that $Z \subseteq U$ and $\mu(U \setminus Z) \leq \varepsilon$.

It easily follows from (S1) - (S4) that the family of all subsets $B$ of $X$ satisfying

(**) “for every $\varepsilon > 0$ there are $U \in cz(X)$ and $Z \in z(X)$ such that $Z \subseteq B \subseteq U$ and $\mu(U \setminus Z) \leq \varepsilon”$

is an algebra containing $z(X)$. Therefore, (***) holds for any set $B$ belonging to $Ba(X)$.

The family $SM(X)$ appears naturally while studying continuous linear functionals on $CV_o(X, E)$. Indeed, the following result holds.

**Lemma 7.2.1.** [Naw89] Let $N(v, W)$ be an $\omega_v$-neighborhood of 0 in $CV_o(X, E)$, where $W$ is balanced. If $T$ is a linear functional on $CV_o(X, E)$ which is bounded on $N(v, W)$, then the function $\mu$ defined on $cz(X)$ by

$$
\mu(U) = \sup\{|T(f)| : f \in N(v, W), f \prec U\}
$$

belongs to $SM(X)$.

**Proof.** Properties (S1) and (S2) are obvious. We shall show that $\mu$ satisfies (S3). Fix $U_1, U_2 \in cz(X)$, $\varepsilon > 0$ and $f \in N(v, W)$ such that $supp(f) \subseteq Z \subseteq U_1 \cup U_2$ for some $Z \in z(X)$. $vf$ vanishes at infinity so there is a compact subset $K_\varepsilon$ of $X$ such that $vf(X \setminus K_\varepsilon) \subseteq \varepsilon W$. Let $K = Z \cap K_\varepsilon$. $U_i$ is a cozero set, so we can find a function $\theta_i \in C(X, I)$ such that $supp(\theta_i) = U_i$, $i = 1, 2$. $K$ is compact, so

$$
\inf\{\langle \theta_1 \lor \theta_2 \rangle(x) : x \in K\} = a > 0.
$$

Let

$$
\theta'_i := (\theta_i \lor \frac{1}{3}a\chi_X) - \frac{1}{3}a\chi_X, i = 1, 2, \text{ and } \theta_o := \frac{2}{3}a\chi_X - (\frac{2}{3}a\chi_X \land \theta_1 \land \theta_2).
$$

Moreover, let $\varphi_i := \theta'_i(\theta_o + \theta'_1 + \theta'_2)^{-1}$ for $i = 0, 1, 2$. We define $f_i := \varphi_if$ for $i = 0, 1, 2$. Then $f = f_o + f_1 + f_2$, $f_i \in N(v, W)$, $f_i \prec U_i$ for $i = 1, 2$ and $f_o \in \varepsilon N(v, W)$. Therefore
2. WEAK APPROXIMATION IN $CV_o(X,E)$

$$|T(f)| \leq |T(f_0)| + |T(f_1)| + |T(f_2)| \leq \varepsilon \delta + \mu(U_1) + \mu(U_2),$$

where $\delta = \sup\{|T(f)| : f \in N(v,W)\}$. This implies (S3).

If (S4) fails to be true, then for some $\varepsilon > 0$ we can find, by induction, a sequence $\{f_n\} \subseteq N(v,W)$ such that

$$\text{supp}(f_n) \cap \text{supp}(f_m) = \emptyset \text{ for } n \neq m \text{ and } T(f_n) > \varepsilon, n, m = 1, 2, \ldots.$$  

However the functions $g_n := f_1 + \ldots + f_n$ belong to $N(v,W)$ and $T(g_n) > \varepsilon n$ for every $n \in \mathbb{N}$. This means that $T$ is unbounded on $N(v,W)$, a contradiction. □

**Notation.** If $\mu \in SM(X)$, we denote by $\mu^*$ the outer measure defined on $X$ by $\mu$; i.e.

$$\mu^*(A) = \inf \{\mu(U) : U \in cz(X), \chi_A \prec U\} \text{ for } A \subseteq X.$$  

**Definition.** Let $V$ be a fixed Nachbin family of weights on $X$. We define $m_V$-topology as the vector topology on $CV_o(X,E)$ which has as a base of neighborhoods of 0 the family of all sets of the form

$$M(\mu, v, W, \varepsilon) = \{f \in CV_o(X,E) : \mu^*(\{x \in X : vf(x) \notin W\}) \leq \varepsilon\},$$

where $\mu \in SM(X)$, $v \in V$, $W \in \mathcal{W}$ and $\varepsilon > 0$. Let $\gamma_V$ be the strongest vector topology on $CV_o(X,E)$ which is weaker than $m_V$ on all $\omega_V$-bounded subsets of $CV_o(X,E)$.

**Lemma 7.2.2.** [Naw89] Let $T$ be such a linear functional on $CV_o(X,E)$ that $\lim_{n} T(f_n) = 0$ for every net $\{f_n\} \subseteq CV_o(X,E)$ which is $\omega_V$-bounded and $\sigma_V$-convergent to 0. Then $T$ is $\gamma_V$-continuous.

**Proof.** It is easily seen that the weakest vector topology on $CV_o(X,E)$ for which $T$ is continuous is weaker than $m_V$ on every $\omega_V$-bounded set. □

**Theorem 7.2.3.** [Naw89] For every Nachbin family $V$, the topology $\gamma_V$ is stronger than the weak topology $w = w(Y,Y^*)$ of the space $Y = (CV_o(X,E), \omega_V)$.

**Proof.** Let us fix an $\omega_V$-continuous linear functional $T$ on $CV_o(X,E)$. We can find an $\omega_V$-neighborhood $N(v,W)$ of 0 such that $W$ is balanced and

$$|T(f)| \leq 1 \text{ for all } f \in N(v,W)$$

Using $T$ and $N(v,W)$, we define $\mu \in SM(X)$ as in Lemma 7.2.1; that is

$$\mu(U) = \sup\{|T(f)| : f \in N(v,W), f \prec U\}, U \in cz(X).$$
Theorem 7.2.3 will be proved if we show that \( T \) is \( \gamma_V \)-continuous.

Let \( \{ f_\alpha \} \) be a \( \omega_V \)-bounded net and \( m_V \)-convergent to 0 in \( CV_o(X, E) \) (cf. Lemma 7.2.3). Fix \( \varepsilon > 0 \). For every \( \alpha \), there are \( U_\alpha \in cz(X) \) and \( Z_\alpha \in z(X) \) such that

\[
A_\alpha := \{ x : vf(x) \notin \varepsilon W \} \subseteq U_\alpha \text{ and } \mu(U_\alpha) \leq \mu^*(A_\alpha) + \varepsilon.
\]

There is \( \theta_\alpha \in C(X, I) \) such that \( \theta_\alpha \prec U_\alpha \) and \( \theta_\alpha(Z_\alpha) = 1 \). Let \( h_\alpha := \theta_\alpha f_\alpha \) and \( k_\alpha := (1 - \theta_\alpha)f_\alpha \). The net \( \{ f_\alpha \} \) is bounded, so there is a \( \delta > 1 \) such that \( \{ f_\alpha \} \subseteq \delta N(v, W) \). Thus, \( \delta^{-1}h_\alpha \in N(v, W) \) and \( h_\alpha \prec U_\alpha \), so that

\[
|T(h_\alpha)| \leq \delta \mu(U_\alpha) \leq \delta (\mu^*(A_\alpha) + \varepsilon).
\]

Moreover, \( k_\alpha \in \varepsilon N(v, W) \), and so

\[
|T(h_\alpha)| \leq \delta (\mu^*(A_\alpha) + \varepsilon) + \varepsilon.
\]

This implies that \( \lim_\alpha T(f_\alpha) = 0 \).

**Notation.** In the sequel we will denote by \( CV_o(X) \otimes_d E \) the subset of \( CV_o(X, E) \) consisting of all functions of the form \( \sum_{i=1}^n \theta_i \otimes a_i \), where \( \theta_i \in CV_o(X) \), \( \text{supp}(\theta_i) \cap \text{supp}(\theta_j) = \emptyset \) if \( i \neq j \), \( a_i \in E \), \( i, j = 1, \ldots, n \), \( n \in \mathbb{N} \).

**Theorem 7.2.4.** [Naw89] Let \( V \) be a Nachbin family of weights on a locally compact Hausdorff space \( X \). For every balanced neighborhood \( W \) of 0 in \( E \) and \( v \in V \), the \( \gamma_V \)-closure of \( N(v, 2W) \cap [CV_o(X) \otimes_d E] \) contains \( N(v, W) \).

In particular, \( CV_o(X) \otimes E \) is \( \gamma_V \) and weakly dense in \( CV_o(X, E) \).

**Proof.** Let us fix \( v, W \) and \( f \in N(v, W) \). The set

\[
A_f := \{ \theta f \in C(X, I) : \text{supp}(\theta) \text{ is compact} \}
\]

is \( \omega_V \)-bounded and contained in \( N(v, W) \). Moreover, \( f \) can be \( \omega_V \)-approximated by elements of \( A_f \) (see Section 5.2). The topology \( \sigma_V \) is weaker than \( \omega_V \), and so \( \gamma_V \) is weaker than \( \omega_V \) on \( A_f \). This implies that \( f \) belongs to the \( \gamma_V \)-closure of \( A_f \). Therefore, for the proof of the theorem, we may assume that \( K = \text{supp}(f) \) is compact.

Let us fix some relatively compact neighborhood \( U \) of \( K \), and let \( D \subseteq C(X, I) \otimes E \) be the set of all functions of the form \( \sum_{i=1}^n \theta_i \otimes f(x_i) \), where \( \theta_i \in C(X, I) \), \( \text{supp}(\theta_i) \subseteq U \), \( \text{supp}(\theta_i) \cap \text{supp}(\theta_j) = \emptyset \), if \( i \neq j \), \( x_i \in U \), \( i, j = 1, \ldots, n \), \( n \in \mathbb{N} \). The set \( f(U) \) is bounded in \( E \) and each function \( v \in V \), being upper semicontinuous, is bounded on \( U \), so \( D \) is \( \omega_V \)-bounded. We shall show that \( f \) can be \( m_V \)-approximated by elements of \( D \cap N(v, 2W) \).
2. WEAK APPROXIMATION IN $CV_o(X, E)$

[Fix any $\gamma_V$-neighborhood $M(\mu, v', W', \varepsilon)$ of 0, where $W'$ is balanced. We may assume that $\sup \{v'(x) : x \in U\} =: a > 0$. The set $f(K)$ is compact, so there exists a finite family $\{B_1, ..., B_n\}$ of pairwise disjoint subsets of $X$ such that $K \subseteq \bigcup_{i=1}^n B_i \subseteq U$, $B_i \in B(X)$, and 

$$f(x) - f(y) \leq \frac{1}{a} W'$$

for every $x, y \in B_i, i = 1, ..., n$.

The submeasure $\mu$ satisfies (**), so we can choose cozero sets $U_1, ..., U_n$ and zero sets $Z_1, ..., Z_n$ such that $Z_i \subseteq B_i \subseteq U_i \subseteq U$ and $\mu(U_i \setminus Z_i) \leq \varepsilon/n, i = 1, ..., n$.

Clearly each $Z_i$ is compact. Using the upper semicontinuity of $v$, we can find pairwise disjoint cozero sets $O_1, ..., O_n$ such that $Z_i \subseteq O_i \subseteq U_i$ and $\sup \{v(x) : x \in O_i\} < 2 \sup \{v(x) : x \in Z_i\}$ for $i = 1, ..., n$.

Choose $x_i \in Z_i$ satisfying 

$$\sup \{v(x) : x \in O_i\} < 2v(x_i)$$

and $\theta_i \in C(X, I)$ such that 

$$\text{supp}(\theta_i) = O_i$$

and $\theta_i(Z_i) = 1, i = 1, ..., n$.

We define $g = \sum_{i=1}^n \theta_i \otimes f(x_i)$. Then $g \in D$ and $v(x)\theta_i(x) < 2v(x_i)$ for every $x \in O_i, i = 1, ..., n$. Thus $g \in D \cap N(v, 2W')$. Moreover,

$$\{x \in X : v'(f - g)(x) \notin W'\} \subseteq \bigcup_{i=1}^n \{x \in U_i : v'(f - g)(x) \notin W'\}$$

$$\subseteq \bigcup_{i=1}^n \{x \in U_i : (f - g)(x) \notin a^{-1} W'\} \subseteq \bigcup_{i=1}^n (U_i \setminus Z_i).$$

Therefore,

$$\mu^*(\{x \in X : v'(f - g) \notin W'\}) \leq \sum_{i=1}^n \mu(U_i \setminus Z_i) \leq \varepsilon,$$

and so $f - g \in M(\mu, v', W', \varepsilon)$, as required. The second assertion immediately follows from the first one and Theorem 7.2.3. □

**Open Problem** [Naw89]. It is not known whether Theorem 7.2.4 holds if $X$ is non-locally compact.

**Theorem 7.2.5.** [Naw89] Let $V$ be Nachbin family on a locally compact Hausdorff space $X$. For every Hausdorff TVS $E$, the locally convex topology associated with the weighted topology $\omega_V = \omega_V(X, E)$ coincides with the topology induced on $CV_o(X, E)$ by the weighted topology...
\( \omega_V(X, E^c) \) of the space \( CV_o(X, E^c) \), where \( E^c = (E, \tau^c) \) is the locally convex space associated with \( E = (E, \tau) \).

**Proof.** The topology \( \tau^c \) is weaker than \( \tau \), so that \( CV_o(X, E) \subseteq CV_o(X, E^c) \) and the inclusion mapping is continuous. Therefore,

\[
\omega_{\hat{V}}(X, E) \geq \omega_V(X, E^c)|_{CV_o(X, E)}.
\]

For the proof of the converse inequality let us fix a convex weakly closed \( \omega_{\hat{V}}(X, E) \)-neighborhood \( G \) of 0 in \( CV_o(X, E) \). We can find \( v \in V \) and a balanced neighborhood \( W \) of 0 in \( E \) such that \( \text{conv } N(v, W) \subseteq G \). The set

\[
N := N(v, (1/2) \text{conv } W) = \{ f \in CV_o(X, E) : vf(X) \subseteq (1/2) \text{conv } W \}
\]

is an \( \omega_V(X, E^c)|_{CV_o(X, E)} \)-neighborhood of 0, which is \( \omega_V(X, E) \)-closed. Therefore, \( N \) is closed in the weak topology \( w \) of \((CV_o(X, E), \omega_V)\). Let

\[
N_d := N(v, \text{conv } W) \cap [CV_o(X) \otimes_d E]
\]

and

\[
N_{d,\infty} := \{ f \in CV_o(X) \otimes_d E : vf(X) \subseteq W_\infty \}
\]

where \( W_\infty := \bigcup_{n=1}^{\infty} W_n \) and \( W_n := 2^{-n} \sum_{i=1}^{2^n} W \). Since \( w \leq \gamma_V \), Theorem 7.2.4 shows that

\[
N = \overline{N}^w \subseteq \overline{N}^d \tag{1}
\]

\( W_\infty \) is \( \tau \)-dense in \( \text{conv } W \), and so \( N_{d,\infty} \) is \( \omega_V \)-dense in \( N_d \). Consequently,

\[
N_{d,\infty} \text{ is weakly dense in } N_d. \tag{2}
\]

Let us observe now that

\[
N_{d,\infty} \subseteq \text{conv } N(v, W). \tag{3}
\]

[Indeed, if \( f \in N_{d,\infty} \) then we can find \( \theta_1, ..., \theta_k \in CV_o(X) \) and \( a_1, ..., a_k \in W_\infty \) such that

\[
f = \sum_{i=1}^{k} \theta_i \otimes a_i, \quad \text{supp}(\theta_i) \cap \text{supp}(\theta_j) = \emptyset \text{ if } i \neq j,
\]

\[
\sup \{vf_j(x) : x \in X \} \leq 1, \quad i, j = 1, ..., k.
\]

There are \( a_{i,j} \in W, i = 1, ..., k, j = 1, ..., m, m \in \mathbb{N} \), such that

\[
a_i = \frac{1}{m} \sum_{j=1}^{m} a_{i,j}, \quad i = 1, ..., k.
\]

Put \( f_j := \sum_{i=1}^{k} \theta_i \otimes a_{i,j}, \quad j = 1, ..., m \). Then it is easily seen that \( vf_j(X) \subseteq W \) and \( f = \frac{1}{m} \sum_{j=1}^{m} f_j \). Therefore, \( f \in \text{conv } N(v, W) \).]
Combining (1), (2) and (3), we have

\[ N \subseteq \overline{N}_d^{u} \subseteq \overline{N}_d^{u} \subseteq \overline{\text{conv}}^{u} N(v, W) \subseteq G. \]

Finally, \( G \) is an \( \omega_V(X, E^c)|_{CV_o(X, E)} \) neighborhood of 0. \qed

**Remark 7.2.6.** [Naw89] It follows from the above results that \( \omega_V(X, E) \) and \( \omega_V(X, E^c)|_{CV_o(X, E)} \) produce the same spaces of continuous linear functionals on \( CV_o(X, E) \). Moreover, \( CV_o(X, E) \) is dense in \( CV_o(X, E^c) \). Therefore, we can identify the dual spaces of \( CV_o(X, E) \) and \( CV_o(X, E^c) \). Indeed, the restriction mapping

\[ CV_o(X, E^c)^* \ni T \longrightarrow T|_{CV_o(X, E)} \in CV_o(X, E)^* \]

is an algebraic isomorphism.
Section 7.1. This section contains some results regarding weak approximation in the space $(C_b(X, E), \beta)$, as given in [Naw85]. This requires the measure theoretic terminology of Section A.7. As seen in Section 4.1, $C_b(X) \otimes E$ is $\beta$-dense in $C_b(X, E)$ for a locally convex $E$, and also for some concrete classes of not necessarily locally convex spaces $E$ (such as admissible spaces and spaces having the approximation property); the general case of density problem remains open. In connection with the Riesz-type representations of functionals on $C_b(X, E)$, this had been obstructing research in the sense that the representation of functionals was limited to the subspace $C_b(X) \otimes E$ (see, e.g., [Shu72, KR81, Kat83]). However, using some idea of “submeasure convergence”, Kalton [Kal83] realized that at least if $X$ is compact (and then $\beta$ is the uniform topology), the “representation theory” can avoid the “density problem”. Later Nawrocki [Naw85] proved that a version of this remains valid also if $X$ is an arbitrary completely regular space. This is used to obtain density results in the weak topology, (i.e. the weakest one defined by the duals of $(C_b(X, E), \beta)$). This result is used later in Chapter 8 to obtain a representation of $\beta$-continuous linear functionals on $C_b(X, E)$.

Section 7.2. This section contains some results regarding weak approximation in the space $(CV_0(X, E), \omega_V)$, as given in [Naw89]. The main result in [Naw89] states that, if $X$ is locally compact, then $CV_0(X) \otimes E$ is always weakly dense in $CV_0(X, E)$. This implies that $CV_0(X) \otimes E$ is dense in $CV_0(X, E)$ equipped with the locally convex topology $\omega_V^c$ associated to $\omega_V$ (i.e., the strongest locally convex topology on $CV_0(X, E)$ which is weaker than $\omega_V$), where $E^c$ is the locally convex space associated to $E$. Again, this result is required later in Chapter 8 to obtain a representation of $\omega_V$-continuous linear functionals on $CV_0(X, E)$.
CHAPTER 8

The Riesz Representation type Theorems

In this chapter, we first define and establish existence of the integral of functions in $C_b(X,E)$ with respect to certain $E^*$-valued measures, and then obtain Riesz representation type theorems for the characterization of dual spaces $(C_b(X,E), u)^*$, $(C_{pc}(X,E), \beta)^*$, $(C_{rc}(X,E), \sigma)^*$ and $(CV_o(X,E), \omega_V)^*$. 
1. Vector-valued Measures and Integration

We shall use here the measure theoretic terminology of Section A.7. Let $X$ be a completely regular Hausdorff space and $E$ a real Hausdorff TVS with non-trivial dual $E^*$ and having a base $\mathcal{W}$ of closed balanced shrinkable neighborhoods of 0 in $E$. Let $z(X)$ (resp. $c(X)$, $k(X)$) be the collection of all zero (resp. closed, compact) subsets of $X$. Let $\mathcal{B}(X)$ denote the algebra generated by $z(X)$ and let $\mathcal{B}_0(X)$ (resp. $\mathcal{B}_o(X)$) denote the $\sigma$-algebra generated by $z(X)$ (resp. $c(X)$). The space of all Baire measures is denoted by $M(\mathcal{B}_o(X))$, respectively. Let $M_t(\mathcal{B}_o(X))$ (resp. $M_r(\mathcal{B}_o(X))$, $M_t(\mathcal{B}_0(X))$) denote the space of all bounded real-valued $\sigma$-additive (resp. $\tau$-additive, tight) measures on $\mathcal{B}_o(X)$ (resp. $\mathcal{B}_0(X)$, $\mathcal{B}_0(X)$). Restricting the above measures to $\mathcal{B}_0(X)$, we clearly have $M_t(\mathcal{B}_0(X)) \subseteq M_r(\mathcal{B}_o(X)) \subseteq M_{t}(\mathcal{B}_o(X))$. Note that $M_t(\mathcal{B}_0(X)) \subseteq M_r(\mathcal{B}_0(X))$ need not hold in general as an $m \in M_r(\mathcal{B}_0(X))$ is not necessarily $z(X)$ regular. Every tight $m \in M(\mathcal{B}(X))$ is $\tau$-additive and hence it has a unique extension $\hat{m} \in M_r(\mathcal{B}(X))$.

We shall require the following classical versions of the Riesz representation theorem.

**Theorem 8.1.1.** (a) (Riesz-Alexandroff) $(C_b(X), u)^* = M(\mathcal{B}(X))$ via the linear isomorphism $L \rightarrow \mu$, where $L(g) = \int_X g \, d\mu$ for all $g \in C_b(X)$ and $||L|| = ||\mu||(X)$.

(b) (Riesz-Buck) $(C_b(X), \beta)^* = M_t(\mathcal{B}_o(X))$ via the linear isomorphism $L \rightarrow \mu$, where $L(g) = \int_X g \, d\mu$ for all $g \in C_b(X)$ and $||L|| = ||\mu||(X)$.

**Proof.** (a) See ([Var61] or [Whe83]).

(b) See ([Gil71] or [Sen72]).

**Definition.** For each $W \in \mathcal{W}$, let $M_W(\mathcal{B}(X), E^*)$ denote the set of all finitely-additive $E^*$-valued set functions $m$ on $\mathcal{B}(X)$ such that

(i) for each $a \neq 0$ in $E$, $m_a(F) = m(F)(a)$ ($F \in \mathcal{B}(X)$) determines an element $m_a$ of $M(\mathcal{B}(X))$;

(ii) there exists a constant $\lambda > 0$ such that $|m|_W(X) \leq \lambda$, where, for each $F \in \mathcal{B}(X)$, we define $|m|_W$ by

$$|m|_W(F) = \sup \left| \sum_i m(F_i)(a_i) \right|,$$

(it would be the same for a balanced $W$) the supremum being taken over all finite partitions $\{F_i\}$ of $F$ into sets in $\mathcal{B}(X)$ (henceforth referred to as a $\mathcal{B}(X)$-partition) and all finite collections $\{a_i\} \subseteq W$. 

Let $M(B(X), E^*) = \bigcup_{W \in W} M_W(B(X), E^*)$. Similarly, we may define the spaces $M_\sigma(B\sigma(X), E^*)$, $M_\tau(B\tau(X), E^*)$ and $M_t(B\tau(X), E^*)$ by replacing $M(B(X))$ by $M_\sigma(B\sigma(X))$, $M_\tau(B\tau(X))$ and $M_t(B\tau(X))$, respectively, in the above definition.

Lemma 8.1.2. [KR81, Kat83] Let $W \in W$. If $m \in M_W(B(X), E^*)$ (resp. $M_{\sigma,W}(B\sigma(X), E^*)$, $M_{\tau,W}(B\tau(X), E^*)$, $M_{t,W}(B\tau(X), E^*)$), then $|m|_W \in M(B(X))$ (resp. $M_{\sigma}(B\sigma(X))$, $(M_t(B\tau(X)))$).

Proof. First, let $m \in M_W(B(X), E^*)$. It follows immediately from the definition that $|m|_W$ is a bounded non-negative set function on $X$, and it is straightforward to show that $|m|_W$ is finitely additive. To show that $|m|_W$ is regular, let $F \in B(X)$ and $\varepsilon > 0$. There exist a $B(X)$-partition $\{F_i\}$ ($1 \leq i \leq n$) of $F$ and a collection $\{a_i\}$ ($1 \leq i \leq n$) of points in $W$ such that

$$|m|_W(F) \leq \sum_{i=1}^{n} |a_i|(F_i) + \varepsilon$$

Since each $a_i$ is regular, there exist zero sets $Z_i (i = 1, ..., n)$ such that $Z_i \subseteq F_i$ and

$$|a_i|(F_j) < |a_i|(Z_j) + \varepsilon/2^i.$$

Let $Z = \bigcup_{i=1}^{n} Z_i$. Then $Z \subseteq F$ and

$$|m|_W(F) \leq \sum_{i=1}^{n} |a_i|(Z_i) + 2\varepsilon \leq |m|_W(F) + 2\varepsilon.$$

Thus $|m|_W \in M(X)$.

Now, let $m \in M_{t,W}(B\tau(X), E^*)$. Then, by the above argument, it easily follows that $|m|_W$ is a bounded, non-negative, finitely-additive set function on $B\tau(X)$. We show that $|m|_W$ is countably additive, as follows. Let $\{A_k : k = 1, 2, ...\}$ be a sequence of disjoint sets in $B\tau(X)$ and suppose that $\bigcup_{k=1}^{\infty} A_k = A$. For any $n \geq 1$,

$$|m|_W(A) \geq |m|_W\left(\bigcup_{k=1}^{n} A_k\right) = \sum_{k=1}^{n} |m|_W(A_k),$$

and so

$$|m|_W(A) \geq \sum_{k=1}^{\infty} |m|_W(A_k).$$
8. THE RIESZ REPRESENTATION TYPE THEOREMS

Let \( \varepsilon > 0 \). Then there exist a \( \mathcal{B}_0(X) \)-partition \( \{F_j : j = 1, \ldots, p\} \) of \( A \) and a collection of points \( \{a_j : j = 1, \ldots, p\} \subseteq W \) such that

\[
|m|_W (A) \leq \sum_{j=1}^{p} m_{a_j}(F_j) + \varepsilon.
\]

Since each \( m_{a_j} \) is countably additive and \( \{F_j \cap A_k : k = 1, 2, \ldots\} \) is a partition of \( F_j \), we have

\[
m_{a_j}(F_j) = \sum_{k=1}^{\infty} m_{a_j}(F_j \cap A_k).
\]

Hence

\[
|m|_W (A) \leq \left( \sum_{j=1}^{p} \sum_{k=1}^{\infty} m_{a_j}(F_j \cap A_k) \right) + \varepsilon \leq \sum_{k=1}^{\infty} |m|_W (A) + \varepsilon.
\]

Since \( \varepsilon \) is arbitrary, it follows from the above inequalities that \( |m|_W \) is countably additive. Next, using the argument of the first part, it easily follows that \( |m|_W \) is \( k(W) \)-regular. Thus \( |m|_W \in M_1(\mathcal{B}_0(X)) \). The proof for other parts is similar to the above one. □

The advantage of taking the base \( \mathcal{W} \) of ‘shrinkable’ closed balanced neighborhoods is that the Minkowski functional \( \rho_W \) of each \( W \in \mathcal{W} \) is continuous and absolutely homogeneous, and further that \( W = \{x \in X : \rho_W(x) \leq 1\} \) (see Section A.4). For this reason, the following approach of defining the integral seems to be simpler than those given in [KR81, Kat83] where the notions of \( F \)-seminorms and bipolars are used, respectively.

Definition. [KR81, Kh95] If \( f \in C_b(X, E) \) and \( W \in \mathcal{W} \), we write

\[
\|f\|_W = \|\rho_W \circ f\| = \sup_{x \in X} |\rho_W(f(x))|.
\]

Let \( m \in M_W(\mathcal{B}(X), E^*) \), \( (W \in \mathcal{W}) \), and \( f \in C_b(X, E) \). Let \( D \) be the collection of all \( \alpha = \{F_1, \ldots, F_n; x_1, \ldots, x_n\} \), where \( \{F_i\} \ (1 \leq i \leq n) \) is a \( \mathcal{B}(X) \)-partition of \( X \) and \( x_i \in F_i \). If \( \alpha_1, \alpha_2 \in D \), define \( \alpha_1 \geq \alpha_2 \) iff each set which appears in \( \alpha_1 \) is contained in some set in \( \alpha_2 \). In this way, \( D \) becomes as indexing set. Let \( S_{\alpha} = \sum_{i=1}^{n} m(F_i)(f(x_i)) \). We now define the integral of \( f \) with respect to \( m \) over \( X \) by

\[
\int_X dm f = \lim_{\alpha \in D} S_{\alpha}.
\]

Regarding the conditions under which this integral exists, we obtain the following lemma.

Lemma 8.1.3. [KR81, Kh95] The integral \( \int_X dm f \), defined above, exists in each of the following cases:

(a) \( f \in C_{pc}(X, E) \);
(b) \( f \in C_b(X,E) \), and \( |m|_W \) is \( \tau \)-additive (in particular, \( |m|_W \) is tight);

(c) \( f \in C_b(X,E) \), and \( |m|_W \) is \( \sigma \)-additive, and the range of \( f \) is measure compact (i.e. \( M_\sigma(f(X)) = M_\tau(f(X)) \)).

Proof. We need to show that, in each case, \( \{S_\alpha : \alpha \in D \} \) is a Cauchy net in \( \mathbb{R} \).

(a) Let \( \varepsilon > 0 \). Choose a balanced shrinkable \( V \in \mathcal{W} \) such that \( V + V \subseteq \varepsilon W \). Since \( f(X) \) is precompact, there exist \( Y_1, \ldots, Y_n \in X \) such that \( f(X) \subseteq \bigcup_{i=1}^n (f(y_i) + V) \). Let \( V_\ell = \{x \in X : \rho_V(f(x) - f(y_\ell)) \leq 1\} \).

Then each \( V_\ell \in \mathcal{B}(X) \). Let \( G_1 = V_1 \) and \( G_\ell = V_\ell \setminus \bigcup_{j=1}^{\ell-1} V_j \) (\( 2 \leq \ell \leq n \)). By keeping those \( G_\ell \)'s which are non-empty, we get, \( \{G_1, \ldots, G_k\} \) say, a \( \mathcal{B}(X) \)-partition of \( X \). Choose \( x_\ell \in G_\ell \) and let \( \alpha_\ell = \{G_1, \ldots, G_k; x_1, \ldots, x_k\} \). Note that if \( x, y \) are in the same \( G_\ell \),

\[
f(x) - f(y) = [f(x) - f(y_\ell)] + [f(y_\ell) - f(x)] \in V + V \subseteq \varepsilon W.
\]

Then, for \( \alpha_1, \alpha_2 \geq \alpha_\ell \), we have

\[
|S_{\alpha_1} - S_{\alpha_2}| \leq |S_{\alpha_1} - S_{\alpha_\ell}| + |S_{\alpha_\ell} - S_{\alpha_2}|.
\]

Suppose \( \alpha_1 = \{F_1, \ldots, F_q; z_1, \ldots, z_q\} \), where each \( F_j \) is contained in some \( G_i \) and \( z_j \in F_j \). Now

\[
|S_{\alpha_1} - S_{\alpha_\ell}| = \left| \sum_{j=1}^q m(F_j)(f(z_j)) - \sum_{i=1}^k m(G_i)(f(x_i)) \right|
\]

\[
= \left| \sum_{j=1}^q m(F_j)(f(z_j)) - \sum_{i=1}^k \sum_{j:F_j \subseteq G_i} m(F_j)(f(x_i)) \right|
\]

\[
= \varepsilon \left| \sum_{i=1}^k \sum_{j:F_j \subseteq G_i} m(F_j)[\varepsilon^{-1}(f(z_j) - f(x_i))] \right| \leq \varepsilon |m|_W(X).
\]

Similarly, we can prove that \( |S_{\alpha_2} - S_{\alpha_\ell}| \leq \varepsilon |m|_W(X) \). Thus

\[
|S_{\alpha_2} - S_{\alpha_1}| \leq 2\varepsilon |m|_W(X),
\]

and \( \{S_\alpha : \alpha \in D \} \) is a Cauchy net in \( \mathbb{R} \).

(b) Let \( \varepsilon > 0 \), and choose an open balanced shrinkable set \( V \in \mathcal{W} \) with \( V + V \subseteq \varepsilon W \). The collection \( \mathcal{V} = \{f^{-1}(f(y) + V) : y \in X\} \) is a cover of \( X \) consisting of cozero sets. Labelling \( \mathcal{V} \) as \( \{V_\lambda : \lambda \in I\} \), we make \( I \) into a directed set by saying that \( \lambda \geq \gamma \Leftrightarrow V_\lambda \subseteq V_\gamma \). By the \( \tau \)-additivity
of \(|m|_W\), there exist \(y_1, \ldots, y_k \in X\) such that \(|m|_W (X \setminus \bigcup_{i=1}^{k} V_{\lambda_i}) < \varepsilon\), where
\[
V_{\lambda_i} = f^{-1}(f(y_i) + V) = \{x \in X : \rho_V(f(x) - f(y_i)) < 1\}.
\]
Define \(G_1 = V_{\lambda_1}, G_i = (V_{\lambda_i} \setminus \bigcup_{j=1}^{i-1} V_{\lambda_j}) (2 \leq i \leq k),\) and \(G_{k+1} = X \setminus \bigcup_{i=1}^{k} V_{\lambda_i}\). Assuming that \(G_i\)'s are non-empty, choose \(x_i \in G_i\) and let \(\alpha_o = \{G_1, \ldots, G_{k+1}; \ x_1, \ldots, x_{k+1}\}\).

Let \(\alpha_1, \alpha_2 \geq \alpha_o\). Suppose \(\alpha_1 = \{F_1, \ldots, F_q; \ z_1, \ldots, z_q\}\), where each \(F_j\) is contained in some \(G_i\) and \(z_j \in F_j\). Then
\[
|S_{\alpha_1} - S_{\alpha_o}| \leq \sum_{i=1}^{k} \sum_{j, F_j \subseteq G_i} m(F_j)(f(z_j) - f(x_i)) + \sum_{j, F_j \subseteq G_{k+1}} m(F_j)(f(z_j)) + \sum_{j, F_j \subseteq G_{k+1}} m(F_j)(f(x_{k+1}))
\leq \varepsilon(|m|_W (X) + 2\|f\|_W).
\]
Similarly, we obtain \(|S_{\alpha_2} - S_{\alpha_o}| \leq \varepsilon(|m|_W (X) + 2\|f\|_W).\) Consequently, \(\{S_{\alpha} : \alpha \in D\}\) is a Cauchy net in \(\mathbb{R}\).

(c) Suppose \(|m|_W\) is \(\sigma\)-additive, and let \(\mu(A) = |m|_W (f^{-1}(A))\) for every Baire set \(A\) of \(f(X)\). Then \(\mu\) is also \(\sigma\)-additive. Now taking \(\mathcal{V}\) as in part (b) and using the fact that \(\mu\) is also \(\tau\)-additive (since \(f(X)\) is measure compact), we can complete the proof by the argument of part (b).

Remark. Part (b) and (c) were proved in ([Fon74], Lemma 3.11) assuming \(E\) a normed space.

**Lemma 8.1.4.** [KR81] Let \(m \in M_W(X, E^*) (W \in \mathcal{W})\) and \(f \in C_b(X, E)\). If the integral \(\int_X dm f\) exists, then
\[
\left| \int_X df \right| \leq \int_X (\rho_W \circ f) d|m|_W \leq \|f\|_W |m|_W (X).
\]

**Proof.** For any \(\varepsilon > 0\), there exist a \(\mathcal{B}(X)\)-partition, \(\{F_i : 1 \leq i \leq n\}\) say, of \(X\), and points \(x_i \in F_i\) such that

\[
\left| \int_X df \right| \leq \sum_{i=1}^{n} m(F_i)(f(x_i)) + \varepsilon
\]
and
\[
\sum_{i=1}^{n} (\rho_W \circ f)(x_i) |m|_W (F_i) \leq \int_X (\rho_W \circ f) d|m|_W + \varepsilon.
\]
Let $H_1$ (resp. $H_2$) be the set of $i \in \{1, \ldots, n\}$ such that $\rho_W(f(x_i)) \neq 0$ (resp. $\rho_W(f(x_i)) = 0$). We note that, if $j \in H_2$, then $\rho_W(tf(x_i)) = 0$ for all $t > 0$. Then
\[
\left| \int_X df \right| \leq \sum_{i \in H_1} (\rho_W \circ f)(x_i) \left| m(F_i) \left( \frac{f(x_i)}{(\rho_W \circ f)(x_i)} \right) \right| + \sum_{i \in H_2} \frac{\varepsilon}{|m|_W(X)} \left| m(F_i) \left( \frac{|m|_W(X)f(x_i)}{\varepsilon} \right) \right| + \varepsilon
\leq \sum_{i \in H_1} (\rho_W \circ f)(x_i) \left| m(F_i) \left( \frac{f(x_i)}{(\rho_W \circ f)(x_i)} \right) \right| + 2\varepsilon.
\]

We note that, if $|m|_W(X) = 0$, then the inequality we are seeking to establish holds trivially. It follows that
\[
\left| \int_X df \right| \leq \sum_{i \in H_1} (\rho_W \circ f)(x_i) |m|_W(F_i) + 2\varepsilon
\leq \int_X (\rho_W \circ f)d|m|_W + 2\varepsilon,
\]
and so, since $\varepsilon$ is arbitrary,
\[
\left| \int_X df \right| \leq \int_X (\rho_W \circ f)d|m|_W.
\]
The other inequality is straightforward to prove.

**Note.** It is easy to verify that, if $m \in M(B(X), E^*)$, $g \in C_b(X)$, and $a \in E$, then
\[
\int_X dm(g \otimes a) = \int_X gdm_a.
\]

**Remark.** An other equivalent approach to define $|m|_W$, due to Katsaras [Kat83], is as follows: For $p$ a continuous seminorm on $E$, we define $M_p(B(X), E^*)$ to be the set $M_W(B(X), E^*)$ where $W = W_p = \{a \in E : p(a) \leq 1\}$. For an $m \in M_p(B(X), E^*)$, we define $m_p$ to be the measure $|m|_W$.

Let now $W \in W$ be balanced, and let $W^{00}$ be the bipolar of $W$ with respect to the pair $\langle E, E^* \rangle$. Let $q = q_W$ be the Minkowski functional of the absolutely convex set $W^{00}$. We have the following:

**Lemma 8.1.5.** [Kat83] A set function $m : B(X) \to E^*$ belongs to $M_W(B(X), E^*)$ iff $m \in M_q(B(X), E^*)$. Moreover, we have $m_q = |m|_W$.

**Proof.** Since $W \subseteq \{a \in E : q(a) \leq 1\}$, it follows that $m \in M_W(B(X), E^*)$, whenever $m \in M_q(B(X), E^*)$, and that $|m|_W \leq m_q$. 
On the other hand, let $m \in M_W(\mathcal{B}(X), E^*)$ and let $A \in \mathcal{B}(X)$. For each $\varphi \in E^*$, we define $|\varphi|_W = \sup \{|\varphi(a)| : a \in W\}$. Then $W^0 = \{\varphi \in E^* : |\varphi|_W \leq 1\}$. It is easy to see that $|\varphi|_W = \sup \{|\varphi(a)| : q(a) \leq 1\}$. From the definition

$$|m|_W(A) = \sup \left\{ \sum_{i=1}^n |m(A_i)|_W : \{A_i\}_{i=1}^n \text{ a } \mathcal{B}(X)-\text{partition of } A \right\},$$

it follows now easily that $m_q(A) \leq |m|_W(A)$. 

2. Integral Representation Theorems

We first characterize the $u$-dual of $C_{pc}(X, E)$ via the integral representation. Recall that $C_b(X) \otimes E$ is weakly dense in $(C_{pc}(X, E), u)$ (see Chapter 7).

**Theorem 8.2.1.** [Kat83, Kh95] $(C_{pc}(X, E), u)^* = M(B(X), E^*)$ via the linear isomorphism $L \to m$ given by

$$L(f) = \int_X dm f \quad (f \in C_{pc}(X, E)).$$

Furthermore, if $L$ is represented as in (1) with

$$m \in M_W(B(X), E^*)(W \in W),$$

then $\|L\|_W = |m|_W (X)$, where

$$\|L\|_W = \sup \{|L(f)| : f \in C_{pc}(X, E), \|f\|_W \leq 1\}.$$

**Proof.** Let $m \in M_W(B(X), E^*) (W \in W)$, and suppose that $L$ is the linear functional on $C_{pc}(X, E)$ defined by (1). Then, by Lemma 8.1.4,

$$|L(f)| \leq \|f\|_W |m|_W (X) \leq |m|_W (X),$$

whenever $f \in C_{pc}(X, E)$ with $f(X) \subseteq W$. Hence $L \in (C_{pc}(X, E), u)^*$. We now show that $\|L\|_W = |m|_W (X)$. Clearly, $\|L\|_W \leq |m|_W (X)$. For any $\varepsilon > 0$, there exist a $B(X)$-partition $\{F_1, \ldots, F_n\}$ of $X$ and a collection $\{a_1, \ldots, a_n\} \subseteq W$ such that

$$|m|_W (X) \leq \left| \sum_{i=1}^n m(F_i)(a_i) \right| + \varepsilon.$$

There exist zero sets $Z_i \subseteq F_i$ such that $|m_{a_i}|(Z_i \setminus F_i) < \frac{\varepsilon}{n}$. Since $C_b(X)$ separates the zero sets in $X$, the closures $\overline{Z_1}, \ldots, \overline{Z_n}$ in $\beta X$ are pairwise disjoint. Since the cozero sets in $\beta X$ form a base for open sets, there exist pairwise disjoint cozero sets $\hat{U}_1, \ldots, \hat{U}_2$ in $\beta X$ with $\overline{Z_i} \subseteq \hat{U}_i$. Each $\hat{U}_i \cap X$ is a cozero set in $X$ containing $Z_i$; there is a cozero set $V_i$ in $X$ with

$$Z_i \subseteq V_i \subseteq \hat{U}_i \cap X \text{ and } |m_{a_i}|(V_i \setminus Z_i) < \frac{\varepsilon}{n}.$$

Choose $h_i \in C_b(X)$ with $0 \leq h_i \leq 1$, $h_i(Z_i) = 1$, and $h_i = 0$ on $X \setminus V_i$. Let $h = \sum_{i=1}^n h_i \otimes a_i$. Then $h \in C_{pc}(X, E)$ and $\|h\|_W \leq 1$. Now

$$L(h) = \sum_{i=1}^n L(h_i \otimes a_i) = \sum_{i=1}^n \int_{Z_i} dm a_i + \sum_{i=1}^n \int_{V_i \setminus Z_i} h_i dm a_i;$$
For each $a \in W$, $h \in m$.

So

$$\sum_{i=1}^{n} m_{a_{i}}(F_{i}) \leq \sum_{i=1}^{n} m_{a_{i}}(F_{i} \setminus Z_{i}) + \sum_{i=1}^{n} m_{a_{i}}(Z_{i})$$

$$\leq \varepsilon + |L(h)| + \sum_{i=1}^{n} |m_{a_{i}}|(V_{i} \setminus Z_{i})$$

$$\leq \|L\|_{W} + 2\varepsilon.$$  

Thus $|m|_{W}(X) \leq \|L\|_{W}$.

Conversely, suppose that $L \in (C_{\text{pc}}(X, E), u)^{\ast}$. Then there exist a $W \in \mathcal{W}$ and $r > 0$ such that

$$|L(f)| \leq r \|f\|_{W} \quad (f \in C_{\text{pc}}(X, E)).$$

For each $a \neq 0$ in $E$, let $L_{a}(g) = L(g \otimes C_{b}(X))$. By (2), we have $|L_{a}(g)| \leq r \|g\|_{\rho_{W}}(a)$ for all $g \in C_{b}(X)$, and so $L_{a} \in (C_{b}(X), u)^{\ast}$. Hence, by Theorem 8.1.1(a), there exists an $m_{a} \in M(\mathcal{B}(X))$ such that $L_{a}(g) = \int_{X} g dm_{a}$ ($g \in C_{b}(X)$) and $\|L_{a}\| = |m_{a}|(X)$.

For each $F \in \mathcal{B}(X)$, define $m(F)(a) = m_{a}(F)$ ($a \in E$). Then $|m(F)(a)| \leq |m_{a}|(X) \leq r \rho_{W}(a)$, and consequently $m$ is a finitely additive $E^{\ast}$-valued set function on $\mathcal{B}(X)$. Next $|m|_{W}(X) \leq r$, as follows. Let $\{F_{1}, ..., F_{p}\}$ be a $\mathcal{B}(X)$-partition of $X$ and $\{a_{1}, ..., a_{p}\} \subseteq W$, and let $\varepsilon > 0$. By using the argument as the one used earlier, there exists an $h \in C_{\text{pc}}(X, E)$ with $\|h\|_{W} \leq 1$ such that

$$\left| \sum_{i=1}^{p} m(F_{i})(a_{i}) \right| \leq |L(h)| + 2\varepsilon \leq r + 2\varepsilon.$$  

So $|m|_{W}(X) \leq r$ and hence $m \in M_{W}(\mathcal{B}(X), E^{\ast})$.

Now for any $f = \sum_{i=1}^{k} f_{i} \otimes a_{i}$ ($f_{i} \in C_{b}(X)$, $a_{i} \in E$) in $C_{b}(X) \otimes E$,

$$L(F) = \sum_{i=1}^{k} L(f_{i} \otimes a_{i}) = \sum_{i=1}^{k} \int_{X} f_{i} dm_{a_{i}} = \sum_{i=1}^{k} \int_{X} dm f_{i} \otimes a_{i} = \int_{X} dm f.$$  

Since $\int_{X} C_{b}(X) \otimes E$ is weakly dense in $(C_{\text{pc}}(X, E), u)$, the above holds for all $f \in C_{\text{pc}}(X, E)$.

Finally, $m$ is unique, as follows. Suppose there is a $\mu \in M(\mathcal{B}(X), E^{\ast})$ such that $L(f) = \int_{X} d\mu f$ for all $f \in C_{\text{pc}}(X, E)$. In particular, for any $g \in C_{b}(X)$ and $a \in E$, $\int_{X} dm g \otimes a = \int_{X} d\mu_{a} g \otimes a$. Hence $\int_{X} g dm_{a} = \int_{X} g dm a$ for all $g \in C_{b}(X)$, and so by Theorem 8.1.1(a), $m_{a} = \mu_{a}$. Thus, for any $F \in \mathcal{B}(X)$ and any $a \in E$, $m(F)(a) = m_{a}(F) = \mu_{a}(F)$. It follows that $m(F) = \mu(F)$, and so $m = \mu$. \qed
2. INTEGRAL REPRESENTATION THEOREMS

The following representation theorem gives a characterization for the
dual of $\mathcal{C}(X,E)$, $\beta$.

**Theorem 8.2.2.** [KR81, Kat83] $(\mathcal{C}(X,E), \beta)^* = \mathcal{M}(\mathcal{B}(X), E^*)$
via the linear isomorphism $L \mapsto m$ given by

$$L(f) = \int_X df \quad (f \in \mathcal{C}(X,E)).$$

Further, if $L$ is represented as in $(\beta)$ with $m \in \mathcal{M}(\mathcal{B}(X), E^*) (W \in W)$, then

$$\|L\|_W = \|m\|_W (X).$$

**Proof.** Let $m \in \mathcal{M}(\mathcal{B}(X), E^*) (W \in W)$, and suppose that $L$ is
defined by (3). Then, by Lemma 8.1.2, $\|m\|_W \in \mathcal{M}(\mathcal{B}(X))$. It follows
from Theorem 8.1.1 that the equation

$$L_W(g) = \int_X g dm \quad (g \in \mathcal{C}(X)).$$

defines a $\beta$-continuous linear functional $L_W$ on $\mathcal{C}(X)$. Hence there exists
a $\varphi \in B_o(X)$, $0 \leq \varphi \leq 1$, such that $|L_W(g)| \leq 1$ whenever $g \in \mathcal{C}(X)$
with $\|\varphi g\| \leq 1$. For any $f \in N(\varphi, W)$, $||\varphi (\rho_W \circ f)|| = ||\rho_W (\varphi f)|| \leq 1
and so, by (4) and Lemma 8.1.4,

$$|L(f)| \leq \int_X (\rho_W \circ f) f dm = L_W(\rho_W \circ f) \leq 1.$$

Thus $L$ is $\beta$-continuous.

Conversely, let $L \in (\mathcal{C}(X,E), \beta)$. Then there exists a $\theta \in B_o(X)$
and $W \in W$ such that

$$|L(f)| \leq 1 \text{ for all } f \in N(\theta, W).$$

(5)

For each $a \neq 0$ in $E$, let $L_a(g) = L(g \otimes a) \ (g \in \mathcal{C}(X))$. It easily
follows from (5) that $L_a \in (\mathcal{C}(X), \beta)^*$ and so by Theorem 8.1.1, there
exists a unique $m_a \in \mathcal{M}(X)$ such that

$$L_a(g) = \int_X g dm_a \quad (g \in \mathcal{C}(X)).$$

For each $F \in \mathcal{B}(X)$, $m(F) (a) = m_a(F) \ (a \in E)$. Since $\beta \leq u$, $L$
is $u$-continuous and so there exists a $W \in W$ such that $|L(f)| \leq 1$
whenever $f \in N(1,W)$. Consider $g \in \mathcal{C}(X)$ with $||g|| \leq 1$. Then
$\rho_W (g(x)a) \leq \rho_W (a)$ for all $x \in X$, and so $g \otimes a \in N(1,W)$ whenever
$\rho_W (g(x)a) \leq \rho_W (a)$ for all $a \in X$. Thus $|L_a|| \leq \rho_W (a)$, and so from the
inequalities

$$|m(F)(a)| \leq |m_a(F)| \leq |m_a| = ||L_a|| \leq \rho_W (a),$$
the continuity of \( m(F) \) follows. Consequently \( m \) is a finitely additive \( E^* \)-valued set function on \( Bo(X) \) such that, for each \( a \neq 0 \) in \( E \), \( m_a \in M_t(X) \).

Next \( |m|_W (X) < \infty \), as follows. Let \( \{ F_j : j = 1, ..., p \} \) be a \( Bo(X) \)-partition of \( X \) and \( a_1, ..., a_p \in W \), and let \( \varepsilon > 0 \). Each \( m_a \) is regular and so there exist compact sets \( K_j \subseteq F_j \) with \( |m_a|((F_j \setminus K_j) < \varepsilon/2p \), and open sets \( V_j \supseteq K_j \) with \( |m_a|(F_j \setminus K_j) < \varepsilon/2p \) for \( j = 1, ..., p \); since \( K_j \)'s are disjoint compact sets and \( X \) is completely regular, the \( V_j \)'s may be chosen pairwise disjoint. Choose functions \( g_j (1 \leq j \leq p) \) in \( C_b(X) \), \( 0 \leq g_j \leq 1 \), such that \( g_j = 1 \) on \( K_j \) and \( \text{supp}(g_j) \subseteq V_j \). Let \( h = \sum g_j \otimes a_j \). Then \( h \in C_p(X, E) \) and \( \|h\|_W = 1 \), and so \( |L(h)| \leq 1 \). By using the above inequalities as in the proof of Theorem 8.2.1, we have \( |m|_W (X) \leq \|L\|_W \).

Thus \( m \in M_t(Bo(X), E^*) \).

Consider any \( f = \sum_{i=1}^k f_i \otimes a_i \) \( f_i \in C_b(X) \), \( a_i \in E \) in \( C_b(X) \otimes E \). Then

\[
L(f) = \sum_{i=1}^k L(f_i \otimes a_i) = \sum_{i=1}^k f_i dm_{a_i} = \sum_{i=1}^k \int_X dm(f_i \otimes a_i) = \int_X dm f.
\]

Since \( C_b(X) \otimes E \) is weakly dense in \( (C_b(X, E), \beta) \), the above holds for all \( f \in C_b(X, E) \). The proof for uniqueness of \( m \) and for \( \|L\|_W = |m|_W (X) \) are the same as in Theorem 8.2.1 with slight modifications.

We next characterize the dual of \( (C_{rc}(X, E), \sigma) \), \( \sigma \) being the \( \sigma \)-compact-open topology. Recall that \( C_{rc}(X, E) \) denotes the subspace of \( C_b(X, E) \) which consists of those functions \( f \) such that \( f(X) \) is relatively compact. Each \( f \) in \( C_b(X) \) or \( C_{rc}(X, E) \) has a continuous extension \( \hat{f} \) to \( \beta X \), the Stone-Cech compactification of \( X \). Let \( S \) denote the set of all \( \sigma \)-compact subsets of \( X \) and, for any \( A \in S \), let \( \hat{A} \) denote the \( \beta X \)-closure of \( A \). Let \( M_{t,W}(Bo(\hat{A}), E^*) (W \in \mathcal{W}) \) denote the set of all \( m \in M_{t,W}(Bo(\beta X), E^*) \) such that the support of \( |m|_W \) is contained in some \( \hat{A} \); let

\[
M_t(Bo(\hat{A}), E^*) = \bigcup_{W \in \mathcal{W}} M_{t,W}(Bo(\hat{A}), E^*).
\]

**Theorem 8.2.3.** [KR91] \( (C_{rc}(X, E), \sigma)^* = \bigcup_{A \in S} M_{t,W}(Bo(\hat{A}), E^*) \) via the linear isomorphism \( L \to m \) given by

\[
L(f) = \int_{\beta X} dm \hat{f} \quad (f \in C_{rc}(X, E)).
\]

Moreover, for some \( W \in \mathcal{W} \), \( \|L\|_W = |m|_W (\beta X) \), where \( \|L\|_W = \{ |L(f)| : f \in C_{rc}(X, E), \|\rho_W \circ f\|_X \leq 1 \} \).
Proof. Using the Gulick’s version [Gu72] of the Riesz representation theorem that \((C_b(X), \sigma)^* = \bigcup_{A \in S} \mathcal{M}_t(\mathcal{B}o(\bar{A}))\), the proof follows as in the above theorem with appropriate modifications. The details may be found in ([KR91], Theorem 4.4). □

Finally, we state a characterize the dual of the weighted space \((CV_o(X, E), \omega_V)\). Here we follow the argument due to Nawrocki [Naw89]. We recall from Remark 7.2.6 that the topologies \(\omega_V(X, E)\) and \(\omega_V(X, E^c)|_{CV_o(X, E)}\) produce the same spaces of continuous linear functionals on \(CV_o(X, E)\). Moreover, \(CV_o(X, E)\) is dense in \(CV_o(X, E^c)\). Therefore, we can identify the dual spaces of \(CV_o(X, E)\) and \(CV_o(X, E^c)\). Indeed, the restriction mapping \(CV_o(X, E^c)^* \ni T \longrightarrow T|_{CV_o(X, E)} \in CV_o(X, E)^*\) is an algebraic isomorphism. Therefore, using Theorem 7.2.5 and the known representation of the dual space of \(CV_o(X, E)\), where \(E\) is locally convex (see [Pro71b] or ([Pro77], Theorem 5.42)), we obtain the following result.

Proposition 8.2.4. [Naw89] Let \(X\) be a locally compact space, \(V\) any Nachbin family on \(X\) and \(E\) a TVS with \(E^*\) separates the points of \(E\). The dual space of \((CV_o(X, E), \omega_V)\) is isomorphic to

\[
VM_t(\mathcal{B}o(X), E^*) \equiv \{vm : v \in V, \ m \in M_t(\mathcal{B}o(X), E^*)\},
\]

the space of all \((E^*, \sigma(E^*, E))\)-valued bounded tight measures on \(X\). □
3. Notes and Comments

Section 8.1. Various classes of $E^*$-valued measures on a topological space $X$ are considered. The existence of the integral of functions in $C_b(X, E)$ with respect to these measures is also given. These results are taken from the papers [Shu72b, KR81, Kat83, Kh95]. Earlier results in the locally convex setting were obtained in [Kat74, Fon74].

Section 8.2. The Riesz representation type theorems for the characterization of dual spaces $(C_b(X, E), u)^*$, $(C_{pc}(X, E), \beta)^*$, $(C_{rc}(X, E), \sigma)^*$ and $(CV_o(X, E), \omega_V)^*$ are obtained via the integral representations. In fact, each of these dual spaces is identified with a suitable class of $E^*$-valued measures on $X$. These results are also taken from the papers [Shu72b, KR81, Kat83, Kh95]. Theorem 8.2.2 has been obtained, independently, in [KR81, Kat83]. For earlier work in the locally convex setting, several useful references can be found in [Shu72b]. Theorem 8.2.4 was obtained in [SumW70] for $E$ a scalar field, in [Pro71b] for $E$ a locally convex space and in [Naw89] for $E$ a general TVS.

There is an extensive work (not considered here) on Bochner and Bartle–Dunford–Schwartz integration processes for vector-valued functions and also on the Riesz representation of operators $T : C_o(X, E) \to E$ for $E$ a Banach or locally convex space (see [Mez02, Mez08, Khu08]). More recent developments in this area have been made in the papers [DL13, Thom12]. These latter papers deal with the theory of integration of scalar functions with respect to a measure with values in a TVS $E$, not necessarily locally convex. They focus on the extension of such integrals from bounded measurable functions to the class of integrable functions, proving adequate convergence theorems, and establishing usable integrability criteria.
CHAPTER 9

Weighted Composition Operators

In this chapter, we give some characterizations of multiplication and composition operators on the weighted function spaces $CV_o(X,E)$ and $CV_b(X,E)$ induced by scalar-, vector-, and operator-valued maps. We also include characterizations of, compact, equicontinuous, precompact and bounded multiplication/composition operators.

As before, the topology of bounded convergence $t_u$ (resp. the topology of pointwise convergence $t_p$) on $CL(E)$ is the linear topology which has a base of neighborhood of 0 consisting of all sets of the form

$$U(D,G) = \{ T \in CL(E) : T(D) \subseteq G \}$$

$$= \{ T \in CL(E) : \|T\|_{D,G} \leq 1 \},$$

where $D$ is a bounded (respectively, finite) subset of $E$, and $G \in \mathcal{W}_E$ is shrinkable. (Here $\|T\|_{D,G} = \sup \{ \rho_G(T(a)) : a \in D \}$.)
1. Multiplication Operators on $CV_o(X, E)$

Let $X$ be a completely regular Hausdorff space, $E$ a Hausdorff TVS, and $V$ a Nachbin family of weights on $X$. It is assumed throughout that the Nachbin family $V$ on $X$ satisfies the following conditions:

(* ) $V > 0$;

(** ) $CV_o(X)$ does not vanish on $X$, i.e., corresponding to each $x \in X$, there exists an $h_x \in CV_o(X)$ such that $h_x(x) \neq 0$.

The above conditions primarily serve to exclude trivial cases along with certain unnecessary pathological situations. If $V > 0$, then condition (**) holds, in particular, when each $v \in V$ vanishes at infinity or $X$ is locally compact. [First suppose that $V \subseteq S^+_0(X)$, and let $x \in X$. Choose $v \in V$ such that $v(x) \neq 0$. Since $X$ is completely regular, there exists an $f \in C_b(X)$ such that $f(x) = 1$. Since $v \in S^+_0(X)$, it easily follows that $vf \in CV_o(X)$ and $v(x)f(x) \neq 0$. Next, suppose that $X$ is locally compact, and let $x \in X$. There exists an $f \in C_{\text{oo}}(X) \subseteq C_o(X)$ such that $f(x) = 1$. Since $V > 0$, choose $v \in V$ such that $v(x) \neq 0$. Then clearly $vf \in CV_o(X)$ and $v(x)f(x) \neq 0$.]

We further mention a useful fact that if $CV_o(X)$ does not vanish on $X$, then, for any $x \in X$ and any open neighborhood $U$ of $x$ in $X$, we can choose a $f \in CV_o(X)$ with

$$0 \leq f \leq 1, f(X\setminus U) = 0 \text{ and } f(x) = 1.$$ 

This follows from Lemma 4.1.2, due to Nachbin [Nac67], by taking $\mathcal{M} = CV_o(X) \subseteq C(X)$, $K = \{x\}$, and $U = U_i$ for all $i = 1, \ldots, n$.

If $CV_o(X)$ does not vanish on $X$ and $E$ is a non-trivial TVS, then $CV_o(X) \otimes E$ and hence $CV_b(X, E)$ and $CV_b(X, E)$ also do not vanish on $X$. [In fact, for any $x \in X$, choose $\varphi$ in $CV_o(X)$ with $\varphi(x) \neq 0$. Then, if $a \neq 0$ in $E$, the function $\varphi \otimes a$ belongs to $CV_o(X) \otimes E$ and clearly $(f \otimes a)(x) = \varphi(x)a \neq 0$.]

**Note.** Earlier in Section 1.2, we have seen by an example that $CV_b(X, E)$ and hence $CV_o(X, E)$ may be trivial for relatively nice $X$ (e.g. $X = \mathbb{Q}$). This situation justifies to assume the above conditions (*) and (**) throughout.

**Definition.** Let $F(X, E)$ be the vector space of all functions from $X$ into $E$. If $\theta : X \to \mathbb{C}$ and $\psi : X \to E$, $E$ a topological algebra, are given maps, the scalar multiplication on $E$ and the multiplication on $F$ give rise to linear mappings $M_\theta$ and $M_\psi$ from $CV_o(X, E)$ into $F(X, E)$ defined by $M_\theta(f) = \theta f$ and $M_\psi(f) = \psi f$, both pointwise; that is,

$$(\theta f)(x) = \theta(x)f(x) \text{ and } (\psi f)(x) = \psi(x)f(x), x \in X.$$
Also if $\pi : X \to CL(E)$, $E$ a TVS, we may define $M_{\pi}$ from $CV_o(X, E)$ into $F(X, E)$ by $M_{\pi}(f) = \pi f$, where

$$(\pi f)(x) = \pi_x(f(x)) := \pi(x)(f(x)), \quad x \in X.$$ 

If $M_{\theta}$, $M_\psi$ and $M_\pi$ map $CV_o(X, E)$ (resp. $CV_o(X, E)$) into itself and are continuous, they are called *multiplication operators* on $CV_o(X, E)$ (resp. $CV_o(X, E)$) induced by $\theta$, $\psi$ and $\pi$, respectively.

We first give necessary and sufficient conditions for $M_{\theta}$, $\theta : X \to \mathbb{C}$, to be the multiplication operator on the weighted space $CV_o(X, E)$. These results hold also for the space $CV_o(X, E)$ with slight modification in the proofs and are therefore omitted.

For any $\theta : X \to \mathbb{C}$, we let $V|\theta| = \{v|\theta| : v \in V\}$.

**Theorem 9.1.1.** [SM91, KT97] For a mapping $\theta : X \to \mathbb{C}$, the following are equivalent:

(a) $M_{\theta}$ is a multiplication operator on $CV_o(X, E)$;

(b) (i) $\theta$ is continuous and (ii) $V|\theta| \leq V$.

Proof. Let $\mathcal{W}$ be a base of closed, balanced and shrinkable neighborhoods of $0$ in $E$.

(a) $\Rightarrow$ (b) To prove (i), let $\{x_\alpha\}$ be a net in $X$ with $x_\alpha \to x \in X$. By assumption (**), there exists an $h \in CV_o(X)$ such that $h(x) \neq 0$. Since $M_{\theta}$ is a self-map on $CV_o(X, E)$, it follows that the function $\theta h$ from $X$ into $\mathbb{C}$ is continuous. Hence $\theta(x_\alpha)h(x_\alpha) \to \theta(x)h(x)$ and consequently $\theta(x_\alpha) \to \theta(x)$.

(ii) To show that $V|\theta| \leq V$, let $v \in V$. By continuity of $M_{\theta}$, given $G \in \mathcal{W}$, there exist $u \in V$ and $H \in \mathcal{W}$ such that

$$M_{\theta}(N(u, H)) \subseteq N(v, G).$$

(1)

Without loss of generality, we may assume that $G \cap H$ is a proper subset of $E$. Choose $a \in E \setminus (G \cup H)$, and put $t = \frac{2u(a)}{\rho_H(a)}$. We claim that $v|\theta| \leq 2tu$.

Fix $x_\alpha \in X$. We shall consider two cases: $u(x_\alpha) \neq 0$ and $u(x_\alpha) = 0$.

Suppose that $u(x_\alpha) \neq 0$, and let $\varepsilon = u(x_\alpha)$. Then $D = \{x \in X : u(x) < 2\varepsilon\}$ is an open neighborhood of $x_\alpha$. Taking $M = CV_o(X)$ with assumption (**) in Lemma 4.1.2, there is an $h \in CV_o(X)$ with

$$0 \leq h \leq 1, \quad h(x_\alpha) = 1 \quad \text{and} \quad h(X \setminus D) = 0.$$ 

Define $f = \frac{1}{2\varepsilon \rho_H(a)}(h \otimes a)$. Since $\rho_H$ is homogeneous, for any $x \in X$,

$$\rho_H[u(x)f(x)] = \rho_H[u(x)\frac{1}{2\varepsilon \rho_H(a)}h(x)a] = \frac{u(x)h(x)}{2\varepsilon}.$$ 


If \( x \in D \), \( u(x) < 2 \varepsilon \) and if \( x \in X \setminus D \), \( h(x) = 0 \); hence
\[
\rho_H[u(x)f(x)] < \frac{2 \varepsilon h(x)}{2 \varepsilon} \leq 1 \quad \text{if} \ x \in D, \\
\rho_H[u(x)f(x)] = 0 < 1 \quad \text{if} \ x \in X \setminus D.
\]

Since \( H = \{ b \in E : \rho_H(b) \leq 1 \} \), we have \( f \in N(u, H) \). Hence, by (1), \( \theta f \in N(v, G) \). This implies that, for any \( x \in X \),
\[
\rho_G\left[ \frac{1}{2 \varepsilon \rho_H(a)} \theta(x)v(x)h(x)a \right] \leq 1, \\
or \quad v(x)h(x)|\theta(x)| \leq \frac{2 \varepsilon \rho_H(a)}{\rho_G(a)} = 2t \varepsilon = 2tu(x_o),
\]

In particular, \( v(x_o)|\theta(x_o)| \leq 2tu(x_o) \).

Now suppose that \( u(x_o) = 0 \) but \( v(x_o)|\theta(x_o)| > 0 \). Put
\[
\varepsilon = \frac{1}{2t}v(x_o)|\theta(x_o)|.
\]

Let \( D = \{ x \in X : u(x) < \varepsilon \} \), and choose an \( h \in CV_o(X) \) with \( 0 \leq h \leq 1 \), \( h(x_o) = 1 \) and \( h(X \setminus D) = 0 \). Define \( g = \frac{1}{\varepsilon \rho_H(a)}(h \otimes a) \). Since \( \rho_H \) is homogeneous, for any \( x \in X \),
\[
\rho_H[u(x)g(x)] = \rho_H[u(x)\frac{1}{\varepsilon \rho_H(a)}h(x)a] = \frac{u(x)h(x)}{\varepsilon}.
\]

If \( x \in D \), \( u(x) < \varepsilon \) and if \( x \in X \setminus D \), \( h(x) = 0 \); hence
\[
\rho_H[u(x)f(x)] < \frac{\varepsilon h(x)}{\varepsilon} \leq 1 \quad \text{if} \ x \in D \\
\rho_H[u(x)f(x)] = 0 < 1 \quad \text{if} \ x \in X \setminus D.
\]

Hence \( g \in N(u, H) \) and so, by (1), \( \theta g \in N(v, G) \). This implies that, for any \( x \in X \),
\[
\rho_G\left[ \frac{1}{\varepsilon \rho_H(a)} \theta(x)v(x)h(x)a \right] \leq 1, \\
or \quad v(x)h(x)|\theta(x)| \leq \frac{\varepsilon \rho_H(a)}{\rho_G(a)} = t \varepsilon = \frac{1}{2}v(x_o)|\theta(x_o)|.
\]

In particular, \( v(x_o)|\theta(x_o)| \leq \frac{1}{2}v(x_o)|\theta(x_o)| \), which is impossible unless \( v(x_o)|\theta(x_o)| = 0 \).

\((b) \Rightarrow (a)\) We first show that \( M_\theta \) maps \( CV_o(X, E) \) into itself. Let \( f \in CV_o(X, E) \), and let \( v \in V \) and \( G \in \mathcal{W} \). Choose \( u \in V \) such that
v|θ| ≤ u. There exists a compact set K ⊆ X such that u(x)f(x) ∈ G for all x ∈ X\K. Then, since G is balanced,

\[ v(x)M_θ(f)(x) = v(x)θ(x)f(x) ∈ G \]

for all x ∈ X\K. Hence vM_θ(f) vanishes at infinity; further, since θ is continuous, M_θ(f) ∈ CV_o(X, E). To prove the continuity of M_θ, let \{f_α\} be a net in CV_o(X, E) with f_α → 0. Let v, G and u be chosen as above. Choose an index α_0 such that f_α ∈ N(u, G) for all α ≥ α_0. Then it follows that θf_α ∈ N(v, G) for all α ≥ α_0. Thus M_θ(f_α) → 0. So M_θ is continuous at 0 and hence, by linearity, on CV_o(X, E).

We next consider the case of the operator M_ψ, where ψ : X → E and E a topological algebra. Recall that a topological algebra E is called \textit{locally idempotent} if it has a base \( \mathcal{W} \) of neighborhoods of 0 consisting of idempotent sets. If G ∈ \( \mathcal{W} \) is idempotent and shrinkable, then its Minkowski functional \( ρ_G \) is continuous and submultiplicative: \( ρ_G(ab) ≤ ρ_G(a)ρ_G(b) \) for all a, b ∈ E; further, if E has an identity e, \( ρ_G(e) ≥ 1 \).

\textbf{Theorem 9.1.2.} [SM91, KT97] Let E be a locally idempotent topological algebra with identity e and \( \mathcal{W} \) a base of neighborhoods of 0. Then, for a mapping ψ : X → E, the following are equivalent:

(a) \( M_ψ \) is a multiplication operator on CV_o(X, E).

(b) (i) ψ is continuous and (ii) \( Vρ_G ◦ ψ ≤ V \) for every G ∈ \( \mathcal{W} \).

Proof. We may assume that \( \mathcal{W} \) consists of closed, balanced, shrinkable and idempotent sets.

(a) ⇒ (b) (i) Let \( \{x_α\} \) be a net in X with \( x_α → x \in X \). Choose an h ∈ CV_o(X) with h(x) ≠ 0. Since \( M_ψ \) is a self-map on CV_o(X, E), it follows that, for any a ∈ E, the function ψ(h ◦ a) from X into E is continuous. Hence h(x_α)ψ(x_α) → h(x)ψ(x) consequently ψ(x_α) → ψ(x). This proves the continuity of ψ.

(ii) Let v ∈ V and G ∈ \( \mathcal{W} \). There exist u ∈ V and H ∈ \( \mathcal{W} \) such that

\[ M_ψ(N(u, H)) ⊆ N(v, G). \]

(2)

Without loss of generality, we may assume that H is a proper subset of E. We claim that \( vρ_G ◦ ψ ≤ 2ρ_H(e)u \).

Fix \( x_α \in X \). First suppose that u(x_α) ≠ 0, and let \( ε = u(x_α) \). Then

\[ D = \{x \in X : u(x) < 2ε\} \]

is an open neighborhood of \( x_α \) and so there exists an h ∈ CV_o(X) such that 0 ≤ h ≤ 1, h(x_α) = 1, and h(X\D) = 0. Define \( f = \frac{1}{2ερ_H(e)}(h ◦ e) \). Then, for any \( x ∈ X \),

\[ ρ_H(u(x)f(x)) = ρ_H(u(x)h(x)e/2ερ_H(e)) ≤ 1; \]
that is, \( f \in N(u, H) \). Hence, by (2), \( \psi f \in N(v, G) \). This implies that, for any \( x \in X \),
\[
v(x)h(x)\rho_G(\psi(x)) \leq 2\varepsilon \rho_H(e).
\]
In particular, \( v(x_0)\rho_G(\psi(x_0)) \leq 2\rho_H(e)u(x_0) \).

Next suppose that \( u(x_0) = 0 \), but \( v(x_0)\rho_G(\psi(x_0)) > 0 \). Put \( \varepsilon = \frac{v(x_0)\rho_G(\psi(x_0))}{2\rho_H(e)} \). Let \( D = \{ x \in X : u(x) < \varepsilon \} \), and choose an \( h \in CV_o(X) \) as above. Define \( g = \frac{1}{\rho_H(e)}(h \otimes e) \). Then \( g \in N(u, H) \) and so by (2), \( \phi g \in N(v, G) \). From this we obtain
\[
v(x_0)\rho_G(\psi(x_0)) \leq \rho_H(e)\varepsilon = v(x_0)\rho_G(\psi(x_0))/2,
\]
which is impossible unless \( v(x_0)\rho_G(\psi(x_0)) = 0 \).

(b) \( \Rightarrow \) (a) We first show that \( M_\psi \) maps \( CV_o(X, E) \) into itself. Let \( f \in CV_o(X, E) \), and let \( v \in V \) and \( G \in \mathcal{W} \). Choose \( u \in V \) such that \( v \rho_G \circ \psi \leq u \). There exists a compact set \( K \subseteq X \) such that \( u(x)f(x) \in G \) for all \( x \in X\setminus K \). Since \( \rho_G \) is submultiplicative, for any \( x \in X\setminus K \),
\[
\rho_G(v(x)\psi(x)f(x)) \leq v(x)\rho_G(\psi(x))\rho_G(f(x)) \leq u(x)\rho_G(f(x)) \leq 1;
\]
hence \( M_\psi(f) \in CV_o(X, E) \). Using again the submultiplicity of \( \rho_G \), the continuity of \( M_\psi \) follows in the same way as in the proof of Theorem 9.1.1.

We mention that Theorem 9.1.2 also remains true if we replace \( CV_o(X, E) \) by \( CV_b(X, E) \).

We now apply the above results to the cases: \( V = S_o^+(X) \) and \( V = S_o^+(X) \).

**Theorem 9.1.3.** [SM91, KT97] (i) If \( \theta : X \to \mathbb{C} \) is a continuous mapping, then \( M_\psi \) is a multiplication operator on \( (C(X, E), k) \).

(ii) If \( E \) is a locally idempotent topological algebra with identity \( e \) and \( \psi : X \to E \) a continuous mapping, then \( M_\psi \) is a multiplication operator on \( (C(X, E), k) \).

**Proof.** (i) In view of Theorem 9.1.1, we only need to verify that \( V \ | \theta | \leq V \), where \( V = S^+_o(X) \). Let \( v \in V \). Choose a compact set \( K \subseteq X \) with \( v(x) = 0 \) for all \( x \in X\setminus K \). Let \( s = \sup\{ |\theta(x)| : x \in K \} \), \( t = \sup\{ v(x) : x \in K \} \) and \( u = st_{XK} \). Then \( u \in V \) and clearly \( v(x) |\theta(x)| \leq u(x) \) for all \( x \in X \).

(ii) Let \( \mathcal{W} \) be a base of neighborhoods of 0 in \( E \) consisting of closed, balanced, shrinkable and idempotent sets. In view of Theorem 9.1.2, we only need to verify that \( V \rho_G \circ \psi \leq V \) for every \( G \in \mathcal{W} \), where \( V = S_o^+(X) \). Let \( v \in V \) and \( G \in \mathcal{W} \). Choose a compact set \( K \subseteq X \) with \( v(x) = 0 \) for all \( x \in X\setminus K \). Let \( s = \sup\{ \rho_G(\psi(x)) : x \in \)
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$K \setminus K$, $t = \sup \{v(x) : x \in K \}$ and $u = st\chi_K$. Then $u \in V$ and clearly $v(x)\rho_G(\psi(x)) \leq u(x)$ for all $x \in X$.

The following example shows that the above result need not hold when $(C(X, E), k)$ is replaced by $(C_b(X, E), \beta)$.

**Example 9.1.4.** Let $X = \mathbb{R}^+$, $E = \mathbb{C}$, and $V = S^+_0(X)$. Let $	heta = \psi : X \to \mathbb{C}$ be given by $\theta(x) = x^2 (x \in X)$, and let $v \in V$ be given by $v(x) = \frac{1}{x} (x \in X)$. Then $v(x) |\theta(x)| = x$ for all $x \in X$. Since each $u \in V$ is a bounded function, $v |\theta| \leq u$ is not true for every $u \in V$; i.e.$V |\theta| \leq V$ does not hold and so, by Theorem 9.1.1, $M_\theta$ is not a multiplication operator on $(C_b(X), \beta)$. The same is also true for the space $(C_b(X), u)$, where $u$ is the uniform topology, since $\beta \leq u$. However, if $\theta$ and $\psi$ are “bounded” continuous functions, then it is easily seen that $M_\theta$ and $M_\psi$ are always multiplication operators on $CV_b(X, E)$ for any Nachbin family $V$.

We now consider multiplication operators induced by operator-valued mappings and give necessary and sufficient conditions for $M_\pi$, $\pi : X \to CL\alpha(E)$, to be the multiplication operator on the weighted space $CV_b(X, E)$. We then use this characterization to show that, in the case of $E$ a locally bounded space, $M_\pi$ is a multiplication operator if $\pi$ is a bounded mapping or $V = S^+_\infty(X)$.

**Theorem 9.1.5.** [SM91, KT97] Let $X$ be a $k$-space and $E$ a TVS. Then, for any continuous mapping $\pi : X \to CL\alpha(E)$, the following are equivalent:

(a) $M_\pi$ is a multiplication operator on $CV_b(X, E)$.

(b) For every $v \in V$ and $G \in \mathcal{W}$, there exist $u \in V$ and $H \in \mathcal{W}$ such that

$v(x)\rho_G(\pi_x(y)) \leq u(x)\rho_H(y)$ for all $x \in X$ and $y \in E$.

**Proof.** We may assume that $\mathcal{W}$ consists of closed, balanced and shrinkable sets.

(a) $\Rightarrow$ (b) Let $v \in V$ and $G \in \mathcal{W}$. By hypothesis, there exist $u \in V$ and $H \in \mathcal{W}$ such that

$M_\pi(N(u, H)) \subseteq N(v, G)$. \hspace{1cm} (3)

We claim that

$v(x)\rho_G(\pi_x(y)) \leq 2u(x)\rho_H(y)$

for all $x \in X$ and $y \in E$. Let $x_0 \in X$ and $y_0 \in E$. Then we consider four cases:

(I) $u(x_0)\rho_H(y_0) \neq 0$.

(II) $u(x_0) = 0$, $\rho_H(y_0) \neq 0$. 


(III) \[ u(x_o) \neq 0, \quad \rho_H(y_o) = 0. \]

(IV) \[ u(x_o) = 0, \quad \rho_H(y_o) = 0. \]

**Case I.** Suppose \( u(x_o)\rho_H(y_o) \neq 0 \), and let \( \varepsilon = u(x_o)\rho_H(y_o) \). Then the set \( D = \{ x \in X : u(x)\rho_H(y_o) < 2\varepsilon \} \) is an open neighborhood of \( x_o \), and so there exists an \( h \in CV_b(X) \) such that \( 0 \leq h \leq 1 \), \( h(x_o) = 1 \), and \( h(X \setminus D) = 0 \). Define \( g = \frac{1}{2\varepsilon}(h \otimes y_o) \). Then, for any \( x \in X \),

\[
\rho_H(u(x)g(x)) = \frac{1}{2\varepsilon}u(x)h(x)\rho_H(y_o) \leq 1;
\]

that is, \( g \in N(u, H) \). Hence, by (3), \( M_\pi(g) \in N(v, G) \). This implies that, for any \( x \in X \),

\[
v(x)h(x)\rho_G(\pi_x(y_o)) \leq 2u(x)\rho_H(y_o).
\]

In particular, by taking \( x = x_o \), we have

\[
v(x)\rho_G(\pi_{x_o}(y_o)) \leq 2u(x_o)\rho_H(y_o).
\]

**Case II.** Suppose \( u(x_o) = 0, \quad \rho_H(y_o) \neq 0 \), but \( v(x_o)\rho_G(\pi_{x_o}(y_o)) > 0 \). Put \( \varepsilon = v(x_o)\rho_G(\pi_{x_o}(y_o))/2 \). Then the set \( D = \{ x \in X : u(x)\rho_H(y_o) < \varepsilon \} \) is an open neighborhood of \( x_o \), and so there exists an \( h \in CV_b(X) \) such that \( 0 \leq h \leq 1 \), \( h(x_o) = 1 \), and \( h(X \setminus D) = 0 \). Define \( g = \frac{1}{\varepsilon}(h \otimes y_o) \). Then, for any \( x \in X \),

\[
\rho_H(u(x)g(x)) = \frac{1}{\varepsilon}u(x)h(x)\rho_H(y_o) \leq 1;
\]

that is, \( g \in N(u, H) \). Hence, by (3), \( M_\pi(g) \in N(v, G) \). This implies that, for any \( x \in X \),

\[
v(x)h(x)\rho_G(\pi_x(y_o)) \leq \varepsilon = \frac{1}{2}v(x)\rho_G(\pi_{x_o}(y_o)).
\]

In particular, by taking \( x = x_o \), we have

\[
v(x)\rho_G(\pi_{x_o}(y_o)) \leq \frac{1}{2}v(x)\rho_G(\pi_{x_o}(y_o)),
\]

which is impossible unless \( v(x_o)\rho_G(\pi_{x_o}(y_o)) = 0 \).

(III). Suppose \( u(x_o) \neq 0, \quad \rho_H(y_o) = 0, \quad \rho_H(y_o) > 0 \). Put \( \varepsilon = \frac{1}{2}v(x_o)\rho_G(\pi_{x_o}(y_o)) \). Then the set \( D = \{ x \in X : u(x) < \varepsilon + u(x_o) \} \) is an open neighborhood of \( x_o \), and so there exists an \( h \in CV_b(X) \) such
that 0 ≤ h ≤ 1, h(x_o) = 1, and h(X \ D) = 0. Define g = h \otimes y_o. Clearly $g \in CV_b(X, E)$ and

$$0 \leq \rho_G \circ g \leq \rho_G(y_o), \rho_G \circ g(x_o) = \rho_G(y_o), \text{ and } \rho_G \circ g(X \ D) = 0.$$ 

Consider $g_1 = \frac{2}{\varepsilon}$. Then, for any $x \in X$,

$$\rho_H(g_1(x)) = \rho_H\left(\frac{1}{\varepsilon} h(x)y_o\right) = \frac{1}{\varepsilon} h(x)\rho_H(y_o) = 0 \in H$$

Hence $g_1 \in N(u, H)$ and so, by (3), $M_u(g_1) \in N(v, G)$. Therefore, for any $x \in X$

$$v(x)\rho_G(\pi_x(g_1(x))) \leq 1, \text{ or } v(x)\rho_G(\pi_x(h(x)y_o)) \leq \varepsilon = \frac{1}{2} v(x)\rho_G(\pi_x(y_o)).$$

In particular, since $h(x_o) = 1, v(x_o)\rho_G(\pi_x(y_o)) \leq \frac{1}{2} v(x_o)\rho_G(\pi_x(y_o))$, a contradiction.

(IV) Suppose $u(x_o) = 0, \rho_H(y_o) = 0$, but $v(x_o)\rho_G(\pi_x(y_o)) > 0$. Put $\varepsilon = \frac{1}{2} v(x_o)\rho_G(\pi_x(y_o))$. Then the set $D = \{x \in X : u(x) < \varepsilon\}$ is an open neighborhood of $x_o$, and so there exists an $h \in CV_b(X)$ such that

$$0 \leq h \leq 1, h(x_o) = 1, \text{ and } h(X \ D) = 0.$$ 

Define $g = h \otimes y_o$. Clearly $g \in CV_b(X, E)$ and, for any $x \in X$,

$$0 \leq \rho_G(g(x)) = h(x)\rho_G(y_o) \leq \rho_G(y_o), \rho_G(x_o) = \rho_G(y_o), \text{ and } \rho_G(g(x)) = 0 \text{ if } x \in X \ D;$$

or $0 \leq \rho_G \circ g \leq \rho_G(y_o), \rho_G \circ g(x_o) = \rho_G(y_o), \text{ and } \rho_G \circ g(X \ D) = 0$.

Consider $g_1 = \frac{2}{\varepsilon}$. Then, for any $x \in X$,

$$\rho_H(g_1(x)) = \rho_H\left(\frac{1}{\varepsilon} h(x)y_o\right) = \frac{1}{\varepsilon} h(x)\rho_H(y_o) = 0 \in H$$

Hence $g_1 \in N(u, H)$ and so, by (4), $M_u(g_1) \in N(v, G)$. Therefore, for any $x \in X$

$$v(x)\rho_G(\pi_x(g_1(x))) \leq 1, \text{ or } v(x)\rho_G(\pi_x(h(x)y_o)) \leq \varepsilon = \frac{1}{2} v(x)\rho_G(\pi_x(y_o)).$$

In particular, since $h(x_o) = 1, v(x_o)\rho_G(\pi_x(y_o)) \leq \frac{1}{2} v(x_o)\rho_G(\pi_x(y_o))$, a contradiction.

(b) ⇒ (a) We first show that $M_\pi$ maps $CV_b(X, E)$ into itself. Let $f \in CV_b(X, E)$. Since $X$ is a $k$-space, it is enough to show that $M_\pi(f)$ is continuous on any given compact subset $K$ of $X$. Let $\{x_\alpha : \alpha \in I\}$ be a net in $K$ with $x_\alpha \to x \in K$. Let $G \in \mathcal{W}$, and let $G_1 \in \mathcal{W}$ with $G_1 + G_1 \subseteq G$. Now $\{f(x_\alpha)\}$, being convergent, is bounded and so, by continuity of $\pi$, there exists an $\alpha_1 \in I$ such that

$$\pi_{x_\beta}(f(x_\alpha)) - \pi_x(f(x_\alpha)) \in G_1, \quad (4)$$

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for all $\alpha \in I$ and all $\beta \geq \alpha_1$. Since $\pi_x : E \to E$ is a continuous linear mapping, there exists an $H \in \mathcal{W}$ such that $\pi_x(y) \in G_1$ for all $y \in H$. Choose an $\alpha_2 \in I$ such that $f(x) \in H$ for all $\alpha \geq \alpha_2$; consequently,

$$\pi_x(f(x_\alpha) - f(x)) \in G_1$$

(5)

for $\alpha \geq \alpha_2$. Choose $\alpha_o \in I$ with $\alpha_o \geq \alpha_1, \alpha_2$. Then, by (4) and (5)

$$\pi_x(f(x_\beta) - f(x)) = \pi_x(f(x_\beta)) - \pi_x(f(x_\beta) + f(x) + f(x)) \in G_1 + G_1 \subseteq G$$

for all $\beta \geq \alpha_o$. Hence $M_\pi(f) \in C(X,E)$. To show that $M_\pi(f) \in CV_b(X,E)$, let $v \in V$ and $G \in W$. By hypothesis, there exist $u \in V$ and $H \in W$ such that

$$v(x)\rho_G(\pi_x(y)) \leq \rho_H(u(x)(y))$$

(6)

for all $x \in X$ and $y \in E$. Choose $\lambda > 0$ such that $u(x)f(x) \in \lambda H$ for all $x \in X$. Then, it follows from (6) that $v(x)\pi_x(f(x)) \in \lambda G$ for all $x \in X$. This proves that $M_\pi(f) \in CV_b(X,E)$.

To establish the continuity of $M_\pi$, let $\{f_\alpha : \alpha \in I\}$ be a net in $CV_b(X,E)$ with $f_\alpha \to 0$, and let $v \in V$ and $G \in W$. Choose $u \in V$ and $H \in W$ as above and which satisfy (6). There exists an $\alpha_o \in I$ such that $f_\alpha \in N(u,H)$ for all $\alpha \geq \alpha_o$. Then, for any $x \in X$ and $\alpha \geq \alpha_o$

$$v(x)\rho_G(\pi_x(f_\alpha(x))) \leq \rho_H(f_\alpha(x)) \leq 1$$

or equivalent $v(x)\pi_x(f_\alpha(x)) \in G$. So $M_\pi$ is continuous at 0 and hence, by its linearity, on $CV_b(X,E)$.

We now deduce that, under same additional conditions on $\pi$, $M_\pi$ is automatically a multiplication operator.

**Corollary 9.1.6. [SM91, KT97]** Let $\pi : X \to CL_\alpha(E)$ be a constant mapping. Then $M_\pi$ is a multiplication operator on $CV_b(X,E)$.

**Proof.** We need to verify that condition (b) of Theorem 9.1.5 holds in this case. Let $v \in V$ and $G \in W$. Choose $T \in CL(E)$ such that $\pi(x) = T$ for all $x \in X$. Since $T$ is continuous and linear, there exist an $m > 0$ and a closed shrinkable $H \in W$ such that

$$v(x)\rho_G(T(y)) \leq m\rho_H(y) \text{ for all } y \in E.$$

Let $u = mv$. Then, for any $x \in X$ and $y \in E$,

$$v(x)\rho_G(\pi(x)(y)) = v(x)\rho_G(T(y)) \leq u(x)\rho_H(y). \quad \Box$$
Corollary 9.1.7. Suppose $E$ is a locally bounded TVS and $\pi : X \to CL_u(E)$ is a continuous and bounded mapping. Then $M_\pi$ is a multiplication operator on $CV_b(X,E)$.

Proof. Let $v \in V$ and $G \in \mathcal{W}$. We may assume that $G$ is a bounded neighborhood of $0$. Choose a closed and shrinkable $H \in \mathcal{W}$ with $H \subseteq G$. Since $\pi(X)$ is bounded in $CL_u(E)$, there exists an $m > 0$ such that $\pi(X) \subseteq mU(H,H)$; i.e.,

$$\rho_H(\pi(x)(y)) \leq m\rho_H(y) \text{ for all } x \in X, y \in H.$$ 

Let $u = mv$. Then, for any $x \in X$ and $y \in E$,

$$v(x)\rho_G(\pi(x)(y)) \leq u(x)\rho_H(y).$$

We now apply Theorem 9.1.5 to the cases $V = S^+_{oo}(X)$ and $V = S^+_0(X)$.

Theorem 9.1.8. [SM91, KT97] Suppose $E$ is a locally bounded TVS and $\pi : X \to CL_u(E)$ is a continuous mapping. Then $M_\pi$ is a multiplication operator on $(C(X,E),k)$.

Proof. We need to verify that condition (a) of Theorem 9.1.5 holds for $V = S^+_{oo}(X)$. Let $v \in V$ and $G \in \mathcal{W}$. Choose a compact $K \subseteq X$ such that $v(x) = 0$ for all $x \in X \setminus K$. We may assume that $G$ is a bounded neighborhood of $0$. Choose a closed and shrinkable $H \in \mathcal{W}$ with $H \subseteq G$. Since $\pi : X \to CL_u(E)$ is continuous, $\pi(K)$ is compact in $CL_u(E)$ and so there exists an $m > 0$ such that $\pi(K) \subseteq mU(H,H)$; i.e.,

$$\rho_H(\pi(x)(y)) \leq m\rho_H(y) \text{ for } x \in K \text{ and } y \in H.$$ 

Let $t = \sup\{v(x) : x \in K\}$, and put $u = mt\chi_K$. Then $u \in V$. Let $x \in X$ and $y \in E$. If $x \in K$, then

$$v(x)\rho_G(\pi(x)(y)) \leq t\rho_H(\pi(x)(y)) \leq u(x)\rho_H(y).$$

If $x \in X \setminus K$, the above holds trivially (since $v(x) = 0$). □

Remark. If $E = \mathbb{C}$, then $CL_u(E) = \mathbb{C}$ and so Example 9.1.4 also shows that the above result need not hold when $(C(X,E),k)$ is replaced by $(C_b(X,E), \beta)$. 
2. Weighted Composition Operators on $CV_o(X,E)$

In this section, we consider the more general notion of weighted composition operators which include the multiplication and composition operators.

**Definition.** Let $F(X,E)$ be the vector space of all functions from a topological space $X$ to a TVS $E$, and let $S(X,E)$ be a vector subspace of $F(X,E)$. Let $L(E)$ be the vector space of all linear maps from $E$ into itself. Then the operator-valued mapping $\pi : X \to L(E)$ and the self-map $\varphi : X \to X$ give rise to a linear mapping $W_{\pi,\varphi} : S(X,E) \to F(X,E)$ defined as

$$W_{\pi,\varphi}(f) = \pi \cdot f \circ \varphi, \ f \in S(X,E);$$

that is,

$$W_{\pi,\varphi}(f)(x) = \pi(x)(f(\varphi(x))) = \pi_x(f(\varphi(x))), \ f \in S(X,E), \ x \in X.$$

(i) In case $S(X,E)$ is equipped with a linear topology, $W_{\pi,\varphi}$ takes $S(X,E)$ into itself and $W_{\pi,\varphi}$ is continuous, we call $W_{\pi,\varphi}$ the *weighted composition operator* on $S(X,E)$ induced by the mappings $\pi$ and $\varphi$.

(ii) If $\varphi : X \to X$ is the identity map, the corresponding operator $W_{\pi,\varphi}$ is the multiplication operator $M_\pi$ on $S(X,E)$ induced by the operator-valued mapping $\pi$.

(iii) Also, if $\pi(x) = I$, the identity operator on $E$, for every $x \in X$, then $W_{\pi,\varphi}$ is the *composition operator* $C_\varphi$ on $S(X,E)$ induced by the self-map $\varphi$ of $X$, where $C_\varphi(f) = f \circ \varphi, \ f \in S(X,E)$.

These operators have been the subject matter of extensive study for the last several decades on different functions spaces; especially on $L_p$-spaces, Bergman spaces, and spaces of continuous functions.

As before, it is assumed that the Nachbin family $V$ satisfies (**) $V > 0$ and (**) $CV_o(X)$ does not vanish on $X$.

**Theorem 9.2.1.** [MS98] Let $X$ be a completely regular $k_R$-space and $E$ a TVS. Let $\pi : X \to CL_u(E)$ and $\varphi : X \to X$ be continuous mappings. Then the following conditions are equivalent:

(a) $W_{\pi,\varphi}$ is a weighted composition operator on $CV_o(X,E)$.

(b) (i) for any $v \in V$ and $G \in \mathcal{W}$, there exist $u \in V$ and $H \in \mathcal{N}$ such that

$$u(\varphi(x))y \in H$$

implies that $v(x)\pi_x(y) \in G$ for all $x \in X, y \in E$;

(ii) for each $v \in V$ and $G \in \mathcal{W}$ and compact set $K \subseteq X$, the set

$$\varphi^{-1}(K) \cap \{x \in X : v(x)\pi_x(y) \notin G\}$$

is compact for each $0 \neq y \in E$. 
Proof. (a) $\Rightarrow$ (b) Suppose that $W_{\pi,\varphi}$ is a weighted composition operator on $CV_o(X,E)$. To establish condition (b)(i), we fix $v \in V$ and $G \in W$. By the continuity of $W_{\pi,\varphi}$, there exist $u \in V$ and $H \in W$ such that

$$W_{\pi,\varphi}(N(u, H) \subseteq N(v, G).$$

Let $\omega = 2u$. Then we claim that

$$\omega(\varphi(x))y \in H \text{ implies that } v(x)\pi_x(y) \in G \text{ for all } x \in X, y \in E.$$  

[Assume that for some $x_o \in X$ and $y_o \in E$, we have $\omega(\varphi(x_o))y_o \in H$ but $v(x_o)\pi_{x_o}(y_o) \notin G$. Let $D = \{x \in X : \omega(x)y_o \in H\}$. Then $D$ is an open neighborhood of $\varphi(x_o)$ and therefore there exists $f \in CV_o(X)$ such that $0 \leq f \leq 1$, $f(\varphi(x_o)) = 1$ and $f(X\setminus D) = 0$. Define $g = 2f \otimes y_o$. Now, it is easy to see that $g \in N(v, H)$ and therefore we have $W_{\pi,\varphi}g \in N(v, G)$. Thus it follows that $v(x_o)\pi_{x_o}(y_o) \in \frac{1}{2}G \subseteq G$, which is a contradiction. This proves our claim.]

To prove (b)(ii), let $v \in V$ and $G \in W$ and compact set $K \subseteq X$. There exists $f \in CV_o(X)$ such that $f(K) = 1$. Fix $0 \neq y \in E$ and define $g = f \otimes y$. Then clearly $W_{\pi,\varphi}g \in CV_o(X,E)$. Now if we put

$$D = \{x \in X : v(x)\pi_x(g(\varphi(x))) \notin G\}$$

and

$$S = \varphi^{-1}(K) \cap \{x \in X : v(x)\pi_x(y) \notin G\},$$

then clearly $D$ is a compact subset of $X$ such that $S \subseteq D$. Thus $S$, being a closed subset of $D$, is compact.

(b) $\Rightarrow$ (a) Suppose that conditions (b)(i) and (b)(ii) hold. First of all we show that $W_{\pi,\varphi}$ is an into map. [Let $f \in CV_o(X,E)$ and let $K \subseteq X$ be a compact set. Let $\{x_\alpha : \alpha \in I\}$ be a net in $K$ such that $x_\alpha \to x$. Fix $G \in W$. Choose balanced $H \in W$ such that $H + H \subseteq G$. Let

$$B = \{f(\varphi(x_\alpha)) : \alpha \in I\},$$

a neighborhood in $E$. Since $\pi : X \to CL_u(E)$ is continuous and $U(B,H)$ is a neighborhood of 0 in $CL_u(E)$, there exists $\alpha_o \in I$ such that

$$(\pi_{x_\alpha} - \pi_x)f(\varphi(x_\alpha)) \in H \text{ for all } \alpha \geq \alpha_o.$$ 

Again, since $\pi_x(f(\varphi(x))) \to \pi_x(f(\varphi(x)))$ in $E$, there exists $\alpha_o \in I$ such that

$$\pi(x)(f(\varphi(x_\alpha))) - \pi_x(f(\varphi(x))) \in H \text{ for all } \alpha \geq \alpha_o.$$ 

Let $\alpha_1 \in I$ be such that $\alpha_1 \geq \alpha_o$ and $\alpha_1 \geq \alpha_o$. Then, for every $\alpha \geq \alpha_1$, we have

$$\pi_{x_\alpha}(f(\varphi(x_\alpha))) - \pi_x(f(\varphi(x))).$$
prove that the set $S \in \{ j \}$ there exist $u$

Now, if we put $K$ a balanced $G$ $i$

X of

This proves that $W$

Again, condition (b)(ii) implies that for each $i$

Further, it implies that $x$ is impossible. Thus $x$

To show that $W$ is continuous, fix $v \in V$ and $G \in \mathcal{W}$. Then, by condition (i), there exist $u \in V$ and $H \in \mathcal{W}$ such that $u(\varphi(x))y \in H$
implies that \( v(x) \pi_x(y) \in G \) for all \( x \in X \) and \( y \in E \). We claim that \( W_{\pi,\varphi}(N(u,H)) \subseteq N(v,G) \). Let \( f \in CV_o(X,E) \) be such that \( f \in N(u,H) \). Then we have \( u(\varphi(x))f(\varphi(x)) \in H \) for all \( x \in X \). Further, it implies that \( v(x) \pi_x(f(\varphi(x))) \in G \) for all \( x \in X \). That is \( W_{\pi,\varphi}f \in N(v,G) \). This proves that \( W_{\pi,\varphi} \) is continuous at the origin in \( CV_o(X,E) \) and hence continuous on \( CV_o(X,E) \). Thus \( W_{\pi,\varphi} \) is a weighted composition operator on \( CV_o(X,E) \).

In the next theorem, we shall characterize the weighted composition operators \( W_{\pi,\varphi} \) on \( CV_o(X,E) \) induced by the mappings \( \pi : X \to E \) and \( \varphi : X \to X \), where \( E \) is a topological algebra.

**Theorem 9.2.2.** [MS98] Let \( X \) be a completely regular \( k_\mathbb{R} \)-space, and let \( E \) be a topological algebra with hypocontinuous multiplication and identity \( e \). Let \( \varphi : X \to X \) be a continuous map. Then the following conditions are equivalent:

(a) \( W_{\pi,\varphi} \) is a weighted composition operator on \( CV_o(X,E) \).

(b) (i) \( \pi : X \to E \) is continuous;
(ii) for any \( v \in V \) and \( G \in \mathcal{W} \) there exist \( u \in V \) and \( H \in \mathcal{W} \) such that \( u(\varphi(x))y \in H \) implies that \( v(x) \pi_x(y) \in G \) for all \( x \in X \), \( y \in E \);
(iii) for any \( v \in V \), \( G \in \mathcal{W} \) and compact set \( K \subseteq X \), the set \( \varphi^{-1}(K) \cap \{ x \in X : v(x) \pi_x(y) \notin G \} \) is compact for each \( y(\neq 0) \in E \).

**Proof.** (a) \( \Rightarrow \) (b) If \( W_{\pi,\varphi} \) is a weighted composition operator on \( CV_o(X,E) \), then we need to prove condition (b(i)), whereas the proof of conditions (b)(ii) and (b)(iii) can be obtained with slight modifications as in the proof of conditions (b)(i) and (b)(ii) of Theorem 9.2.1. Fix \( x_o \in X \). Then there exists \( f \in CV_o(X,E) \) such that \( f(\varphi(x_o)) \neq 0 \). Also, there exists an open neighborhood \( D \) of \( x_o \) in \( X \) such that \( f(\varphi(x)) \neq 0 \) for all \( x \in D \). Let \( h : D \to \mathbb{C} \) be defined as \( h(x) = (f(\varphi(x)))^{-1} \) for all \( x \in D \). Let \( g : X \to E \) be defined as \( g(x) = f(x)e \) for all \( x \in X \). Then clearly \( g \in CV_o(X,E) \) and therefore \( W_{\pi,\varphi}g \in CV_o(X,E) \). Now, since the mapping \( \pi \cdot g \circ \varphi \cdot h \) is continuous at \( x_o \), it follows that the mapping \( \pi \) is continuous at \( x_o \). This proves that the mapping \( \pi \) is continuous on \( X \).

(b) \( \Rightarrow \) (a) Suppose that conditions (b)(i) through (b)(iii) hold. Let \( f \in CV_o(X,E) \) and let \( K \subseteq X \) be a compact set. Fix \( x_o \in K \) and \( G \in \mathcal{W} \). Then there exists \( H \in \mathcal{W} \) such that \( H + H \subseteq G \). Let \( B_1 = f(\varphi(K)) \) and \( B_2 = \{ \pi(x_o) \} \). Then \( B_1, B_2 \in b(E) \). Since the multiplication in \( E \) is hypocontinuous, there exist \( H_1, H_2 \in \mathcal{W} \) such that \( H_1 B_1 \subseteq H \) and \( H_2 B_2 \subseteq H \). Since the map \( \pi \) is continuous at \( x_o \), there exists an open
neighborhood $D_1$ of $x_o$ in $K$ such that
\[ \pi(x) - \pi(x_o) \in H_1 \text{ for all } x \in D_1. \]
Again, since $f \circ \varphi$ is continuous at $x_o$, there exists an open neighborhood $D_2$ of $x_o$ in $K$ such that
\[ f(\varphi(x)) - f(\varphi(x_o)) \in H_2 \text{ for all } x \in D_2. \]
Let $D = D_1 \cap D_2$. Then, for any $x \in D$, we have
\[ \pi(x)f(\varphi) - \pi(x_o)f(\varphi(x_o)) = (\pi(x) - \pi(x_o)f(\varphi(x)) + \pi(x_o)(f(\varphi(x)) - f(\varphi(x_o))) \in H_1B_1 + H_2B_2 \subseteq H + H \subseteq G. \]
This proves that $\pi \cdot f \circ \varphi$ is continuous at $x_o$ in $K$ and hence on $X$. Thus $W_{\pi,\varphi}f \in C(X,E)$. Again with slight modifications in the proof of Theorem 9.2.1, it can be shown that $W_{\pi,\varphi}f \in CV_o(X,E)$ and $W_{\pi,\varphi}$ is continuous on $CV_o(X,E)$.

\[\text{Remark 9.2.3.}\]
In case $X$ is a completely regular $k_{\mathcal{R}}$-space and $E$ is a topological algebra with hypocontinuous multiplication not necessarily containing an identity $e$, then the continuous mappings $\pi: X \to E$ and $\varphi: X \to X$ induce the weighted composition operator $W_{\pi,\varphi}$ on $CV_o(X,E)$ iff conditions (b)(ii) and (b)(iii) of Theorem 9.2.2. are satisfied.

\[\text{Theorem 9.2.4.}\] [MS98] Let $X$ be a completely regular space and $E$ be a TVS, and let $\varphi: X \to X$ be a mapping. Then the following are equivalent:

(a) $\varphi$ induces the composition operator $C_\varphi$ on $CV_o(X,E)$;
(b) $\varphi$ induces the composition operator $C_\varphi$ on $CV_o(X,E)$ and $CV_o(X,E)$ is invariant under $C_\varphi$;
(c) (i) $\varphi$ is continuous,
(ii) for any $v \in V$ and $G \in \mathcal{W}$ there exist $u \in V$ and $H \in \mathcal{W}$ such that
\[ u(\varphi(x))y \in H \text{ implies that } v(x)y \in G \text{ for all } x \in X, y \in E, \]
(iii) for any $v \in V$, $G \in \mathcal{W}$ and compact set $K \subseteq X$, the set
\[ \varphi^{-1}(K) \cap \{x \in X : v(x)y \notin G\} \] is compact for each $y(\neq 0) \in E$.

\[\text{Proof.}\] (a) $\Rightarrow$ (c) In view of assumption (**), it follows that if $\varphi: X \to X$ is any function for which the corresponding composition map $C_\varphi$ induced on $CV_o(X,E)$ has its range contained in $C(X,E)$, then $\varphi$ is necessarily continuous. Hence (c)(i) holds.
To prove condition (c)(ii), we fix $v \in V$ and $G \in \mathcal{W}$. Since $C_\varphi$ is continuous on $CV_0(X, E)$, there exist $u \in V$ and $H \in \mathcal{W}$ such that

$$C_\varphi(N(u, H)) \subseteq N(v, G).$$

Let $\omega = 2u$. We claim that $\omega(\varphi(x))y \in H$ implies that $v(x)y \in G$ for all $x \in X$ and $y \in E$. Assume that for some $x_o \in X$ and $y_o \in E$, we have $\omega(\varphi(x_o))y_o \in H$ but $v(x_o)y_o \notin G$. Let $D = \{x \in X : \omega(x)y_o \in H\}$. Then $D$ is an open neighborhood of $\varphi(x_o)$ and therefore there exists $f \in CV_0(X)$ such that $0 \leq f \leq 1$, $f(\varphi(x_o)) = 1$ and $f(X \setminus D) = 0$. Define $g = 2f \otimes y_o$. Then $g \in N(u, H)$ and thereby $C_\varphi g \in N(v, G)$. Further, it implies that $v(x_o)y_o \notin G$, which is impossible. Thus condition c(ii) is satisfied.

To see condition (c)(iii), let $v \in V$ and $G \in \mathcal{W}$ and compact set $K \subseteq X$. Then there exists $f \in CV_0(X)$ such that $f(K) = 1$. Fix $y_o(\neq 0) \in E$. Define $g = f \otimes y_o$. Let

$$S = \{x \in X : v(x)g(\varphi(x)) \notin G\}$$

and

$$M = \varphi^{-1}(K) \cap \{x \in X : u(x)y_o \notin G\}.$$ 

Then obviously $S$ is a compact subset of $X$ and $M \subseteq S$. Thus $M$ is compact, being a closed subset of $S$. This proves condition (c).

(c) $\Rightarrow$ (b) To show that $C_\varphi$ is a composition operator on $CV_0(X, E)$, we fix $v \in V$ and $G \in \mathcal{W}$. By condition (c)(ii), there exist $u \in V$ and $H \in \mathcal{W}$ such that $u(\varphi(x))y \in H$ implies that $v(x)y \in G$, for all $x \in X$ and $y \in E$. Now we claim that $C_\varphi(N(u, H)) \subseteq N(v, G)$. [Let $f \in N(u, H)$. Then since $u(\varphi(x))f(\varphi(x)) \in H$ for all $x \in X$, it follows that $v(x)f(\varphi(x)) \in G$ for all $x \in X$. That is, $C_\varphi f \in N(v, G)$. This proves the continuity of $C_\varphi$ at the origin in $CV_0(X, E)$ and hence $C_\varphi$ is continuous on $CV_0(X, E)$.

To complete the proof of condition (b), it remains to show that $CV_0(X, E)$ is invariant under $C_\varphi$. [Fix $f \in CV_0(X, E)$. Let $v \in V$ and $G \in \mathcal{W}$. Consider the set $S = \{x \in X : v(x)f(\varphi(x)) \notin G\}$. Choose a balanced $G_1 \in \mathcal{W}$ with $G_1 + G_1 \subseteq G$. According to condition (c)(ii), there exist $u \in V$ and $H \in \mathcal{W}$ such that

$$u(\varphi(x))y \in H \text{ implies that } v(x)y \in G_1 \text{ for all } x \in X, y \in E.$$ 

Let $K = \{x \in X : u(x) f(x) \notin H\}$. Then $K$ is a compact subset of $X$ and $\varphi(S) \subseteq K$. Let $\lambda = \sup\{u(x) : x \in K\}$. Then obviously $\lambda > 0$. Since $f(\varphi(S))$ is precompact in $E$, there exists a finite set $\{f(\varphi(x_i))\}_{i=1}^n$
in \( f(\varphi(S)) \) such that
\[
f(\varphi(S)) \subseteq \bigcup_{i=1}^{n} \{ f(\varphi(x_i)) + \frac{1}{2\lambda} H \}.
\]
In view of condition (c)(iii), it follows that for each \( i \in \{1, 2, \ldots, n\} \), the set
\[
A_i = \varphi^{-1}(K) \cap \{ x \in X : v(x)f(\varphi(x_i)) \notin G_1 \}
\]
is compact. If we put \( A = \bigcup_{i=1}^{n} A_i \), then \( A \) is a compact subset of \( X \). Now, to show that \( S \subseteq A \), let \( y \in S \). Then there exists \( j \in \{1, 2, \ldots, n\} \) such that \( f(\varphi(y)) - f(\varphi(x_j)) \in \frac{1}{2\lambda} H \). Further, it implies that \( u(\varphi(y))[f(\varphi(y)) - f(\varphi(x_i))] \in \frac{1}{2} H \). By hypothesis, it follows that
\[
v(\varphi(y))[f(\varphi(y)) - f(\varphi(x_j))] \in G_1.
\]
Now, if \( v(y)(f(\varphi(x_i)) \in G_1, v(y)f(\varphi(y)) = v(y)[f(\varphi(y)) - f(\varphi(x_j))] + v(y)(f(\varphi(x_i)) \in G_1 + G_1 \subseteq G \), a contradiction (since \( y \in S \)). Hence \( v(y)(f(\varphi(x_i)) \notin G_1 \), showing that \( y \in A \). Thus \( S \), being a closed subset of \( A \), is compact. This proves that \( C_{\varphi}f \in CV_o(X, E) \).]
\( (b) \Rightarrow (a) \). This is clear.

**Remark 9.2.5.** If \( \varphi : X \to X \) is a surjective homeomorphism, then condition (c)(iii) is automatically satisfied and such a function will therefore induce a composition operator on \( CV_o(X, E) \) as soon as condition (c)(ii) is satisfied.
3. Compact Weighted Composition Operators

This section contains characterization of compact weighted composition operators and, in particular, of precompact, equicontinuous and bounded multiplication operators on the weighted space $CV_o(X, E)$, where $E$ is a non-trivial TVS.

As before, it is assumed that the Nachbin family $V$ satisfies $(*) V > 0$ and $(**)$ $CV_o(X)$ does not vanish on $X$.

**Theorem 9.3.1.** [MS98] Let $X$ be a locally compact space and $E$ a TVS. Let $V \subseteq S^+_0(X)$ and $\pi : X \to CL_u(E)$ and $\varphi : X \to X$ be mappings with $\varphi$ continuous, and suppose that $W_{\pi,\varphi}$ is a weighted composition operator on $CV_o(X, E)$ Then the following conditions are equivalent:

(a) $W_{\pi,\varphi}$ is a compact operator on $CV_o(X, E)$.

(b) (1) $\pi : X \to CL_u(E)$ is continuous.

(2) For every $x \in X$, $\pi_x := \pi(x)$ is a compact operator on $E$.

(3) $\varphi$ is locally constant on $X \setminus z(\pi)$, where $z(\pi) = \{x \in X : \pi(x) = 0\}$.

*Proof.* (a) $\Rightarrow$ (b) Suppose $W_{\pi,\varphi}$ is a compact operator on $CV_o(X, E)$. Then conditions (b)(1)-(3) hold, as follows.

(b) (1) Let $x_o \in X$, $B \in b(E)$, $G \in W$. Since $X$ is locally compact, choose a neighborhood $K_o$ of $\varphi(x_o)$ such that $cl-K_o$ is compact. Choose $f \in CV_o(X)$ such that $f(cl-K_o) = 1$. Let $D_o = \varphi^{-1}(K_o)$. For each $y \in B$, define $g_y = f \otimes y$. Then the set $M = \{g_y : y \in B\}$ is bounded in $CV_o(X, E)$. Further, it implies that the set $W_{\pi,\varphi}(M)$ is relatively compact in $CV_o(X, E)$. Now, by the Arzela-Ascoli Theorem 3.1.4, the set $W_{\pi,\varphi}(M)$ is equicontinuous at $x_o$. Thus there exists a neighborhood $D_1$ of $x_o$ such that

$$\pi_x(g_y(\varphi(x))) - \pi_x(g_y(\varphi(x_o))) \in G$$

for all $x \in D_1$, $y \in B$. Let $D = D_o \cap D_1$. Then we have

$$\pi_x(y) - \pi_x(y) \in G$$

for all $x \in D$, $y \in B$.

That is,

$$\pi_x - \pi_x \in N(B, G)$$

for all $x \in D$.

This proves that $\pi : X \to CL_u(E)$ is continuous at $x_o$ and hence on $X$.

(b)(2) Let $x_o \in X$, $B \in b(E)$. We need to show that $\pi_{x_o}(B)$ is relatively compact in $E$. Choose $f \in CV_o(X)$ such that $f(\varphi(x_o)) = 1$. For each $y \in B$, define $g_y = f \otimes y$. Let $F = \{g_y : y \in B\}$. Then it follows that $W_{\pi,\varphi}(F)$ is relatively compact in $CV_o(X, E)$. So, by Theorem 3.1.4,
the set $W_{\pi,\varphi}(F)(x_o)$ is relatively compact in $E$. But
\[
W_{\pi,\varphi}(F)(x_o) = \{\pi_{x_o}(g_y(\varphi(x_o))) : y \in B\} = \\
\{\pi_{x_o}(f((\varphi(x_o))y) : y \in B\} = \pi_{x_o}(B);
\]
hence $\pi_{x_o}(B)$ is relatively compact in $E$. This proves that $\pi_{x_o}$ is a compact operator on $E$.

(b)(3) Suppose $\varphi$ is not locally constant on $X \setminus z(\pi)$. Then there exists $x_o \in X \setminus z(\pi)$ such that $\varphi$ is not locally constant on any open neighborhood $D$ of $x_o$. Let $B$ be any open base at $x_o$. The set $B$ can be ordered by the relation $D_1 \leq D_2$ if $D_1 \subseteq D_2$. Thus there exists a net $\{x_D : D \in B\}$ in $X$ such that $x_D \to x_o$ such that $\varphi(x_D) \neq \varphi(x_o)$. For each $D \in B$, we choose $f_D \in C_b(X)$ such that $f_D(\varphi(x_o)) = 1$ and $f_D(\varphi(x_D)) = 0$. Again choose $h \in CV_o(X)$ such that $h(\varphi(x_o)) = 1$. Let $y \in E$ be such that $\pi_{x_o}(y) \neq 0$. Then there exists $H \in W$ such that $\pi_{x_o}(y) \notin H$. For each $D \in B$, define $g_D = (h_D) \otimes y$. Since $CV_o(X)$ is a $C_b(X)$-module, it easily follows that the set $M = \{g_D : D \in B\}$ is bounded in $CV_o(X, E)$ and consequently the set $W_{\pi,\varphi}(M)$ is relatively compact in $CV_o(X, E)$. Again, by Theorem 3.1.4, the set $W_{\pi,\varphi}(M)$ is equicontinuous at $x_o$. Thus, for a given net $x_D \to x_o$ and $H \in W$, we have
\[
\sup\{\rho_H[\pi_{x_D}(g_D(\varphi(x_D))) - \pi_{x_o}(g_D(\varphi(x_o)))] : D \in B\} \to 0.
\]
That is, $\rho_H(\pi_{x_o}(y)) \to 0$, which is impossible because $\pi_{x_o}(y) \notin H$. This proves that $\varphi$ is locally constant on $X \setminus z(\pi)$.

(b) $\Rightarrow$ (a)

Suppose that the conditions (b)(1)-(3) hold. Let $F \subseteq CV_o(X, E)$ be a bounded set. We need to show that $W_{\pi,\varphi}(F)$ is compact in $CV_o(X, E)$.

It is enough to show that the set $cl-W_{\pi,\varphi}(F)$ satisfies conditions (i)-(iii) of the Arzela-Ascoli Theorem 3.1.4.

(i) Clearly, $cl-W_{\pi,\varphi}(F)$ is $\beta$-closed. Since the closure of a bounded set is bounded, it follows that $cl-W_{\pi,\varphi}(F)$ is $\beta$-bounded.

(ii) Let $x_o \in X$ and $B = \{f(\varphi(x_o)) : f \in F\}$. Then clearly the set $B$ is bounded in $E$. Since $\pi_{x_o}$ is a compact operator, it readily follows that
\[
cl-W_{\pi,\varphi}(F)(x_o) = cl-\pi_{x_o}(f(\varphi(x_o))) : f \in F\} = cl-\pi_{x_o}(B),
\]
which is relatively compact in $E$.

(iii) To show that $cl-W_{\pi,\varphi}(F)$ is equicontinuous on each compact subset of $X$, it suffices that $W_{\pi,\varphi}(F)$ is equicontinuous on each compact subset of $X$. Suppose $K$ is any compact subset of $X$. Let $x_o \in K$ and $G \in W$. Consider the set $S = \{f(\varphi(x)) : x \in X, f \in F\}$. Then $S$ is bounded in $E$. In case $x_o \in z(\pi)$, it is easy to see that $W_{\pi,\varphi}(F)$ is equicontinuous at $x_o$. Now suppose $x_o \notin z(\pi)$. Then, by condition (3),
there exists an open neighborhood $D_1(x_o)$ of $x_o$ such that $\varphi$ is constant on $D_1(x_o)$. Since $\pi : X \to CL_u(E)$ is continuous, there exists an open neighborhood $D_2(x_o)$ of $x_o$ such that

\[ \pi_x - \pi_{x_o} \in N(S,G) \text{ for all } x \in D_2(x_o). \]

Let $D = D_1(x_o) \cap D_2(x_o)$. Then $D$ is a neighborhood of $x_o$ in $K$ such that

\[ \pi_x(f(\varphi(x))) - \pi_{x_o}(f(\varphi(x_o))) \in G \text{ for all } x \in D \text{ and } f \in F. \]

So $W_{\pi,\varphi}(F)$ is equicontinuous at $x_o$ and hence on $K$. □

**Theorem 9.3.2.** [MS98] Let $X$ be a locally compact space and $E$ a Hausdorff topological algebra with hypocontinuous multiplication containing an identity $e$. Let $V \subseteq S_0^+(X)$ and $\pi : X \to CL_u(E)$ and $\varphi : X \to X$ be mappings with $\varphi$ continuous, and suppose that $W_{\pi,\varphi}$ is a weighted composition operator on $CV_o(X,E)$. Then the following conditions are equivalent:

(a) $W_{\pi,\varphi}$ is a compact weighted composition operator on $CV_o(X,E)$.

(b) (1) $\pi : X \to CL_u(E)$ is continuous;
(2) For every $x \in X$, $\pi_x := \pi(x)$ is a compact operator on $E$;
(3) $\varphi$ is locally constant on $X \setminus z(\pi)$.

Proof. The proof can be obtained by slight modification in the proof of Theorem 9.2.2 and Theorem 9.3.1. □

**Corollary 9.3.3.** [MS98] Let $X$ be a locally compact space and $E$ a noncompact TVS, and let $V \subseteq S_0^+(X)$. Then there is no compact composition operator on $CV_o(X,E)$.

Proof. We first note if $\pi : X \to CL_u(E)$ is the map given by $\pi(x) = I$ for all $x \in X$, then the weighted composition operator $W_{\pi,\varphi}$ becomes the composition operator $C_\varphi$. Since $E$ is an infinite dimensional TVS, the identity operator $I$ on $E$ is not compact. Consequently, the condition (b) of Theorem 9.3.2 cannot hold. □

**Remarks.** In Theorems 9.3.1 and 9.3.2, if we take $U$ as a system of weights on $X$ such that $U \subseteq V$, then the results hold for the space $CU_o(X,E)$.

(ii) Theorems 9.3.1 and 9.3.2 make sure that there are non-zero compact weighted composition operator on these spaces of continuous functions, where as it is not the case with $L_p$-spaces. However, there need not be non-zero compact weighted composition operator on $L_p$-spaces (with non-atomic measures) [Tak92].

**Example.** Let $X = \mathbb{N}$ with the discrete topology, and let $E = C_b(\mathbb{R})$. Let $\pi : \mathbb{N} \to CL(E)$ be defined by $\pi(n) = \frac{1}{n} A_n$ for all $n \in \mathbb{N}$, where
$A_n : E \to E$ is defined by $A_n f(t) = f(n)$ for all $f \in E$ and $t \in \mathbb{R}$. Clearly, $\pi(n)$ is a compact operator on $\mathbb{R}$. Let $\varphi : \mathbb{N} \to \mathbb{N}$ be defined by $\varphi(n) = n^2, n \in \mathbb{N}$. Let $\omega : \mathbb{N} \to \mathbb{R}^+$ be defined by $\omega(n) = \frac{1}{n}, n \in \mathbb{N}$. Consider $V = \{\lambda \omega : \lambda \geq 0\}$. Now, it is easy to see that the mappings $\pi$ and $\varphi$ satisfy all the conditions of Theorem 9.3.1 and therefore $W_{\pi, \varphi}$ is a compact weighted composition operator on $CV_b(X, E)$.

We next consider the particular case of multiplication operators but characterize more general notions of equicontinuous, precompact and bounded multiplication operator $M_\pi$ on $CV_0(X, E)$ and $CV_b(X, E)$. Recall that a linear map $T : CV_b(X, E) \to CV_b(X, E)$ is said to be compact (resp. precompact, equicontinuous, bounded) if it maps some neighborhood in $CV_b(X, E)$ into a compact (resp. precompact, equicontinuous, bounded) subset of $CV_b(X, E)$.

**Lemma 9.3.4.** [Oub02, AAK09] For any $v \in V, G \in W$ and $x \in X$, 

$$\frac{1}{v(x)} = \sup \{\rho_G(f(x)) : f \in N(v, G)\}$$

$$= \sup \{\rho_G(f(x)) : f \in CV_b(X, E), \|f\|_{v, G} \leq 1\}. \quad (1)$$

**Proof.** Let $x \in X, v \in V, G \in W$. There exist $f \in CV_b(X, E)$ and $H \in W$ such that $(\rho_H \circ f)(x) = 1$. Choose $a \in E$ with $\rho_G(a) = 1$.

**Case I.** Suppose $v(x) = 0$. For each $n \geq 1$, set

$$U_n := \{y \in X : v(y) < \frac{1}{n} \text{ and } 1 - \frac{1}{n} < \rho_H(f(y)) < 1 + \frac{1}{n}\};$$

and consider $h_n \in C_b(X)$ with

$$0 \leq h_n \leq n, \quad h_n(x) = n, \quad \text{and } \text{supp}(h_n) \subseteq U_n.$$ 

Clearly, the function $g_n := \frac{n}{n+1} h_n \rho_H \circ f \otimes a \in CV_b(X, E)$ and

$$\|g_n\|_{v, G} = \sup \{v(y) \frac{n}{n+1} h_n(y) \rho_H(f(y)) \rho_G(a) : y \in X\}$$

$$\leq \frac{1}{n} \cdot \frac{n}{n+1} \cdot n \cdot (1 + \frac{1}{n}) \cdot 1 = 1,$$

hence,

$$\sup \{\rho_G(f(x)) : f \in CV_b(X, E), \|f\|_{v, G} \leq 1\} \geq \sup \{\rho_G(g_n(x)) : n \in \mathbb{N}\}$$

$$= \sup \{\frac{n}{n+1} h_n(x) \rho_H(f(x)) \rho_G(a) : n \in \mathbb{N}\}$$

$$= \sup \{\frac{n}{n+1} \cdot n \cdot 1 : n \in \mathbb{N}\} = \infty = \frac{1}{v(x)}.$$
Case II. Suppose \( v(x) \neq 0 \). For \( n > \frac{1}{v(x)} \), let

\[
F_n := \{ y \in X : \frac{v(x)}{v(x) + \frac{1}{2n}} \leq \rho_H(f(y)) \leq \frac{v(x)}{v(x) - \frac{1}{2n}} \},
\]

\[
U_n := \{ y \in X : \frac{v(x)}{v(x)} < \rho_H(f(y)) < \frac{v(x)}{v(x) - \frac{1}{n}} \}.
\]

Then choose \( h_n, k_n \in C_b(X) \) with

\[
0 \leq h_n \leq \frac{1}{v(x) + \frac{1}{n}}, \quad h_n(x) = \frac{1}{v(x) + \frac{1}{n}} \text{ on } F_n, \text{ and } \supp(h_n) \subseteq U_n,
\]

\[
0 \leq k_n \leq 1, \quad k_n(x) = 1, \quad \text{and } \supp(k_n) \subseteq J_n := F_n \cap \{ y \in X : v(y) < v(x) + \frac{1}{n} \}.
\]

The function \( g_n := \frac{v(x) - \frac{1}{n}}{v(x)} h_n k_n \rho_H \circ f \otimes a \in CV_b(X, E) \) and

\[
\| g_n \|_{v,G} = \sup \{ v(y) \rho_G(g_n(y)) : y \in X \}
\]

\[
= \sup \{ v(y) \frac{v(x) - \frac{1}{2n}}{v(x)} h_n(y) k_n(y) \rho_H(f(y)) \rho_G(a) : y \in X \}
\]

\[
= \sup \{ v(y) \frac{v(x) - \frac{1}{2n}}{v(x)} h_n(y) k_n(y) \rho_H(f(y)) \rho_G(a) : y \in J_n \}
\]

\[
< (v(x) + \frac{1}{2n}) \frac{v(x) - \frac{1}{2n}}{v(x)} + \frac{1}{v(x) + \frac{1}{2n}} \cdot \frac{1}{v(x) - \frac{1}{2n}} = 1;
\]

hence

\[
\sup \{ \rho_G(f(x)) : f \in CV_b(X, E), \| f \|_{v,G} \leq 1 \}
\]

\[
\geq \sup \{ \rho_G(g_n(x)) : n \in \mathbb{N} \}
\]

\[
= \sup \{ \frac{v(x) - \frac{1}{2n}}{v(x)} \cdot \frac{1}{v(x) - \frac{1}{2n}} \cdot 1 \cdot 1 : n \in \mathbb{N} \} = \frac{1}{v(x)}.
\]

On the other hand,

\[
\sup \{ \rho_G(f(x)) : f \in CV_b(X, E), \| f \|_{v,G} \leq 1 \}
\]

\[
= \frac{1}{v(x)} \sup \{ v(y) \rho_G(f(y)) : f \in CV_b(X, E), \| f \|_{v,G} \leq 1 \}
\]

\[
\leq \frac{1}{v(x)} \sup_{y \in X} \{ \sup v(y) \rho_G(f(y)) : f \in CV_b(X, E), \| f \|_{v,G} \leq 1 \}
\]
\[\frac{1}{v(x)} \sup \{\|f\|_{v,G} : f \in CV_b(X, E), \|f\|_{v,G} \leq 1\} = \frac{1}{v(x)}\]

Thus, in each case, (1) holds. \(\square\)

Recall that if \(A \subseteq X\), then a point \(x \in A\) is called an \textit{isolated point} of \(A\) if \(x\) is not a limit point of \(A\), i.e. if there exists a neighborhood \(U\) of \(x\) in \(X\) such that \(U \cap A = \{x\}\). Consequently, if \(X\) has no isolated point, then, for each \(x \in X\),

**Theorem 9.3.5.** [Oub02, AAK09] Let \(\pi : X \rightarrow CL(E)\) be a map such that \(M_{\pi}(CV_b(X, E)) \subseteq C(X, E)\) and suppose \(X\) has no isolated points. If \(M_{\pi}\) is equicontinuous on \(CV_b(X, E)\), then \(M_{\pi} = 0\).

**Proof.** Suppose \(M_{\pi}\) is equicontinuous on \(CV_b(X, E)\) but \(M_{\pi}(f_o) \neq 0\) for some \(f_o \in CV_b(X, E)\). Then there exists \(x_o \in X\) with \(\pi_{x_o}(f_o(x_o)) \neq 0\). Since \(M_{\pi}\) is equicontinuous, there exist some \(v \in V\) and balanced \(G \in W\) such that \(M_{\pi}(N(v, G))\) is equicontinuous on \(X\) and in particular at \(x_o\). We may assume that \(f_o \in N(v, G)\) (since \(N(v, G)\) is absorbing). Hence, for every balanced \(H \in W\), there exists a neighborhood \(D\) of \(x_o\) in \(X\) such that

\[\pi_{y}(f(y)) - \pi_{x_o}(f(x_o)) \in H\] for all \(y \in D\) and \(f \in N(v, G)\).

Since \(x_o\) is not isolated, there exists some \(y \in D\) with \(y \neq x_o\). Choose then \(g_y \in C_b(X)\) with \(0 \leq g_y \leq 1\), \(g_y(y) = 0\), and \(g_y(x_o) = 1\). Then \(g_y f_o \in N(v, G)\) and so

\[\pi_{y}(g_y f_o(y)) - \pi_{x_o}(g_y f_o(x_o)) \in H;\]

that is, \(\pi_{x_o}(f_o(x_o)) \in H\). Since \(H \in W\) is arbitrary and \(E\) is Hausdorff (i.e. \(\bigcap_{U \in W} U = \{0\}\)), we have \(\pi_{x_o}(f_o(x_o)) = 0\). This is the desired contradiction. \(\square\)

**Corollary 9.3.6.** [Oub02, AAK09] Let \(\pi : X \rightarrow CL(E)\) be a map such that \(M_{\pi}(CV_b(X, E)) \subseteq C(X, E)\). If \(X\) is a \(V_{\mathbb{R}}\)-space without isolated points, then \(M_{\pi}\) is precompact iff \(M_{\pi} = 0\).

**Proof.** Suppose \(M_{\pi}\) is precompact. Then, by Theorem 3.2.1, \(M_{\pi}\) is equicontinuous. Hence, by Theorem 9.3.5, \(M_{\pi} = 0\). The converse is trivial. \(\square\)

**Theorem 9.3.7.** [Oub02, AAK09] Let \(\pi : X \rightarrow CL(E)\) be such that \(M_{\pi}(CV_b(X, E)) \subseteq C(X, E)\). Then \(M_{\pi}\) is a bounded multiplication operator on \(CV_b(X, E)\) iff there exist \(v \in V\) and \(G \in W\) such that for any \(u \in V\), \(H \in W\), there exists \(\lambda > 0\) such that

\[\lambda u(x)a \in G\] implies that \(u(x)\pi_{x}(a) \in H\), \(x \in X\), \(a \in E\),
or equivalently

\[ u(x)\rho_H(\pi_x(a)) \leq \lambda v(x)\rho_G(a), \ x \in X, \ a \in E. \]  

(A)

Proof. Similar to that of Theorem 9.1.1. □
4. Notes and Comments

Section 9.1. Singh and Summers [SS88] have studied the notion of composition operators on $CV_o(X, \mathbb{C})$. Later, Singh and Manhas [SM91, SM92] made an analogous study of multiplication operators $M_\theta, M_\psi$ and $M_\pi$ from $CV_b(X, E)$ (resp. $CV_o(X, E)$) into itself, assuming $E$ a locally convex space or a locally $m$-convex algebra. These operators are induced by certain maps $\theta : X \to \mathbb{C}$ and $\psi : X \to E$ and $\pi : X \to CL(E)$. This section contains results from the papers [KT97, KT02, MS98] which generalize the results of above authors to the case when $E$ is a TVS or topological algebra which is not necessarily locally convex. (See [SK97, Oub05]).

Section 9.2. This section contains characterizations of the more general notion of weighted composition operators $W_{\pi,\varphi}$ from $CV_b(X, E)$ (resp. $CV_o(X, E)$) into itself, due to Manhas and Singh [MS98]. Here $\varphi : X \to X$ and $\pi : X \to CL(E)$.

Section 9.3. A systematic study of compact weighted composition operators on spaces of continuous functions was initiated by Kamowitz [Kam81] on the Banach algebra $C(X)$ for a compact Hausdorff space $X$. This study has been further extended to function algebra setting and Banach lattices by Feldman [Fel90], Singh and Summers [SS87] and Takagi [Tak88]. Jamison and Rajagopalan [JR88] and Takagi [Tak91] have further generalized these results to the spaces $C(X, E)$. Finally, in [MS97] Manhas and Singh have obtained a study of compact and weakly compact weighted composition operators on the weighted locally convex spaces $CV_o(X, E)$, where $V \subseteq S^+_o(X)$ is a Nachbin family on $X$ and $E$ is a quasi-complete locally convex space. This section contains characterization of compact Weighted Composition Operators $W_{\pi,\varphi}$ and also of precompact, equicontinuous and bounded multiplication operators on the non-locally convex weighted spaces $CV_o(X, E)$, as given in [MS98, Oub02, AAK09] for $E$ a non-trivial TVS. More general results for operators $W_{\pi,\varphi}$ from $CV_b(X, E)$ into $CU_b(Y, E)$ can be found in [Oub05]; here $X$ and $Y$ are completely regular spaces and $\varphi : Y \to X$ and $\pi : Y \to CL(E)$ are the corresponding maps maps.
The General Strict Topology

The general strict topology was defined by Busby in [Bus68] in the case of $Y$ a $C^*$-algebra and $A$ a closed two-sided ideal in $Y$ as the one given by the seminorms $x \to \max\{\|ax\|, \|xa\|\}$ for $a \in A, x \in Y$. Independently, this topology was studied in greater detail and in a more general setting of $Y$ a Banach left $A$-module by Sentilles and Taylor in an interesting paper [ST69] as the one given by the seminorms $x \to \|ax\|$ for $a \in A, x \in Y$. More recently, Khan-Mohammad-Thaheem [KMT05] have considered the case of $Y$ a locally idempotent $F$-algebra which contains $A$ as a closed two-sided ideal, while Shantha [Shan04, Shan05] has taken $Y$ a locally convex module over a locally convex algebra $A$.

In this chapter, we begin with the study of general strict and related topologies on a topological left $A$-module $Y$, where $Y$ is Hausdorff TVS and $A$ a Hausdorff topological algebra having a two-sided bounded approximate identity. These results were originally given in the papers [ST69, Shan04, KMT05, SumW72] in particular cases and extended recently in [Kh07, Kh08]. We shall also consider essentiality of topological left $A$-modules $(C_b(S, E), u)$ and $(C(S, E), k)$. Finally, we study the properties of the topological module $\text{Hom}_A(A, Y)$ of continuous homomorphisms.
1. Strict Topology on Topological Modules

Recall that if $Y$ is a TVS and $A$ be a topological algebra, both over
the same field $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$), then $Y$ is called a topological left $A$-module
if it is a left $A$-module and the module multiplication $(a,x) \rightarrow a.x$ from
$A \times Y$ into $Y$ is separately continuous. If $b(A)$ (resp. $b(Y)$) denote the
collection of all bounded sets in $A$ (resp. $Y$), then module multiplication
$(a,x) \rightarrow ax$ is called $b(A)$- (resp. $b(Y)$-) hypocontinuous if, given any
neighborhood $G$ of 0 in $Y$ and any $D \in b(A)$ (resp. $B \in b(Y)$), there
exists a neighborhood $H$ of 0 in $Y$ (resp. $V$ of 0 in $A$) such that $D.H \subseteq G$
(resp. $V.B \subseteq G$). We mention that if $(Y,\tau)$ is an ultrabarrelled topological
left $A$-module, then the module multiplication is $b(A)$-hypocontinuous.
(See section A.6).

Definition. Let $A$ be a topological algebra and $(Y,\tau)$ a TVS which
is a topological left $A$-module, and let $\mathcal{W}$ be a base of neighborhoods
of 0 in $Y$. For any bounded set $D \subseteq A$ and $G \in \mathcal{W}$, we set

$$M(D,G) = \{ x \in Y : D.x \subseteq G \}.$$ 

The uniform topology $u = u_A$ (resp. strict topology $\beta = \beta_A$) on $Y$
is defined as the linear topology which has a base of neighborhoods of 0
consisting of all sets of the form $M(D,G)$, where $D$ is a bounded (resp.
finite) subset of $A$ and $G \in \mathcal{W}$. If $A$ has a two-sided approximate identity
$\{ e_\lambda : \lambda \in I \}$, the compact-open topology $k$ on $Y$ is defined just as above
with $D$ being a finite subset of $\{ e_\lambda : \lambda \in I \}$.

As mentioned earlier, $\beta$ was introduced by Sentilles and Taylor ([ST69],
p.145) in the case of $Y$ a left Banach $A$-module as the one given by the
seminorms $x \rightarrow \| a.x \|$ for $a \in A, x \in Y$.

As a classical example, if $A = C_o(S)$ and $Y = C_b(S,E)$ with $S$ a
locally compact space and $E$ a TVS and $Y$ endowed with $\tau = u$, the
uniform topology and $A$ endowed with the sup-norm topology and has
an approximate identity. In this particular case, $\beta$ is the strict topology
and $k$ is the topology of uniform convergence on compact subsets of $S$.

Another instance of the above is that $A$ is a topological algebra and
$Y = CM_d(A)$, the set of all double multipliers $(S,T)$, where $A$ is a
topological algebra and $S,T : A \rightarrow A$ are continuous linear mappings
satisfying $aS(b) = T(a)b$ for all $a,b \in A$ [Jo64, KMT99]. We shall
consider more cases of essential topological left $A$-modules $(C_b(S,E),u)$
and $(C(S,E),k)$ in the later part of this section.

In the sequel, we shall assume, unless stated otherwise, that $Y$ is a
topological left $A$-module, where $Y = (Y,\tau)$ is a Hausdorff TVS with
1. Strict Topology on Topological Modules

Let \( W \) a base of neighborhoods of 0 and \( A \) a Hausdorff topological algebra with \( \{e_\lambda : \lambda \in I\} \) a bounded two-sided approximate identity. We shall also assume that \( A \) is faithful in \( Y \) in the sense that, for any \( x \neq 0 \) in \( Y \), \( a.x = 0 \) for all \( a \in A \) implies that \( x = 0 \) (cf. [Jo64, ST69]). In some results, the assumption of ultrabarrelledness on \( A \) or \( Y \) is required essentially to apply the uniform boundedness principle.

**Theorem 10.1.1.** [Kh07] Let \((Y, \tau)\) be a topological left \( A \)-module with \( A \) having a two-sided approximate identity \( \{e_\lambda : \lambda \in I\} \). Then:

(a) \( k \leq \beta \leq u \).
(b) \( k, \beta, \) and \( u \) are Hausdorff.
(c) For each \( x \in Y \), \( e_\lambda x \xrightarrow{\beta} x \).
(d) \( \bigcup_{\lambda \in I} e_\lambda Y \) is \( \beta \)-dense in \( Y \); in particular, \( Ye \) is \( \beta \)-dense in \( Y \).

If, in addition the module multiplication is \( b(A) \)-hypocontinuous, then:

(e) \( u \leq \tau \).
(f) \((Y, k), (Y, \beta) \) and \((Y, u) \) are topological \( A \)-modules.

**Proof.** (a) Since \( \{e_\lambda : \lambda \in I\} \subseteq A \), it follows that \( k \leq \beta \). Since every finite set in \( A \) is bounded, we have \( \beta \leq u \).

(b) We only need to show that \( k \) is Hausdorff. Let \( x \in \bigcap_{\lambda \in I} \bigcap_{G \in W} M(e_\lambda, G) \). Since \( \tau \) is Hausdorff, \( \bigcap_{G \in W} G = \{0\} \) and so \( e_\lambda x = 0 \) for all \( \lambda \in I \). Then, for any \( a \in A \), \( ae_\lambda \to a \) in \( A \) and so, by separate continuity of module multiplication,

\[
a.x = \lim_{\lambda} (ae_\lambda).x = \lim_{\lambda} a.(e_\lambda x) = 0.
\]

Since \( A \) is faithful in \( Y \), we have \( x = 0 \). Thus \((Y, k)\) is Hausdorff.

(c) Let \( x \in Y \), and let \( D \) be a finite set in \( A \) and \( G \in W \). Since \( ae_\lambda \to a \) in \( A \) for all \( a \in D \) and \( D \) is finite, by separate continuity of module multiplication, there exists \( \lambda_0 \in I \) such that for all \( \lambda \geq \lambda_0 \) and \( a \in D \)

\[
(ae_\lambda - a).x \in G, \quad \text{or} \quad a.(e_\lambda x - x) \in G.
\]

Hence \( e_\lambda x - x \in M(D, G) \) for all \( \lambda \geq \lambda_0 \). So \( e_\lambda x \xrightarrow{\beta} x \).

(d) Let \( x \in Y \). Since \( \{e_\lambda x : \lambda \in I\} \subseteq \bigcup_{\lambda \in I} e_\lambda Y \) and, by (c), \( e_\lambda x \xrightarrow{\beta} x \), we have \( x \in \text{\beta-cl}(\bigcup_{\lambda \in I} e_\lambda Y) \). Thus \( \bigcup_{\lambda \in I} e_\lambda Y \) is \( \beta \)-dense in \( Y \).

(e) We need to verify that, for any net \( \{x_\alpha : \alpha \in J\} \) in \( Y \), \( x_\alpha \xrightarrow{\tau} 0 \) implies that \( x_\alpha \xrightarrow{u} 0 \). Let \( D \) be a bounded set in \( A \) and \( G \in W \). By \( b(A) \)-hypocontinuity, choose balanced \( H \in W \) such that \( DH \subseteq G \). Since
$D$ is bounded, there exists $r > 1$ such that $D \subseteq rV$. Since $x_\alpha \xrightarrow{\tau} 0$, there exists $\alpha_o \in J$ such that

$$x_\alpha \in H \text{ for all } \alpha \geq \alpha_o.$$ 

Then, for any $a \in D$ and $\alpha \geq \alpha_o$,

$$a.x_\alpha \in D.H \subseteq G;$$

that is, $x_\alpha \in M(D,G)$ for all $\alpha \geq \alpha_o$. Hence $x_\alpha \xrightarrow{u} 0$, and so $u \leq \tau$.

(f) We prove the result only for $(Y,u)$ as the proofs for $(Y,\beta)$ and $(Y,k)$ are similar to it. First, let $\{b_\alpha : \alpha \in J\} \subseteq A$ with $b_\alpha \to 0$ and fix $x(\neq 0)$ in $Y$. To show that $b_\alpha x \xrightarrow{u} 0$ in $Y$, let $D$ be a bounded subset of $A$ and $G \in \mathcal{W}$. By $b(A)$-hypocontinuity, choose balanced $H \in \mathcal{W}$ such that $D.H \subseteq G$. Since $b_\alpha \to 0$ in $A$, by separate continuity of module multiplication, $b_\alpha.x \xrightarrow{\tau} 0$ in $Y$ and so there exists $\alpha_o \in J$ such that

$$b_\alpha.x \in H \text{ for all } \alpha \geq \alpha_o.$$ 

Hence, for any $a \in D$ and $\alpha \geq \alpha_o$, $ab_\alpha.x \in D.H \subseteq G$; that is, $b_\alpha.x \in M(D,G)$, for all $\alpha \geq \alpha_o$. Thus $b_\alpha.x \xrightarrow{u} 0$.

Next, let $\{x_\alpha : \alpha \in J\} \subseteq Y$ with $x_\alpha \xrightarrow{u} 0$ and fix $a(\neq 0)$ in $A$. Let $D$ be a bounded set in $A$ and $G \in \mathcal{W}$. Since the map $R_\alpha : A \to A$ given by $R_\alpha(b) = ba, b \in A,$ is linear and continuous (by separate continuity of multiplication in $A$), it follows that $Da$ is bounded in $A$. Since $x_\alpha \xrightarrow{u} 0$, there exists $\alpha_o \in I$ such that for all $\alpha \geq \alpha_o$

$$x_\alpha \in M(Da,G), \text{ or } a.x_\alpha \in M(D,G).$$

Hence $a.x_\alpha \xrightarrow{u} 0$ in $Y$. Consequently, $(Y,u)$ is a left semitopological $A$-module.

\textbf{Theorem 10.1.2.} [Kh07] Let $(Y,\tau)$ be a topological left $A$-module with $b(Y)$-hypocontinuous module multiplication and $A$ having a two-sided approximate identity $\{e_\lambda : \lambda \in I\}$. Then:

(p) $k = \beta$ on $\tau$-bounded sets.

(q) If a sequence $\{x_\alpha\} \subseteq Y$ is $\tau$-bounded and $k$-convergent, then it is $\beta$-convergent.

\textbf{Proof.} (p) Let $S$ be a $\tau$-bounded in $Y$, and let $\{x_\alpha : \alpha \in J\}$ be a net in $S$ with $x_\alpha \xrightarrow{k} x \in S$. Let $D$ be a finite set in $A$ and $G \in \mathcal{W}$. Choose balanced $H \in \mathcal{W}$ with $H + H + H \subseteq G$. By $b(Y)$-hypocontinuity, there exists a neighborhood $V$ of 0 in $A$ such that $V.S \subseteq H$. Since $D$ is finite, there exists $\lambda_o \in I$ such that
ae_{\lambda_0} - a \in V \text{ for all } a \in D.

Since \( x_\alpha \xrightarrow{k} x_o \), \( e_{\lambda_0}x_\alpha \xrightarrow{\tau} e_{\lambda_0}x_o \) and so, for each \( a \in D \), \( a.(e_{\lambda_0}x_\alpha) \xrightarrow{\tau} a.(e_{\lambda_0}x_o) \). Since \( D \) is finite, choose \( \alpha_o \in J \) such that

\[ a.(e_{\lambda_0}x_\alpha - e_{\lambda_0}x_o) \in H \text{ for all } \alpha \geq \alpha_o \text{ and } a \in D. \]

Then, for any \( a \in D \) and \( \alpha \geq \alpha_o \),

\[
(a.x_\alpha - x) = (a - ae_{\lambda_0}).x_\alpha + ae_{\lambda_o}.(x_\alpha - x) + (ae_{\lambda_o} - a).x
\in V.S + H + V.S \subseteq H + H + H \subseteq G;
\]

that is, \( x_\alpha - x \in M(D, G) \) for all \( \alpha \geq \alpha_o \). Hence \( x_\alpha \xrightarrow{\beta} x \), and so \( \beta \leq k \) on \( S \). Thus \( k = \beta \) on \( S \).

(q) Suppose that \( \{x_n\} \) is \( \tau \)-bounded and \( x_n \xrightarrow{k} x \in Y \). By (1), \( k = \beta \) on the \( \tau \)-bounded set \( \{x, x_n : n \geq 1\} \); hence \( x_n \xrightarrow{\beta} x \). □

**Lemma 10.1.3.** [Kh07] Let \( Y \) be a topological left \( A \)-module with \( A \) ultrabarralled. Let \( S \) be a \( \beta \)-bounded set in \( Y \). Then, for any bounded set \( D \subseteq A \), \( D.S \) is \( \tau \)-bounded in \( Y \).

**Proof.** For any \( x \in Y \), define \( R_x : A \rightarrow Y \) by \( R_x(a) = a.x, a \in A \). Clearly, each \( R_x \) is linear and also continuous (by separate continuity of module multiplication). We next show that \( \{R_x : x \in S\} \) is pointwise bounded in \( \text{CL}(A, Y) \). Let \( a \in A \) and \( G \in W \). Since \( S \) in \( \beta \)-bounded, there exists \( r > 0 \) such that \( S \subseteq rM(a, G) \). So

\[
\{R_x(a) : x \in S\} = \{a.x : x \in S\} \subseteq rG,
\]

showing that \( \{R_x : x \in S\} \) is pointwise bounded. Since \( A \) is ultrabarralled, by the principle of uniform boundness, \( \{R_x : x \in S\} \) is equicontinuous and hence uniformly bounded in \( \text{CL}(A, Y) \). Since \( D \) is bounded in \( A \), \( D.S = \bigcup_{x \in S} R_x(D) \) is \( \tau \)-bounded in \( Y \). □

**Theorem 10.1.4.** [Kh07] Let \( (Y, \tau) \) be a topological left \( A \)-module with \( A \) ultrabarralled. Then:

(i) \( \beta \) and \( u \) have the same bounded sets in \( Y \).

(ii) If \( (Y, \beta) \) is ultrabornological, then \( \beta = u \) on \( Y \).

If, in addition, \( A \) has a two-sided approximate identity \( \{e_\lambda : \lambda \in I\} \), then:

(iii) \( k = \beta \) on \( u \)-bounded sets.

(iv) A sequence \( \{x_n\} \subseteq Y \) is \( \beta \)-convergent iff it is \( u \)-bounded and \( k \)-convergent.
Proof. (i) Since $\beta \leq u$, every $u$-bounded set is $\beta$-bounded. Now, let $S$ be any $\beta$-bounded set in $Y$. By Lemma 10.1.3, for any bounded set $D \subseteq A$, $D.S$ is $\tau$-bounded in $Y$ or, equivalently, $S$ is $u$-bounded.

(ii) By (i), the identity map $i : (Y, \beta) \to (Y, u)$ takes bounded sets into bounded sets. Since $Y$ is ultrabornological, $i$ is continuous. Hence $u \leq \beta$.

(iii) Let $S$ be a $u$-bounded set in $Y$, and let $\{x_\alpha : \alpha \in J\} \subseteq S$ with $x_\alpha \to x_0 \in S$. Let $D$ be a finite set in $A$ and $G \in W$. Choose a balanced $H \in W$ with $H \subseteq G$. As in the proof of Lemma 10.1.3, by the principle of uniform boundedness, $\{R_x : x \in S\}$ is equicontinuous in $CL(A, Y)$. Hence there exists a neighborhood $V$ of 0 in $A$ such that $V.S \subseteq H$. Choose $\lambda_o \in I$ such that $ae_{\lambda_o} - a \in V$ for all $a \in D$.

Since $x_\alpha \to x_0$, $e_{\lambda_o}.x_\alpha \to e_{\lambda_o}.x_0$ and so by separate continuity, for each $a \in D$,

$$ae_{\lambda_o}.x_\alpha \to ae_{\lambda_o}.x_0.$$ 

Since $D$ is finite, choose $\alpha_o \in J$ such that $ae_{\lambda_o}.(x_\alpha - x_0) \in H$ for all $\alpha \geq \alpha_o$ and $a \in D$.

Hence, for any $a \in D$ and $\alpha \geq \alpha_o$,

$$a.(x_\alpha - x_0) = (a - ae_{\lambda_o}).x_\alpha + ae_{\lambda_o}.(x_\alpha - x_0) + (ae_{\lambda_o} - a).x_0 \in V.S + H + V.S \subseteq H + H \subseteq G.$$ 

So $x_\alpha \beta \to x_o$ and hence $\beta \leq k$ on $S$.

(iv) Let $\{x_n\} \subseteq Y$ with $x_n \beta \to x \in Y$. Then it is $\beta$-bounded and hence, by part (i), $u$-bounded. Further, since $k \leq \beta$, $x_n \to x$. Conversely, suppose that $\{x_n\}$ is $u$-bounded and $x_n \to x \in Y$. By (iii), $k = \beta$ on the $u$-bounded set $\{x, x_n : n \geq 1\}$; hence $x_n \beta \to x$. □

The next result characterizes the equivalence of bounded sets in the topologies $\beta, u, \tau$ on $Y$.

**Theorem 10.1.5.** [Kh07] Let $(Y, \tau)$ be a topological left $A$-module with $A$ ultrabarralled. Then the following are equivalent:

(a) $u$ and $\tau$ have the same bounded sets in $Y$.

(b) $\beta$ and $\tau$ have the same bounded sets in $Y$.

If the module multiplication is either $b(A)$-hypocontinuous or $b(Y)$ hypocontinuous, then each of (a) and (b) is equivalent to:

(c) The $\beta$-closure of a $\tau$-bounded set is $\tau$-bounded.
Proof. (a) ⇔ (b) since β and u have the same bounded sets by Theorem 10.1.4(i).

(b) ⇒ (c). Let S be a τ-bounded set in Y, and let $D = \{e_{\lambda} : \lambda \in I\}$. Then $D.S$ is τ-bounded in Y (cf. [Mal86], p. 28). [Indeed, let $G \in W$. By b(Y)-hypocontinuity, there exists a neighborhood $V$ of 0 in $A$ such that $V.S \subseteq G$. Since $D$ is a bounded set in $A$, there exists $r > 0$ such that $D \subseteq rV$. Hence $D.S \subseteq rV.S \subseteq rG$.] Further, by b(A)-hypocontinuity and Theorem 10.1.1((a), (e)), $\beta \leq \tau$ and so $D.S$ and hence $\beta$-cl$(D.S)$ is τ-bounded. By (b), $\beta$-cl$(D.S)$ is τ-bounded. Now, for any $x \in S, e_{\lambda}.x \xrightarrow{\beta} x$ (by Theorem 10.1.1(c)); since each $e_{\lambda}.x \in D.S$, we have $x \in \beta$-cl$(D.S)$. Hence $S \subseteq \beta$-cl$(D.S)$ and so $\beta$-cl$(S)$ is τ-bounded.

(c) ⇒ (a). Since the module multiplication is b(A)-hypocontinuous, by Theorem 10.1.1(e), $u \leq \tau$ and so every τ-bounded set is u-bounded. Now, let S be a u-bounded set in Y. Then, if $D = \{e_{\lambda} : \lambda \in I\}$, $D.S$ is τ-bounded. [Indeed, let $G \in W$. Since $D$ is a bounded set in $A$, $M(D,G)$ is a u-neighborhood of 0 in Y and so there exists $r > 0$ such that $S \subseteq rM(D,G) = M(D,rG)$; that is, $D.S \subseteq rG$.] By (c), $\beta$-cl$(D.S)$ is τ-bounded. But, again by Theorem 10.1.1(c), $S \subseteq \beta$-cl$(D.S)$; hence S is τ-bounded.

□

We now consider a characterization of the equivalence of the topologies $\beta, u$, and $\tau$ on Y.

Theorem 10.1.6. [Kh07] Let $(Y, \tau)$ be a topological left $A$-module with Y an F-space and A an F-algebra. Consider the following conditions:

(a) There exists $a \in A$ such that the map $x \rightarrow a.x$ is an isomorphism of Y onto Y.

(b) $\beta = \tau$ on Y.

(c) $(Y, \beta)$ is metrizable.

(d) $(Y,\beta)$ is ultrabornological.

(e) $\beta = u$ on Y.

(f) For each $x \in Y, e_{\lambda}.x \xrightarrow{\beta} x$.

Then (a) ⇒ (b) ⇒ (c) ⇒ (d) ⇒ (e); if $A$ has a two-sided approximate identity, (e) ⇒ (f). If $(Y,\tau)$ is a Banach algebra and $A$ has a bounded two-sided approximate identity, then (f) ⇒ (a); that is, (a) − (f) are equivalent:

Proof. (a) ⇒ (b). Suppose (a) holds, and let $x_{a} \xrightarrow{\beta} x$ in Y. Now $\phi : x \rightarrow a.x$ is a continuous, linear and one-one map from $(Y,\tau)$ onto $(Y,\tau)$. As a consequence of the open mapping theorem, the inverse map
\[ \phi^{-1} : a.x \to x \text{ is continuous. Since } a.x \xrightarrow{\tau} a.x, \text{ we have } \phi^{-1}(a.x) \xrightarrow{\tau} \phi^{-1}(a.x); \text{ that is, } x \xrightarrow{\tau} x. \text{ So } \tau \leq \beta. \]

(b) \Rightarrow (c). This is obvious since \((Y, \tau)\) is metrizable.

(c) \Rightarrow (d) since every metrizable TVS is ultrabornological.

(d) \Rightarrow (e). This follows from Theorem 10.1.4(ii).

(e) \Rightarrow (f). This follows from Theorem 10.1.1(c).

Now suppose that \((Y, \tau)\) is a Banach algebra and \(A\) has a bounded two-sided approximate identity. Then (f) \Rightarrow (a) follows from \([ST69]\), Theorem 2.4.

□

Remark. We can apply above results to extend an observation on inner derivation, due to Phillips \([Ph95]\). Let \((Y, \tau)\) be a topological algebra and \(A\) a closed two-sided ideal in \(Y\), with \(Y\) not necessarily an \(F\)-algebra. A linear map \(D : A \to A\) is called a derivation on \(A\) if

\[ D(ab) = D(a)b + aD(b), \quad a, b \in A. \]

For any \(x \in Y\), the map \(\delta_x : A \to A\) defined by \(\delta_x(a) = [x, a] = xa - ax, a \in A\), is a derivation on \(A\), called the inner derivation on \(A\). A derivation \(D : A \to A\) is called approximately inner if there exists a net \(\{a_\alpha : \alpha \in J\}\) in \(A\) such that \(D(a) = \lim \alpha \delta_{a_\alpha}(a)\) for all \(a \in A\). We claim that, for any \(x \in Y, \delta_x\) is approximately inner. Indeed, by Theorem 10.1.1(d), \(A\) is \(\beta\)-dense in \(Y\) and so there exists a net \(\{a_\alpha : \alpha \in J\}\) in \(A\) such that \(a_\alpha \xrightarrow{\beta} x\). Then, for any \(a \in A, \)

\[ \delta_x(a) = \lim_\alpha (a_\alpha a - aa_\alpha) = \lim_\alpha \delta_{a_\alpha}(a). \]

This was observed earlier in \([Ph95],\text{ p. 244}\) in the case of \(Y = M_d(A)\).

□

We now consider the notion of essential modules, due to Rieffel \([Rie67]\).

Definition. Let \((Y, \tau)\) be a topological left \(A\)-module with \(A\) having a left approximate identity \(\{e_\lambda\}\). The essential part \(Y_e\) of \(Y\) is defined as \(Y_e = \{x \in Y : e_\lambda x \xrightarrow{\tau} x\}\). We say that \(Y\) is essential if \(Y = Y_e\).

Clearly, \(Y_e\) is a topological left \(A\)-submodule of \(Y\). Note that, for any \(a \in A\) and \(x \in Y, e_\lambda(a.x) = (e_\lambda a).x \xrightarrow{\tau} a.x\) (by separate continuity of module multiplication) and so \(a.x \in Y_e\); hence \([A.Y] \subseteq Y_e\), where \([A.Y]\) is the vector subspace of \(Y\) spanned by \(A.Y = \{a.x : a \in A, x \in Y\}\). Further, for any \(x \in Y_e\), clearly \(\{e_\lambda x : \lambda \in I\} \subseteq A.Y\) and \(e_\lambda x \xrightarrow{\tau} x\) and so \(x \in \tau-cl[A.Y]\). Thus \([A.Y] \subseteq Y_e \subseteq \tau-cl[A.Y]\).
We can now state an analogue of factorization theorem 10.1.1 for topological modules, again due to Ansari-Piri [AP90]. A topological left $A$-module $Y$ is called $A$-factorable if, for each $x \in Y$, there exist $a \in A$ and $y \in Y$ such that $x = a \cdot y$.

**Theorem 10.1.7.** [AP90] Let $A$ be a fundamental $F$-algebra with a uniformly bounded left approximate identity. If $Y$ is an $F$-space which is an essential topological left $A$-module, then $Y$ is $A$-factorable.

We mention that if $Y$ is $A$-factorable, then $Y$ is essential since $Y = A \cdot Y \subseteq Y_e \subseteq Y$, or that $Y = Y_e$. Note that the above is a so-called left-hand version of the factorization theorem. Clearly its right-hand version also holds.

We next consider some properties of the essential part $Y_e$ of $Y$.

**Theorem 10.1.8.** [Kh07] Let $(Y, \tau)$ be a topological left $A$-module with $Y$ ultrabarralled and $A$ having a bounded left approximate identity $\{e_\lambda : \lambda \in I\}$. Then:

(i) $Y_e$ is $\tau$-closed in $Y$.

(ii) $Y_e = \tau$-cl $[A \cdot Y]$.

**Proof.** (i) Let $x \in \tau$-cl $(Y_e)$. We need to show that $e_\lambda \cdot x \overset{\tau}{\rightarrow} x$. Let $G \in \mathcal{W}$. Choose a balanced $H \in \mathcal{W}$ such that $H + H + H \subseteq G$. For each $\lambda \in I$, define $L_\lambda : Y \rightarrow Y$ by $L_\lambda(y) = L_{e_\lambda}(y) = e_\lambda \cdot y$, $y \in Y$. Since $D = \{e_\lambda : \lambda \in I\}$ is bounded in $A$, it follows from the proof of Lemma 10.1.3 that $\{L_\lambda : \lambda \in I\}$ is pointwise bounded and hence equicontinuous in $CL(Y, Y)$ by the principle of uniform boundedness. There exists $H_1 \in \mathcal{W}$ such that $L_\lambda(H_1) \subseteq H$ for all $\lambda \in I$. Since $x \in \tau$-cl $(Y_e)$, we can choose $x_o \in Y_e$ such that $x - x_o \in H_1 \cap H$.

Since $e_\lambda \cdot x_o \rightarrow x_o$, choose $\lambda_o \in I$ such that $e_\lambda \cdot x_o - x_o \in H$ for all $\lambda \geq \lambda_o$.

Hence, for any $\lambda \geq \lambda_o$,

$$e_\lambda \cdot x - x = e_\lambda \cdot (x - x_o) + (e_\lambda \cdot x_o - x_o) + (x_o - x)$$

$$\in L_\lambda(H_1 \cap H) + H + H_1 \cap H \subseteq H + H + H \subseteq G;$$

that is, $e_\lambda \cdot x \overset{\tau}{\rightarrow} x$ and so $x \in Y_e$. Hence $Y_e$ is $\tau$-closed.

(ii) As seen above, $[A \cdot Y] \subseteq Y_e \subseteq \tau$-cl $[A \cdot Y]$. Hence, by (i),

$Y_e = \tau$
Corollary 10.1.9. Let $Y$ and $A$ be as above. Then $Y$ is essential iff $Y = [A.Y]^\tau$.

Proof. By definition, $Y$ is essential if $Y = \{x \in Y : e_\lambda.x \xrightarrow{\tau} x\}$. Hence, by above theorem, $Y$ is essential iff $Y = [A.Y]^\tau$. □

We mention that if $A$ is a fundamental $F$-algebra with a uniformly bounded left approximate identity and $Y$ is an $F$-space which is an essential topological left $A$-module, then, by Theorem 10.1.7, $Y$ is $A$-factorable and so $Y = A.Y \subseteq Y_e \subseteq Y$, or that $Y = A.Y = Y_e$.

Following Rieffel ([Rie67], p. 454), we may also consider a version of the Theorem 10.1.8 for the $u$-topology on $Y$, as follows.

Theorem 10.1.10. [Kh07] Let $(Y,\tau)$ be an essential topological left $A$-module with $b(A)$-hypocontinuous module multiplication and $A$ having a bounded left approximate identity $\{e_\lambda : \lambda \in I\}$. Then $u-cl[A.Y] = \{x \in Y : e_\lambda.x \xrightarrow{u} x\}$.

Proof. Let $Y_1 = \{x \in Y : e_\lambda.x \xrightarrow{u} x\}$. For any $y \in Y_1$, each $e_\lambda.y \in A.Y_1 \subseteq A.Y$ and $e_\lambda.y \xrightarrow{u} y$, and so clearly $y \in u-cl[A.Y]$; hence $Y_1 \subseteq u-cl[A.Y]$. Now, let $x \in u-cl[A.Y]$. Then there exists a net $\{x_\alpha : \alpha \in J\}$ in $[A.Y]$ such that $x_\alpha \xrightarrow{u} x$. To show that $e_\lambda.x \xrightarrow{u} x$, let $D$ be a bounded subset of $A$ and $G \in \mathcal{W}$. Choose balanced $H \in \mathcal{W}$ such that $H + H + H \subseteq G$. By $b(A)$-hypocontinuity, choose $H_1 \in \mathcal{W}$ with $D.H_1 \subseteq H$. Put $D_1 = D \cup \{e_\lambda : \lambda \in I\}$, a bounded set in $A$. Since $x_\alpha \xrightarrow{u} x$, choose $\alpha_o \in J$ such that

\[ x_\alpha - x \in M(D_1, H_1 \cap H) \quad \text{for all } \alpha \geq \alpha_o. \]

Since $(Y,\tau)$ is essential, $e_\lambda.x_\alpha \xrightarrow{\tau} x_\alpha_o$ and so there exists $\lambda_o \in I$ such that

\[ e_\lambda.x_\alpha_o - x_\alpha_o \in H_1 \quad \text{for all } \lambda \geq \lambda_o. \]

Hence, for any $a \in D$ and $\lambda \geq \lambda_o$,

\[ a.(e_\lambda.x - x) = ae_\lambda.(x - x_\alpha_o) + a.(e_\lambda.x_\alpha_o - x_\alpha_o) + a.(x_\alpha_o - x) \]
\[ \leq a.(H_1 \cap H) + a.H_1 + H_1 \cap H \]
\[ \subseteq H + H + H \subseteq G; \]

that is, $e_\lambda.x \xrightarrow{u} x$, and so $x \in Y_1$. Thus $u-cl[A.Y] = Y_1$. □
2. Essentiality of Modules of Vector-valued Functions

We now present two results on essentiality of topological left $A$-modules of continuous vector-valued functions endowed with the uniform and compact-open type topologies. Recall that if $X$ be a Hausdorff topological space and $E$ a non-trivial Hausdorff TVS with $W_E$ a base of neighborhoods of 0 in $E$, then the uniform topology $u$ on $C_b(X,E)$ is the linear topology which has a base of neighborhoods of 0 consisting of all sets of the form

$$N(X,W) = \{ f \in C_b(X,E) : f(X) \subseteq W \},$$

where $W$ varies over $W_E$.

**Theorem 10.2.1.** [Kh07] Suppose $A$ is a topological algebra with a bounded approximate identity $\{ e_\lambda : \lambda \in I \}$ and $E$ a topological left $A$-modules such that the module multiplication is both $b(A)$-hypocontinuous and $b(E)$-hypocontinuous. Then:

(i) $(C_b(X,E),u)$ and $(C_o(X,E),u)$ are topological left $A$-modules with respect to the module multiplication $(a,f) \mapsto a.f$ as pointwise action:

$$(a.f)(x) = a.f(x), \ a \in A, \ f \in C_b(X,E), \ x \in X.$$  

(ii) If $X$ is locally compact, then $(C_o(X,E),u)$ is essential.

Proof. (i) First, we need to show that if $\{ a_\alpha : \alpha \in J \} \subseteq A$ with $a_\alpha \to 0$ and $f \not\equiv 0 \in C_b(X,E)$, then $a_\alpha.f \nrightarrow 0$ in $C_b(X,E)$. Let $G \in W_E$. Since $f(X)$ is bounded in $E$, by $b(E)$-hypocontinuity, there exists a neighborhood $V$ of 0 in $A$ such that $V.f(X) \subseteq G$. Since $a_\alpha \to 0$ in $A$, there exists $\alpha_0 \in J$ such that $a_\alpha \in U$ for all $\alpha \geq \alpha_0$. Hence

$$a_\alpha.f(X) \not\subseteq V.f(X) \subseteq G \text{ or } a_\alpha.f \notin N(X,G) \text{ for all } \alpha \geq \alpha_0.$$  

Next, let $a \not\equiv 0 \in A$ and $\{ f_\alpha : \alpha \in J \} \subseteq C_b(X,E)$ with $f_\alpha \nrightarrow 0$ in $C_b(X,E)$. Then $a.f_\alpha \nrightarrow 0$ in $C_b(X,E)$, as follows. Let $G \in W_E$. By $b(A)$-hypocontinuity, there exists an $H \in W_E$ such that $aH \subseteq G$. Since $f_\alpha \nrightarrow 0$ in $C_b(X,E)$, there exists an $\alpha_0 \in J$ such that $f_\alpha(X) \not\subseteq H$ for all $\alpha \geq \alpha_0$. Hence

$$a.f_\alpha \notin N(X,aH) \subseteq N(X,G) \text{ for all } \alpha \geq \alpha_0.$$  

(ii) Fix $f \in C_o(X,E)$. We need to show that $e_\lambda.f \nrightarrow f$. Let $G \in W_E$. Choose a balanced $H \in W_E$ with $H+H+H \subseteq G$. Since $D = \{ e_\lambda : \lambda \in I \}$ is bounded in $A$, by $b(A)$-hypocontinuity, there exists $V \in W_E$ such that

$$D.V \subseteq H.$$
Since \( f \in C_0(X,E) \), there exists a compact set \( K \subseteq X \) such that \( f(x) \in V \cap H \) for all \( x \in X \setminus K \).

Since \( f(K) \) is compact, there exists a finite set \( \{z_1, ..., z_n\} \subseteq K \) such that \( f(K) \subseteq \bigcup_{i=1}^n (f(z_i) + V \cap H) \).

Since \( E \) is an essential \( A \)-module, there exists \( \lambda_0 \in I \) such that \( e^\lambda f(z_i) - f(z_i) \in H \) for all \( \lambda \geq \lambda_0 \) and \( i = 1, ..., n \).

Now, let \( x \in X \) and \( \lambda \geq \lambda_0 \). First suppose \( x \in K \). Choose \( i \in \{1, ..., n\} \) such that \( f(x) - f(z_i) \in V \cap H \).

Then
\[
e^\lambda f(x) - f(x) = e^\lambda (f(x) - f(z_i)) + (e^\lambda f(z_i) - f(z_i)) + (f(z_i) - f(x)) \\
\in D.(V \cap H) + H - (V \cap H) \subseteq H + H + H \subseteq G.
\]

If \( x \in X \setminus K \),
\[
e^\lambda f(x) - f(x) \in e^\lambda(V \cap H) - V \cap H \subseteq e^\lambda V - H \subseteq H + H \subseteq G. \quad \square
\]

**Definition.** If \( X \) is a completely regular hemicompact space with \( \{K_n\} \) an increasing sequence of compact sets which form a base for compact sets on \( X \), we define the **compact-open type topology** \( k \) on \( C(X,E) \) as the linear topology which has a base of neighborhoods of 0 consisting of all sets of the form
\[
N(K_n, W) = \{ f \in C(X,E) : f(K_n) \subseteq W \},
\]
where \( n \geq 1 \) and \( W \in \mathcal{W}_E \).

**Theorem 10.2.2.** [Kh07] Let \( X,E \) and \( k \) be as in the above definition. Suppose \( A \) is a topological algebra with a bounded approximate identity \( \{e_\lambda : \lambda \in I\} \) and \( E \) a topological left \( A \)-module such that the module multiplication is both \( b(A) \)- and \( b(E) \)- hypcontinuous. Then:

(i) \( (C(X,E),k) \) is a topological left \( A \)-module with respect to the module multiplication \( (a,f) \rightarrow a.f \).

(ii) \( (C(X,E),k) \) is essential.

**Proof.** (i) First, we need to show that if \( \{b_\alpha : \alpha \in J\} \subseteq A \) with \( b_\alpha \rightarrow 0 \) and \( f \) (\( \neq 0 \)) in \( C(X,E) \), then \( b_\alpha.f \rightarrow 0 \) in \( C(X,E) \). Fix any \( K_m \in \{K_n\} \) and \( G \in \mathcal{W}_E \). By \( b(E) \)-hypcontinuity, there exists a neighborhood \( V \) of 0 in \( A \) with \( V.f(K_m) \subseteq G \). Since \( b_\alpha \rightarrow 0 \) in \( A \), there exists \( \alpha_o \in J \) such that
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\[ \alpha \in V \text{ for all } \alpha \geq \alpha_o. \]  

Hence, for any \( \alpha \geq \alpha_o, b_{\alpha}.f(K_m) \subseteq V.f(K_m) \subseteq G \) and so

\[ b_{\alpha}.f(K_m) \subseteq V.f(K_m) \subseteq G \quad \text{and so} \quad b_{\alpha}.f \in N(K_m, G). \]

Next, let \( b \neq 0 \in A \) and \( \{g_{\alpha} : \alpha \in J\} \subseteq C(X, E) \) with \( g_{\alpha} \overset{k}{\to} 0 \) in \( C(X, E) \). Then \( b.g_{\alpha} \overset{k}{\to} 0 \) in \( C(X, E) \), as follows. Fix any \( K_m \in \{K_n\} \) and \( G \in \mathcal{W}_E \). By \( b(A) \)-hypocontinuity, there exists \( H \in \mathcal{W}_E \) with \( b.H \subseteq G \).

Choose \( \alpha_o \in J \) such that \( g_{\alpha}(K_m) \subseteq H \) for all \( \alpha \geq \alpha_o \). Hence, for any \( \alpha \geq \alpha_o, b.g_{\alpha}(K_m) \subseteq b.H \subseteq G \) and so

\[ b.g_{\alpha} \in N(K_m, b.H) \subseteq N(X, G). \]

(ii) Fix \( f \in C(X, E) \). We need to show that \( e_{\lambda}.f \overset{k}{\to} f \). Let \( K_m \in \{K_n\} \) and \( G \in \mathcal{W}_E \). Choose a balanced \( H \in \mathcal{W}_E \) with \( H + H + H \subseteq G \).

Since \( D = \{e_{\lambda} : \lambda \in I\} \) is bounded in \( A \), by \( b(A) \)-hypocontinuity, there exists \( V \in \mathcal{W}_E \) such that

\[ D.V \subseteq H. \]

Since \( f(K_m) \) is compact, there exists a finite set \( \{z_1, ..., z_n\} \subseteq K_m \) such that

\[ f(K_m) \subseteq \bigcup_{i=1}^n (f(z_i) + V \cap H). \]

Since \( E \) is an essential \( A \)-module, there exists \( \lambda_o \in I \) such that

\[ e_{\lambda}.f(z_i) - f(z_i) \in H \quad \text{for all } \lambda \geq \lambda_o \quad \text{and} \quad i = 1, ..., n. \]

Now, let \( x \in K_m \) and \( \lambda \geq \lambda_o \). Choose \( i \in \{1, ..., n\} \) such that

\[ f(x) - f(z_i) \in V \cap H. \]

Then

\[ e_{\lambda}.f(x) - f(x) = e_{\lambda}.(f(x) - f(z_i)) + (e_{\lambda}.f(z_i) - f(z_i)) + (f(z_i) - f(x)) \]

\[ \in D.(V \cap H) + H + V \cap H \subseteq H + H + H \subseteq G. \]

□
3. Topological Modules of Homomorphisms

In this section, we shall study the properties of the topological module $\text{Hom}_A(A, Y)$ of continuous homomorphisms. These results were originally given in the papers [Rie67, ST69, Shan04] and in the present general form in [Kh08].

Definition: Let $E$ and $Y$ be topological left $A$-modules, where $E$ and $Y$ are TVSs and $A$ a topological algebra. Then a mapping $T: E \to Y$ is called an $A$-module homomorphism if $T(a.x) = a.T(x)$ for all $a \in A$ and $x \in E$ ([Rie67], p. 447). (Similarly, if $E$ and $Y$ are right $A$-modules, then we can define an $A$-module homomorphism as a mapping $T: E \to Y$ satisfying $T(x.a) = T(x).a$ for all $a \in A$ and $x \in E$. We will state results for left modules over $A$, similar results holding, of course, for right modules.)

Our main interest is the study of $A$-module homomorphism from $A$ into $Y$. 

Lemma 10.3.1. [Kh08a] Let $Y$ be a left $A$-module. Suppose that $A$ is faithful in $Y$. Then any $A$-module homomorphism $T: A \to Y$ is homogeneous (that is, $T(\lambda a) = \lambda T(a)$ for all $\lambda \in \mathbb{K}$ and $a \in A$).

Proof. Let $a \in A$ and $\lambda \in \mathbb{K}$. Then, for any $c \in A$,

$$c.T(\lambda a) = T(c. (\lambda a)) = T((\lambda c) a) = (\lambda c). T(a) = c. \lambda T(a)$$

Since $A$ is faithful in $Y$, $T(\lambda a) = \lambda T(a)$. \hfill $\Box$

We now establish the linearity and continuity of an $A$-module homomorphisms using the factorization theorem. The proof of the following theorem is analogous to those given in [Jo66, Cra69, Rie69, SumM72, KMT99].

Theorem 10.3.2. [Kh08a] Let $Y$ be a topological left $A$-module with $Y$ metrizable and $A$ strongly factorable. Then any $A$-module homomorphism $T: A \to Y$ is linear and continuous.

Proof. To show that $T$ is linear, let $a_1, a_2 \in A$ and $\alpha, \beta \in \mathbb{K}$. If we take $\{a_n\} = \{a_1, a_2, 0, 0, ...\}$, then clearly $a_n \to 0$; since $A$ is strongly factorable, there exist $b, c_1, c_2 \in A$ such that $a_1 = c_1 b, a_2 = c_2 b$. So

$$T(\alpha a_1 + \beta a_2) = T((\alpha c_1 + \beta c_2)b) = T((\alpha c_1 + \beta c_2)) T(b) = \alpha T(c_1 b) + \beta T(c_2 b) = \alpha T(a_1) + \beta T(a_2);$$

hence $T$ is linear. Since $Y$ is metrizable, to show that $T$ is continuous, it suffices to show that if $\{a_n\} \subseteq A$ with $a_n \to 0$, then $T(a_n) \to 0$. Using
again the strong factorability of $A$, we can write $a_n = c_n b$, where $b \in A$ and $\{c_n\} \subseteq A$ with $c_n \to 0$. Then

$$T(a_n) = T(c_n b) = c_n T(b) \to 0. T(b) = 0.$$  
(by the separate continuity of module multiplication). Thus $T$ is continuous. \hfill \square

**Definition.** (Cf. [Rie67], p. 447) Let $E$ and $Y$ be topological left $A$-modules, where $E$ and $Y$ are TVSs and $A$ a topological algebra. Let $\text{Hom}_A(E, Y)$ denote the vector space of all continuous linear left $A$-module homomorphisms of $E$ into $Y$. If $E$ is an $A$-bimodule, then defining $(a \ast T)(x) = T(x, a)$, $\text{Hom}_A(E, Y)$ becomes a left $A$-module. In fact, for any $b \in A$, $x \in E$,

$$(a \ast T)(b.x) = T((b.x).a) = T(b.(x.a)) = b.T(x.a) = b.(a \ast T)(x).$$

In particular, $\text{Hom}_A(A, Y)$ is a left $A$-module. Note that if $A$ is commutative, then defining $(T \ast a)(x) = T(a, x)$, $\text{Hom}_A(E, Y)$ becomes a right $A$-module. In fact, for any $b \in A$, $x \in E$,

$$(T \ast a)(b.x) = T(a.(b.x)) = T(b.(a.x)) = b.T(a.x) = b.(T \ast a)(x).$$

**Lemma 10.3.3.** [Kh08a] Let $E$ and $Y$ be topological left $A$-modules with $A$ having an approximate identity $\{e_\lambda : \lambda \in I\}$. If $E$ is an essential $A$-module, then $\text{Hom}_A(E, Y) = \text{Hom}_A(E, Y_e)$. In particular, $\text{Hom}_A(A, Y) = \text{Hom}_A(A, Y_e)$.

**Proof.** Since $Y_e \subseteq Y$, clearly $\text{Hom}_A(E, Y_e) \subseteq \text{Hom}_A(E, Y)$. Now let $T \in \text{Hom}_A(E, Y)$. Then, for any $x \in E$, since $e_\lambda x \to x$,

$$\lim_{\lambda} e_\lambda T(x) = \lim_{\lambda} T(e_\lambda x) = T(x).$$

Therefore $T(x) \in Y_e$, i.e. $T \in \text{Hom}_A(E, Y_e)$. \hfill \square

We mention that $\text{Hom}_A(E, Y)$ has been extensively studied in the case of $E$ and $Y$ as the Banach modules of Banach-valued function spaces $L^1(G, A)$ and $C_0(G, A)$, where $G$ is a locally compact abelian group and $A$ a commutative Banach algebra (see, e.g., [Wen52, Rie69, HL92]. If $E = Y = A$, then $\text{Hom}_A(A, A)$ is the usual multiplier algebra of $A$, and is denoted by $M(A)$ (see section 10.1).

**Definition:** Let $A$ be a Hausdorff topological algebra and $(Y, \tau)$ a Hausdorff TVS which is a topological left $A$-module and has a base $\mathcal{W}_Y$ of neighborhoods of 0 in $Y$. The topology of bounded convergence $t_u = t_{u,A}$ (resp. the topology of pointwise convergence $t_p = t_{p,A}$) on $\text{Hom}_A(A, Y)$
is defined as the linear topology which has a base of neighborhood of 0 consisting of all sets of the form

\[ U(D, G) = \{ T \in \text{Hom}_A(A, Y) : T(D) \subseteq G \}, \]

where \( D \) is a bounded (respectively, finite subset) of \( A \) and \( G \in \mathcal{W}_Y \).

Clearly, \( t_p \leq t_u \). Further, we obtain:

**Lemma 10.3.4.** [Kh08a] Let \((Y, \tau)\) be a topological left \( A \)-module with \((b(A)\)\)-hypocontinuous module multiplication. Then both \((\text{Hom}_A(A, Y), t_u)\) and \((\text{Hom}_A(A, Y), t_p)\) are topological left \( A \)-modules.

**Proof.** We prove the result only for \((\text{Hom}_A(A, Y), t_u)\). For any \( a \in A \) and \( T \in \text{Hom}_A(A, Y) \), the map \((a, T) \rightarrow a * T\) is separately continuous, as follows. First, let \( \{a_\alpha : \alpha \in J\} \) be a net in \( A \) with \( a_\alpha \rightarrow a \in A \), and let \( D \) be a bounded subset of \( A \) and \( G \in \mathcal{W}_Y \). By \((b(A)\)-hypocontinuity, there exists a balanced \( H \in \mathcal{W}_Y \) such that \( D.H \subseteq G \). Since \( T \) is continuous, there exists \( \alpha_0 \in J \) such that

\[ T(a_\alpha) - T(a) \in H \text{ for all } \alpha \geq \alpha_0. \]

Now, for any \( b \in D \) and \( \alpha \geq \alpha_0 \),

\[ (a_\alpha * T - a * T)(b) = T(ba_\alpha) - T(ba) = b[T(a_\alpha) - T(a)] \in D.H \subseteq G; \]

that is, \( a_\alpha * T - a * T \in U(D, G) \) for all \( \alpha \geq \alpha_0 \). Hence \( a_\alpha * T \xrightarrow{t_u} a * T \).

Now, let \( \{T_\alpha : \alpha \in J\} \) be a net in \( \text{Hom}_A(A, Y) \) with \( T_\alpha \xrightarrow{t_u} T \in \text{Hom}_A(A, Y) \), and let \( D \) be a bounded subset of \( A \) and \( G \in \mathcal{W}_Y \). Since the map \( R_\alpha : A \rightarrow A \) given by \( R_\alpha(b) = ba, b \in A \), is linear and continuous (by separate continuity of multiplication in \( A \)), it follows that \( R_\alpha(D) = Da \) is bounded in \( A \). Since \( T_\alpha \xrightarrow{t_u} T \), there exists \( \alpha_0 \in J \) such that

\[ T_\alpha - T \in U(Da, G) \text{ for all } \alpha \geq \alpha_0. \]

Now, for any \( b \in D \) and \( \alpha \geq \alpha_0 \),

\[ (a * T_\alpha - a * T)(b) = (T_\alpha - T)(ba) \in G; \]

that is, \( a * T_\alpha - a * T \in U(D, G) \) for all \( \alpha \geq \alpha_0 \). Hence \( a * T_\alpha \xrightarrow{t_u} a * T. \ \square \)

The following results generalize some results of [Wan61, Bus6, Pom92, KMT99] to modules of continuous homomorphisms.

**Theorem 10.3.5.** [Kh08a] Let \( Y \) be a topological left \( A \)-module. Then:

(i) If \( Y \) is an \( F \)-space and \( A \) is strongly factorable, then both \((\text{Hom}_A(A, Y), t_u)\) and \((\text{Hom}_A(A, Y), t_p)\) are complete.
(ii) If $Y$ is complete and $A$ is ultrabarrelled having a bounded approximate identity, then both $(\text{Hom}_A(A,Y), t_p)$ and $(\text{Hom}_A(A,Y_e), t_p)$ are complete.

Proof. (i) Let $\{T_\alpha : \alpha \in J\}$ be a $t_u$-Cauchy net in $\text{Hom}_A(A,Y)$. Since $t_p \leq t_u$, $\{T_\alpha : \alpha \in J\}$ is an $t_p$-Cauchy net in $\text{Hom}_A(A,Y)$; in particular, for each $a \in A$, $\{T_\alpha(a)\}$ is a Cauchy net in $X$. Consequently, by completeness of $Y$, the mappings $T : A \to Y$, given by $T(a) = \lim_\alpha T_\alpha(a)$ $(a \in A)$, is well-defined. Further, for any $a, b \in A$,

$$T(ab) = \lim_\alpha T_\alpha(ab) = \lim_\alpha a.T_\alpha(b) = a.T(b).$$

Since $Y$ is metrizable and $A$ is strongly factorable, by Theorem 10.3.2, $T \in \text{Hom}_A(A,Y)$. We now show that $T_\alpha \overset{t_u}{\longrightarrow} T$. Let $D$ be a bounded subset of $A$ and take closed $G \in W_Y$. There exists an index $\alpha_o$ such that $T_\alpha(a) - T_\gamma(a) \in G$ for all $a \in D$ and $\alpha, \gamma \geq \alpha_o$.

Since $G$ is closed, fixing $\alpha \geq \alpha_o$ and taking $\lim_\gamma$, we have $T_\alpha(a) - T(a) \in G$ for all $a \in D$.

Hence,

$$T_\alpha(a) - T(a) \in U(D,G) \text{ for all } a \in D.$$ 

Thus $(\text{Hom}_A(A,Y), t_u)$ is complete. By a similar argument, $(\text{Hom}_A(A,Y_e), t_p)$ is also complete.

(ii) We first show that $\text{Hom}_A(A,Y)$ is $t_p$-complete. [Let $\{T_\alpha : \alpha \in J\}$ be a $t_p$-Cauchy net in $\text{Hom}_A(A,Y)$. Then, for each $a \in A$, $\{T_\alpha(a) : \alpha \in J\}$ is a Cauchy net in $Y_e$ and hence in $Y$ for all $a \in A$. Since $(Y, \tau)$ is complete, $\{T_\alpha(a)\}$ is convergent for all $a \in A$. Define $T : A \to Y$ by

$$T(a) = \lim_\alpha T_\alpha(a), \ a \in A.$$ 

Then $T$ is a left $A$-module homomorphism. Also $\{T_\alpha : \alpha \in J\}$ is pointwise bounded. Since $A$ is ultrabarrelled, by the principle of uniform boundedness, $\{T_\alpha : \alpha \in J\}$ is equicontinuous. Therefore, given any closed $G \in W_Y$, there exists a neighborhood $V$ of 0 in $A$ such that

$$T_\alpha(V) \subseteq G \text{ for all } \alpha \in J; \text{ that is } \cup_{\alpha \in J} T_\alpha(V) \subseteq G.$$ 

If $a \in V$, $T_\alpha(a) \in T_\alpha(V) \subseteq G$. Therefore, $\{T_\alpha(a) : \alpha \in J\}$ is a net in $G$ for all $a \in V$. Thus

$$T(a) = \lim_\alpha T_\alpha(a) \in cl-G = G; \text{ that is, } T(V) \subseteq G.$$
Therefore $T : A \to Y$ is continuous, and so $T \in Hom_A(A,Y)$. Thus $Hom_A(A,Y)$ is $tp$-complete. Since $A$ has an approximate identity, by Lemma 10.3.3,

$$Hom_A(A,Y) = Hom_A(A,Y_e).$$

Hence $Hom_A(A,Y_e)$ is also $tp$-complete. \hfill $\Box$

**Definition:** For any $x \in Y$, consider the map $R_x : A \to Y$ given by

$$R_x(a) = a.x, a \in A.$$ 

Clearly, $R_x$ is linear and continuous (by separate continuity of module multiplication); further, for any $a,b \in A$,

$$R_x(ab) = (ab).x = a.(b.x) = a.R_x(b),$$

so that $R_x \in Hom_A(A,Y)$. Now define a map $\mu : Y \to Hom_A(A,Y)$ by

$$\mu(x) = R_x, \quad x \in Y.$$ 

It is easily seen that $\mu(Y) = \{R_x : x \in Y\}$ is a left $A$-submodule of $Hom_A(A,Y)$. [In fact, for any $a \in A$ and $x \in Y$,

$$(a \ast R_x)(b) = R_x(ba) = b.(a.x) = R_{a.x}(b), \quad b \in A,$$

so that $a \ast R_x \in Hom_A(A,Y)$.]

**Theorem 10.3.6.** [Kh08a] Let $Y$ be a topological left $A$-module with $A$ having a two-sided approximate identity $\{e_\lambda : \lambda \in I\}$. Then $\mu(Y_e)$ is $tp$-dense in $Hom_A(A,Y_e)$; in particular, $\mu(Y)$ is $tp$-dense in $Hom_A(A,Y_e)$.

**Proof.** Let $T \in Hom_A(A,Y_e)$. For each $\lambda \in I$, define $x_\lambda = T(e_\lambda)$. Then

$$\lim_\gamma e_\gamma.x_\lambda = \lim_\gamma e_\gamma.T(e_\lambda) = \lim_\gamma T(e_\gamma e_\lambda) = T(e_\lambda) = x_\lambda,$$

and so $x_\lambda \in Y_e$. Now, for any $a \in A$, $a.x_\lambda = a.T(e_\lambda) = T(ae_\lambda) \to T(a)$; hence

$$\mu(x_\lambda)(a) = R_{x_\lambda}(a) = a.x_\lambda = a.T(e_\lambda) \to T(a)$$

Therefore $\mu(x_\lambda) \to_T T$. Thus $T \in tp-cl[\mu(Y_e)]$; that is, $\mu(Y_e)$ is $tp$-dense in $Hom_A(A,Y_e)$. \hfill $\Box$

**Remark:** We mention that $\mu(Y)$ need not be $u$-closed in $Hom_A(A,Y)$ even if $Y = A$ is a metrizable locally $C^*$-algebra or a Banach algebra (see also ([Lar70], p. 17) and ([Ph88], p.187)). We include the following example, due to Wang ([Wan61], p. 1138), shows that $\mu_d(A)$ need not be $u$-closed in $CM_d(A)$, in general. (See also ([Lar70], p. 17) and ([Ph88], p.187)).
Example. Let \( \varphi : \mathbb{R} \to \mathbb{R} \) be a function with \( \varphi(x) \geq 1 \) \( (x \in \mathbb{R}) \) and
\[
\lim_{x \to +\infty} \varphi(x) = \lim_{x \to -\infty} \varphi(x) = +\infty.
\]
Define
\[
A = A_\varphi = \{ f \in C(\mathbb{R}, \mathbb{C}) : \| f \|_\varphi = \sup_{x \in \mathbb{R}} | \varphi(x)f(x) | < \infty \}.
\]
Then \( A \) forms a commutative normed algebra under pointwise operations and the above norm. To see that \( A \) is complete, let \( \{ f_n \} \) be a Cauchy sequence in \( A \). Then \( \{ f_n \} \) and \( \{ f_n \varphi \} \) are Cauchy sequences in the usual supremum norm \( \| \cdot \|_\infty \). Let \( f \) and \( g \) be, respectively, the uniform limits of \( \{ f_n \} \) and \( \{ f_n \varphi \} \). Then
\[
\sup_{x \in \mathbb{R}} | g(x) | < \infty \text{ and } f \varphi = g.
\]
Hence \( A \) is a Banach algebra. Further, \( A \) is faithful. It is apparent that \( A \subseteq C_0(\mathbb{R}, \mathbb{C}) \) but \( A \not= C_0(\mathbb{R}, \mathbb{C}) \). Indeed any real-valued function \( f \in C_0(\mathbb{R}, \mathbb{C}) \) such that \( f(x)\varphi(x) \geq 1 \) belongs to \( C_0(\mathbb{R}, \mathbb{C}) \setminus A \). However, \( A \) with the operator norm is dense in \( C_0(\mathbb{R}, \mathbb{C}) \), and so \( A \) is not complete in the operator norm. For instance, let \( f \in C_0(\mathbb{R}, \mathbb{C}) \) and, since \( C_0(\mathbb{R}, \mathbb{C}) \) is dense in \( C_0(\mathbb{R}, \mathbb{C}) \), there exists a sequence \( \{ f_n \} \subseteq A \) of functions with compact support such that
\[
\lim_{n \to \infty} \| f_n - f \|_\infty = 0.
\]
Then
\[
\sup_{\| g \| \leq 1} \| f_n g - f g \|_\varphi = \sup_{\| g \| \leq 1} \sup_{x \in \mathbb{R}} | (f_n(x) - f(x))g(x)\varphi(x) | 
\leq \| f_n - f \|_\infty \sup_{\| g \| \leq 1} \sup_{x \in \mathbb{R}} | g(x)\varphi(x) | 
\leq \| f_n - f \|_\infty,
\]
showing that \( f \in \text{cl}(A) \). Consequently, \( A \) is not closed in \( C_0(\mathbb{R}, \mathbb{C}) \) in the operator norm. \( \square \)

Theorem 10.3.7. Let \( X \) be a topological left \( A \)-module with \( A \) having a two-sided approximate identity \( \{ e_\lambda : \lambda \in I \} \). If \( (X, \beta) \) is complete, the map \( \mu : X \to \text{Hom}_A(A, X) \) defined by \( \mu(y) = R_y, y \in X, \) is onto.

Proof. By Theorem 10.3.6, \( \mu(X) \) is \( t_p \)-dense in \( \text{Hom}_A(A, X) \). We now show that \( \mu(X) \) is \( t_p \)-closed in \( \text{Hom}_A(A, X) \). Let \( T \in t_p\text{-cl} \mu(X) \). There exists a net \( \{ x_\alpha : \alpha \in J \} \subseteq X \) such that \( R_{x_\alpha} \to T \). Then \( \{ x_\alpha : \alpha \in J \} \) is \( \beta \)-Cauchy in \( X \). [Let \( D \) be a finite subset of \( A \) and \( G \in \mathcal{W}_X \).]
Choose a balanced $H \in W_X$ with $H + H \subseteq G$. Since $R_{x_\alpha} \xrightarrow{t} T$, there exists $\alpha_o \in I$ such that for all $\alpha \geq \alpha_o$,

$$R_{x_\alpha} - T \in N(D, H) \text{ or } R_{x_\alpha}(a) - T(a) \in H \text{ for all } a \in D.$$  

Then, for any $a \in D$ and $\alpha, \gamma \geq \alpha_o$,

$$a.x_\alpha - a.x_\gamma = [R_{x_\alpha}(a) - T(a)] + [T(a) - R_{x_\gamma}(a)] \in H + H \subseteq G.$$  

Since $(X, \beta)$ is complete, $x_\alpha \xrightarrow{\beta} x_o, x_o \in X$. Hence $R_{x_\alpha} \xrightarrow{t} R_{x_o}$. By uniqueness of limit in Hausdorff spaces, $T = R_{x_o} \in \mu(X)$. Thus $\mu(X) = Hom_A(A, X_e)$. □
4. Notes and Comments

Section 10.1. In 1968, Busby [Bus68] also defined a general strict topology on any $B^*$-algebra $Y$ which contains $A$ as a closed two-sided ideal; this is given by the seminorms $x \rightarrow \max\{||ax||, ||ax||\}$ \((a \in A)\). Independently, this topology was studied in greater detail and in a more general setting by Sentilles and Taylor [ST69] where $A$ is a Banach algebra having an approximate identity and $Y$ is a left Banach $A$-module. More recently, Khan-Mohammad-Thaheem [KMT05] have considered the case of $Y$ a locally idempotent $F$-algebra which contains $A$ as a closed two-sided ideal, while Shantha [Sha04] has taken $Y$ a locally convex module over a locally convex algebra $A$. As pointed out in Sentilles and Taylor [ST69], the essential ingredient of arguments used by Buck [Buc58] and later authors is the presence of an approximate identity in $C_0(X)$ or $A$.

This section contains some results from the paper [Kh07] which deal with the study of general strict and related topologies on a topological left $A$-module $Y$, where $Y$ is Hausdorff TVS and $A$ a Hausdorff topological algebra having a two-sided bounded approximate identity. These include generalization of several results of Sentilles and Taylor [ST69] from the setting of Banach left modules to topological left modules obtained by Shantha [Shan04, KMT05].

Section 10.2. This section two results from [Kh07] on the essentiality of topological left $A$-modules of continuous vector-valued functions, which were originally proved in [SumW72] for a particular case.

Section 10.3. In this section, a study is made of the properties of the topological module $Hom_A(A,Y)$ of continuous homomorphisms.

In [Rie67, Rie69], Rieffel made an extensive study of the Banach module $Hom_A(A,X)$ of continuous homomorphisms. Further results in this direction have been obtained in [ST69, Rue77] in connection with the study of general strict topology. More recently, Shantha [Sha04, Shan05] has studied homomorphisms in the case of locally convex modules. This section contains some results from a recent work [Kh08a] which has investigated the extent to which some of the results of above authors are also true in the non-locally convex setting of topological modules. As application, an observation on inner derivation, due to Phillips [Ph95], is presented in a more general setting.

We mention that particular cases of the module $Hom_A(A,X)$ are the algebras $M_l(A), M_r(A), M(A)$ and $M_d(A)$ which consists of all continuous left multipliers, right multipliers, multipliers and double multipliers. There is an extensive literature on multipliers; see, e.g., [Wan61,
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Jo64, Jo66, Tay70, TW70, APT73, Fon75, Lat76, Tom76, Dav77, CL78, Won85, Ph88, Hus89, KMT99, MKT01].
CHAPTER 11

Mean Value Theorem and Almost Periodicity
1. The Mean Value Theorem in TVSs

We present here the mean value theorem and the mean value inequality for class of Gateaux differentiable functions $f: A \subseteq E \to F$, where $E$ and $F$ are TVSs (both over the field $\mathbb{R}$ of real numbers) and $A \subseteq E$ an open set. These results are given in [Vain64] for $E$ and $F$ normed spaces and in the present general form in [Kh97].

**Definition.** A function $f: A \to F$ is said to be Gateaux differentiable at $x_o \in A$ if there exists a mapping from $E$ into $F$, denoted by $f'(x_o)$, such that, given any $z \in E$ and a balanced neighborhood $V$ of 0 in $F$, there exists a $\delta > 0$ satisfying,

$$
\frac{f(x_o + tz) - f(x)}{t} - f'(x_o)(z) \in V, \\
$$

whenever $0 < |t| < \delta$; $f'(x_o)$ is called the Gateaux derivative of $f$ at $x_o$ and we briefly write as,

$$
f'(x_o)(z) = \lim_{t \to 0} \frac{f(x_o + tz) - f(x_o)}{t}, \quad z \in E. \tag{1}
$$

Some authors also require $f'(x_o)$ to be linear, but it is not assumed here. The following example shows that a Gateaux differentiable function needs not being continuous.

**Example.** The function $f: \mathbb{R}^2 \to \mathbb{R}$, given by

$$
f(x) = \frac{a^2b}{a^2 + b^2} \text{ if } x = (a, b) \neq (0, 0), \text{ and } f((0, 0)) = 0,
$$

is not continuous at $x = (0, 0)$, although $f''(0)$ exists. □

We first consider the Lagrange form of the mean value theorem for real-valued functions on $E$. For any $x, y \in E$, the line segment $[x, y]$ joining $x$ and $y$ is defined as the set

$$
[x, y] = \{(1-t)x + ty : 0 \leq t \leq 1\}.
$$

**Theorem 11.1.1.** [Kh97] Let $E$ be a Hausdorff TVS over $\mathbb{R}$, and let $A \subseteq E$ be an open set. Let $g: A \to \mathbb{R}$ be a function continuous and Gateaux differentiable at each point of a segment $[x_o, x_o + h]$ in $A$. Then there exists a $\theta \in (0, 1)$ such that

$$
g(x_o + h) - g(x_o) = g'(x_o + \theta h)(h).
$$

**Proof.** Define $\phi: [0, 1] \to \mathbb{R}$ by

$$
\phi(t) = g(x_o + th), \quad t \in [0, 1].
$$
Then \( \phi \) is differentiable on \([0, 1]\). In fact, using (1), we obtain
\[
[\phi'(t) = \lim_{\Delta t \to 0} \frac{g(x_o + (t + \Delta t)h) - g(x_o + th)}{\Delta t} = g'(x_o + th)(h).
\]
By the classical mean value theorem, there exists a \( \theta \in (0, 1) \) such that
\[
\phi(1) - \phi(0) = \phi'(\theta).
\]
Hence
\[
g(x_o + h) - g(x_o) = \phi(1) - \phi(0) = g'(x_o + \theta h)(h). \quad \square
\]

An exact analogue of Theorem 11.1.1 for vector-valued (or even for complex-valued) functions need not hold as is shown below:

**Example.** Consider the function \( f : [0, 2\pi] \to \mathbb{C} \) defined by
\[
f(x) = \cos x + i \sin x = e^{ix}, \quad x \in [0, 2\pi].
\]
Clearly \( f'(x) = ie^{ix} \). Now \( f(2\pi) - f(0) = 1 - 1 = 0 \), but
\[
(2\pi - 0)f'(c) = 2\pi ie^{ic} \neq 0 \quad \text{for all } 0 < c < 2\pi.
\]
Hence
\[
f(2\pi) - f(0) \neq (2\pi - 0)f'(c) \quad \text{for all } 0 < c < 2\pi.
\]
However, \( f(x) \) does satisfy the Mean Value Inequality for the values \( x_o = 0, x_o + h = 2\pi \) since
\[
|f(2\pi) - f(0)| = 0 \leq \sup\{|2\pi - 0||ie^{ic}| : 0 \leq c \leq 2\pi\} = 2\pi. \quad \square
\]

However, for the class of these functions, the following version of the mean value theorem holds.

**Theorem 11.1.2.** [Kh97] Let \( E \) and \( F \) be a Hausdorff topological vector spaces over \( \mathbb{R} \), and let \( A \subseteq E \) be an open set. Suppose \( F \) has a non-trivial (real) continuous dual \( F^* \), and let \( f : A \to F' \) be a function continuous and Gateaux differentiable at each point of the segment \([x_o, x_o + h]\) in \( A \). Then, given \( u \in F^* \), there exists a \( \theta \in (0, 1) \) such that
\[
< f(x_o + h) - f(x_o), u > = < f'(x_o + \theta h)(h), u >. \quad (2)
\]

**Proof.** Define \( g : [x_o, x_o + h] \to \mathbb{R} \) by
\[
g(x) = < f(x), u >, \quad x \in [x_o, x_o + h].
\]
Then it follows from the linearity and continuity of \( u \) that
\[
g'(x)(h) = < f'(x_o)(h), u >.
\]
By Theorem 11.1.1, there exists a \( \theta \in (0, 1) \) such that
\[
g(x_o + h) - g(x_o) = g'(x_o + \theta h)(h).
\]
This establishes (2). □

The above result clearly reduces to Theorem 11.1.1 when \( F = \mathbb{R} \).

**Theorem 11.1.3.** (Mean Value Inequality) [Kh97] Under the hypothesis of Theorem 11.1.2, given a continuous seminorm \( p \) on \( F \), there exists a \( \theta \in (0, 1) \) such that,
\[
p(f(x_o + h) - f(x_o)) \leq p(f'(x_o + \theta h)(h)).
\]
(3)

**Proof.** By the analytic form of the Hahn-Banach theorem ([Ko69], Theorem 8, §17.3), we can choose a \( u \neq 0 \) ∈ \( F^* \) such that
\[
<y, u > \leq p(y) \quad \text{for all} \quad y \in F
\]
and
\[
<f(x_o + h) - f(x_o), u > = p(f(x_o + h) - f(x_o)).
\]
By Theorem 11.1.2, there exists a \( \theta \in (0, 1) \) such that
\[
<f(x_o + h) - f(x_o), u >=< f'(x_o + \theta h)(h), u > .
\]
Consequently, we obtain (3). □

**Remark.** We have seen above that the function \( f : [0, 2\pi] \to \mathbb{C} \) defined by
\[
f(x) = \cos x + i \sin x = e^{ix}, x \in [0, 2\pi].
\]
does not satisfy Theorem 11.1.1, the Mean Value Theorem, for the values \( x_o = 0, x_o + h = 2\pi \). However, \( f(x) \) does satisfy Theorem 3, the Mean Value Inequality, for the values
\[
x_o = 0, x_o + h = 2\pi
\]
since
\[
|f(2\pi) - f(0)| = 0 \leq \sup\{2\pi - 0||ie^{ic}| : 0 \leq c \leq 2\pi\} = 2\pi.
\]

As an application of the above theorem, we obtain:

**Corollary 11.1.4.** [Kh97] Suppose \( F \) is locally convex and \( A \subseteq E \) an open connected set, and let \( f : A \to F \) be continuous and Gateaux differentiable at each point of \( A \). If \( f'(x_o) = 0 \) for each \( x \in A \), then \( f \) is constant on \( A \).

**Proof.** Fix \( x \in A \), and let \( B = \{x \in A : f(x) = f(x_o)\} \). Clearly, \( B \) is non-empty and closed in \( A \). \( B \) is also open in \( A \), as follows. Let \( x \in B \). Choose a balanced neighborhood \( V \) of 0 in \( E \) such that \( U = x + V \subseteq A \).
Then, for each $y \in U$, $[x,y] \in U$. Hence, for each $y \in U$ and any continuous seminorm $p$ on $F$, it follows from Theorem 11.1.3 and the hypothesis that

$$p(f(y) - f(x)) \leq \sup_{z \in [x,y]} (f'(z)(y - x)) = p(0) = 0.$$ 

Since $F$ is locally convex and Hausdorff, $f(y) = f(x) = f(x_0)$ for all $y \in U$. Hence $U \subseteq B$. Since $A$ is connected, $B = A$. Thus $f(x) = f(x_0)$ for all $x \in A$. □

**Theorem 11.1.5.** (Mean Value Inclusion) [AS67, Kh97] Suppose $F$ is a locally convex space. Then, under the hypothesis of Theorem 2,

$$f(x_0 + h) - f(x_0) \in \overline{co}(B),$$

where $B = \{f'(x_0)(h) : x \in [x_0, x_0 + h]\}$ and $\overline{co}(B)$ is the closed convex hull of $B$.

Proof. Suppose $f(x_0 + h) - f(x_0) \notin \overline{co}(B)$. By ([Ko69], Theorem 2, § 20.7), there exists a $u(\neq 0) \in F^*$ and an $r \in \mathbb{R}$ such that,

$$<y, u> \leq r < f(x_0 + h) - f(x_0), u>$$

for all $y \in \overline{co}(B)$. Define $\phi : [0, 1] \to \mathbb{R}$ by $\phi(t) = < f(x_0 + th), u >$. Then $\phi'(t) = < f'(x_0 + th)(h), u >$. There exists a $\theta \in (0, 1)$ such that $\phi(1) - \phi(0) = \phi'(\theta)$. Then

$$< f(x_0 + h) - f(x_0), u > = \phi(1) - \phi(0) = < f'(x_0 + \theta h)(h), u >$$

which contradicts (4). □

**Remark.** Theorem 11.1.5 need not hold if $F$ is not locally convex, or if $\overline{co}(B)$ is replaced by $\overline{B}$ or $co(B)$. This follows from ([AS67], Examples 1.23-1.25).
2. Almost Periodic Functions with Values in a TVS

In this section, we consider the concept of almost periodicity of functions having values in a topological vector space $E$. The results are taken mainly from [NGu84, KA11].

**Definition:** A subset $P$ of $\mathbb{R}$ is called *relatively dense* in $\mathbb{R}$ if there exists a number $\ell > 0$ such that every interval of length $\ell$ in $\mathbb{R}$ contains at least one point of $P$.

**Definition:** Let $(E, \tau)$ be TVS with a base $W = W_E$ of balanced neighborhoods of 0. A function $f : \mathbb{R} \to E$ is called *almost periodic* if it is continuous and, for each $W \in W$, there exists a number $\ell_W > 0$ such that each interval of length $\ell_W$ in $\mathbb{R}$ contains a point $\tau_W$ such that

$$f(t + \tau_W) - f(t) \in W \quad \text{for all} \quad t \in \mathbb{R}. \quad (*)$$

A number $\tau_W \in \mathbb{R}$ for which $(*)$ holds is called $W$-translation number of $f$.

The above property says that, for each $W \in W$, the function $f$ has a set of $W$-translation numbers $P_{W,f}$ which is relatively dense in $\mathbb{R}$.

The set of all almost periodic functions $f : \mathbb{R} \to E$ is denoted by $AP(\mathbb{R}, E)$. For any $f : \mathbb{R} \to E$ and a fixed $h \in \mathbb{R}$, the $h$-translate of $f$ is defined as the function $f_h : \mathbb{R} \to E$ defined by

$$f_h(t) = f(t + h), \quad t \in \mathbb{R}.$$  

We shall denote $H(f) = \{f_h : h \in \mathbb{R}\}$, the set of all translates of $f$.

**Definition:** Let $X$ be a Hausdorff space and $(E, d)$ a metrizable TVS. The *uniform topology* $u$ on $C_b(X, E)$ is defined as the topology given by the metric $\rho$

$$\rho(f, g) = \sup_{x \in X} d(f(x), g(x)), \quad f, g \in C_b(X, E).$$

**Theorem 11.2.1.** [NGu84, KA11] Let $X$ be a Hausdorff space and $(E, d)$ a metric linear space.

(a) If a sequence $\{f_n\} \subseteq C_b(X, E)$ is $u$-convergent to a function $f : X \to E$, then $f \in C_b(X, E)$.

(b) If $\{f_n\}$ is a $u$-Cauchy sequence in $C_b(X, E)$ and if there is a function $f : X \to E$ such that $f_n(x) \to f(x)$ for each $x \in X$, then $f_n \xrightarrow{u} f$ and $f \in C_b(X, E)$.

(c) If $E$ is complete, then so is $(C_b(X, E), u)$.

Proof. (a) $f$ is continuous as follows. Let $x_o \in X$ and $W \in W$. We show that there exists a neighborhood $G$ of $x_o$ in $X$ such that

$$f(y) \in f(x_o) + W \quad \text{for all} \quad y \in G.$$
Choose a balanced $V \in \mathcal{W}$ such that $V + V + V \subseteq W$. Since $f_n \xrightarrow{u} f$, there exists an integer $N$ such that

$$f_n(x) - f(x) \in V \text{ for all } n \geq N \text{ and all } x \in X.$$  

(A)

Since $f_N$ is continuous at $x_o$, there exists a neighborhood $G$ of $x_o$ in $X$ such that

$$f_N(y) \in f_N(x_o) \text{ for all } y \in G.$$  

(B)

Then, for any $y \in G$, using (A) and (B),

$$f(y) - f(x_o) = (f(y) - f_N(y)) + (f_N(y) - f_N(x_o)) + (f_N(x_o) - f(x_o))$$

$$\in V + V + V \subseteq W.$$  

Thus $f$ is continuous on $X$.

Next, $f$ is bounded as follows. Suppose $f$ is unbounded. Then there exists a $W \in \mathcal{W}$ such that $f(X)$ is not contained in $nW$ for all $n \geq 1$. Hence, for each $n \geq 1$, there exists $x_n \in X$ such that $f(x_n) \notin nW$. Choose a balanced $V \in \mathcal{W}$ such that $V + V \subseteq W$. Since $f_n \xrightarrow{u} f$, there exists an integer $K$ such that

$$f_n(x) - f(x) \in V \text{ for all } n \geq K \text{ and } x \in X.$$  

In particular,

$$f_K(x_n) - f(x_n) \in V \text{ for all } n \geq 1.$$  

Since $f_K$ is bounded, there exists an integer $m > 1$ such that $f_K(X) \subseteq mV$. Then

$$f(x_m) = [f(x_m) - f_K(x_m)] + f_K(x_m) \in V + mV \subseteq mV + mV \subseteq mW,$$

a contradiction. Hence $f$ is bounded.

(b) Let $\{f_n\}$ be a $u$-Cauchy sequence in $C_b(X, E)$. By hypothesis, for each $x \in X$, $f(x) := \lim_{n \to \infty} f_n(x)$ exists in $E$. Then $f_n \xrightarrow{u} f$, as follows. Let $W \in \mathcal{W}$ be closed. Since $\{f_n\}$ is $u$-Cauchy, there exists an integer $J$ such that

$$f_n(x) - f_m(x) \in W \text{ for all } n, m \geq J \text{ and } x \in X.$$  

Fix any $n \geq J$. Since $W$ is closed and, for each $x \in X$, $f_m(x) \longrightarrow f(x)$, we have

$$f_n(x) - f(x) \in W \text{ for } x \in X.$$  

Thus $f_n(x) - f(x) \in W$ for all $n \geq J$ and all $x \in X$; i.e.

$$f_n - f \in N(X, W) \text{ for all } n \geq J.$$  

This proves that $f_n \xrightarrow{u} f$. By (a), $f \in C_b(X, Y)$.

(c) Let $\{f_n\}$ be a $u$-Cauchy sequence in $C_b(X, E)$. Since $p \leq u$, $\{f_n\}$ is a $p$-Cauchy sequence in $C(X, E)$; that is, for each $x \in X$, $\{f_n(x)\}$ is
a Cauchy sequence in $E$. Since $E$ is complete, for each $x \in X$, $f(x) := \lim_{n \to \infty} f_n(x)$ exists in $E$. Then, by (b), $f_n \xrightarrow{u} f$ and that $f$ is continuous and bounded. Hence $(C_b(X,E),u)$ is complete. \( \square \)

**Theorem 11.2.2.** [NGu84, KA11] Let $E$ be a TVS. Let $f \in AP(\mathbb{R},E)$. Then:

(a) $f$ has precompact range $f(\mathbb{R})$; hence $f$ is bounded.

(b) $f$ is uniformly continuous on $\mathbb{R}$.

**Proof.** (a) Let $W \in \mathcal{W}$. Choose a balanced $V \in \mathcal{W}$ such that $V + V \subseteq W$. Since $f$ is almost periodic, there exists a number $\ell = \ell_V > 0$ such that each interval of length $\ell$ in $\mathbb{R}$ contains a point $\tau_V$ such that $f(t + \tau_V) - f(t) \in V$ for all $t \in \mathbb{R}$. 

By continuity of $f$, the set $f[0, \ell]$ is compact in $E$ and hence precompact. So there exist $x_1, ..., x_n \in f[0, \ell]$ such that $f[0, \ell] \in \bigcup_{i=1}^{n}(x_i + V)$. 

We claim that $f(\mathbb{R}) \subseteq \bigcup_{i=1}^{n}(x_i + W)$. [Take an arbitrary $t \in \mathbb{R}$. By (1), there exists $\tau \in [-t, -t + \ell]$ such that $f(t + \tau) - f(t) \in V$. By (2), choose $x_k \in \{x_1, ..., x_n\}$ such that $f(t + \tau) - x_k \in V$. Then $f(t) - x_k = [f(t) - f(t + \tau)] + [f(t + \tau) - x_k] \in -V + V \subseteq W$; and therefore $f(t) \in x_k + W$.] Thus $f(\mathbb{R})$ is precompact in $E$.

(b) Let $W \in \mathcal{W}$. Choose a balanced $V \in \mathcal{W}$ such that $V + V + V \subseteq W$. There exists a number $\ell = \ell_V > 0$ such that each interval of length $\ell$ in $\mathbb{R}$ contains a point $\tau_V$ such that $f(t + \tau_V) - f(t) \in V$ for all $t \in \mathbb{R}$.

Now $f$, being almost periodic, is continuous on $\mathbb{R}$. Then $f$ is uniformly continuous on the compact set $[-1, 1 + \ell_V]$, so there exists $\delta = \delta_V > 0$ (we may assume $0 < \delta < 1$ without loss of generality) such that $f(s) - f(t) \in V$ for all $s, t \in [-1, 1 + \ell_V]$ with $|s - t| < \delta$. (4)

Let $a, b \in \mathbb{R}$ with $|a - b| < \delta$ and assume $a < b$. We claim that $f(a) - f(b) \in W$. [We may assume that $a < b$. Choose a $\tau_V \in [-a, -a + \ell_V]$ satisfying (3). Then $a + \tau_V \in a + [-a, -a + \ell_V] = [0, 0 + \ell_V] \subseteq [-1, 1 + \ell_V]$;]
since $0 < b - a < \delta < 1,$

$b + \tau_V \in b + [-a, -a + \ell_V] = [b - a, b - a + \ell_V] = [0, 1 + \ell_V] \subseteq [-1, 1 + \ell_V].$

Also, $|(a + \tau_V) - (b + \tau_V)| = |a - b| < \delta,$ so by (4),

$$f(a + \tau_V) - f(b + \tau_V) \in V.$$  (5)

Therefore, by (3) and (5),

$$f(a) - f(b) = [f(a) - f(a + \tau_V)] + [f(a + \tau_V) - f(b + \tau_V)]$$

$$+ [f(b + \tau_V) - f(b)]$$

$$\in -V + V + V \subseteq W.$$  (6)

Thus $f$ is uniformly continuous on $\mathbb{R}$. \hfill \Box

**Remark.** Clearly, by Theorem 11.2.2(a), $AP(\mathbb{R}, E) \subseteq C_b(\mathbb{R}, E)$.

**Theorem 11.2.3.** [NGu84, KA11] Let $E$ be a TVS. If $\{f_n\}$ is a sequence in $AP(\mathbb{R}, E)$ such that $f_n \xrightarrow{u} f$, then $f \in AP(\mathbb{R}, E)$.

**Proof.** Clearly, by Theorem 11.2.2(b), $f$ is continuous on $\mathbb{R}$. Let $W \in \mathcal{W}$. Choose a balanced $V \in \mathcal{W}$ such that $V + V + V \subseteq W$. Since $f_n \xrightarrow{u} f$, there exists an integer $N \geq 1$ such that such that

$$f_n(t) - f(t) \in V \text{ for all } t \in \mathbb{R} \text{ and } n \geq N.$$  (6)

Since $f_N$ is almost periodic, there exists a number $\ell = \ell_V > 0$ such that each interval of length $\ell$ in $\mathbb{R}$ contains a point $\tau = \tau_V$ such that

$$f_N(t + \tau) - f_N(t) \in V \text{ for all } t \in \mathbb{R}.$$  (7)

Then, by (6) and (7), for any $t \in \mathbb{R}$

$$f(t + \tau) - f(t) = [f(t + \tau) - f_N(t + \tau)] + [f_N(t + \tau) - f_N(t)]$$

$$+ [f_N(t) - f(t)]$$

$$\in -V + V + V \subseteq W.$$  (7)

Since the set $P_{V,f_N}$ of almost periods of $f_N$ is relatively dense, we take $P_{V,f} = P_{V,f_N}$. Hence $f$ is also almost periodic.

**Theorem 11.2.4.** [NGu84, KA11] Let $E$ be a TVS. If $f : \mathbb{R} \rightarrow E$ is almost periodic, then the functions (i) $\lambda f$ $(\lambda \in \mathbb{K})$, (ii) $\overline{f(t)}$ and (iii) $f_h(t) = f(t + h)$ $(h \in \mathbb{R})$ are also almost periodic.

**Proof.** (i) This is trivial if $\lambda = 0$. Suppose $\lambda \neq 0$. Let $W \in \mathcal{W}$ be balanced. Then $V = \lambda^{-1}W \in \mathcal{W}$. Since $f$ is almost periodic, there exists a number $\ell = \ell_V > 0$ such that each interval of length $\ell$ in $\mathbb{R}$ contains a point $\tau_V$ such that

$$f(t + \tau_V) - f(t) \in V \text{ for all } t \in \mathbb{R}.$$
Then

\[(\lambda f)(t + \tau) - (\lambda f)(t) = \lambda [f(t + \tau) - f(t)] \in \lambda V = W \text{ for all } t \in \mathbb{R}.
\]

Hence \( \lambda f \) is almost periodic.

(ii) Let \( W \in \mathcal{W} \) be balanced. There exists a number \( \ell = \ell_W > 0 \) such that each interval of length \( \ell \) in \( \mathbb{R} \) contains a point \( \tau \) such that

\[ f(t + \tau) - f(t) \in W \text{ for all } t \in \mathbb{R}. \]

Put \( s = -t \); we get:

\[ \overline{f}(s - \tau) - \overline{f}(s) = f(-s + \tau) - f(-s) = f(t + \tau) - f(t) \in W. \]

Therefore \( \overline{f}(s - \tau) - \overline{f}(s) \in W \) for every \( s \in \mathbb{R} \). This shows that \( \overline{f} \) is almost periodic with \(-\tau\) as a \( W \)-translation number.

(iii) Let \( W \in \mathcal{W} \) be balanced, and let \( h \in \mathbb{R} \). There exists a number \( \ell = \ell_W > 0 \) such that each interval of length \( \ell \) in \( \mathbb{R} \) contains a point \( \tau \) such that

\[ f(t + \tau) - f(t) \in W \text{ for all } t \in \mathbb{R}. \]

Replacing \( t \) by \( t + h \), we have

\[ f(t + h + \tau) - f(t + h) \in W \text{ for all } t \in \mathbb{R}, \]

or

\[ f_h(t + \tau) - f_h(t) \in W \text{ for all } t \in \mathbb{R}. \quad \square \]

**Theorem 11.2.5.** [NGu84, KA11] Let \( E \) and \( F \) be a TVSs, and let \( f \in \text{AP}(\mathbb{R}, E) \). If \( g : \overline{f(\mathbb{R})} \to F \) is any continuous function, then the composed function \( g \circ f \in \text{AP}(\mathbb{R}, F) \).

**Proof.** Let \( U \in \mathcal{W}_F \) be balanced. Since \( \overline{f(\mathbb{R})} \) is compact, \( g \) is uniformly continuous on \( \overline{f(\mathbb{R})} \) and so there exists a \( W \in \mathcal{W}_E \) such that

\[ g(x) - g(y) \in U \text{ for all } x, y \in \overline{f(\mathbb{R})} \text{ with } x - y \in W. \]

Since \( f \) is almost periodic, there exists a number \( \ell = \ell_W > 0 \) such that each interval of length \( \ell \) in \( \mathbb{R} \) contains a point \( \tau \) such that

\[ f(t + \tau) - f(t) \in W \text{ for all } t \in \mathbb{R}. \]

Consequently,

\[ g[f(t + \tau)] - g[f(t)] \in U \text{ for all } t \in \mathbb{R}. \]

Thus \( g \circ f \in \text{AP}(\mathbb{R}, F). \quad \square \)

We now proceed to establish the Bochner’s criteria for almost periodicity. As a first step, we obtain:
2. ALMOST PERIODIC FUNCTIONS WITH VALUES IN A TVS

**Theorem 11.2.6.** [NGu84, KA11] Let $E$ be a TVS and $f : \mathbb{R} \to E$ a continuous function. If the set of translates $H(f) = \{ f_h : h \in \mathbb{R} \}$ is $u$-sequentially compact in $C_b(\mathbb{R}, E)$, then $f$ is almost periodic.

*Proof.* Suppose $f$ is not almost periodic. Then there exists a $W \in \mathcal{W}$ such that for every $\ell > 0$, there exists an interval of length $\ell$, $[-a, -a + \ell]$ (say) which contains no $W$-translation number of $f$. Consequently, for every $h \in [-a, -a + \ell]$, there exists $t_h \in \mathbb{R}$ such that

$$
 f(t_h + h) - f(t_h) \notin W.
$$

Let us consider $h_1 \in \mathbb{R}$ and an interval $(a_1, b_1)$ with $b_1 - a_1 > 2|h_1|$ which contains no $W$-translation number of $f$. Now let $h_2 = \frac{a_1 + b_1}{2}$. Since

$$
 -\frac{b_1 - a_1}{2} < h_1 < \frac{b_1 - a_1}{2},
$$

$h_2 - h_1 \in (a_1, b_1)$ and therefore $h_2 - h_1$ cannot be a $W$-translation number of $f$. Let us consider another interval $(a_2, b_2)$ with $b_2 - a_2 > 2(|h_1| + |h_2|)$, which contains no $W$-translation number of $f$. Let $h_3 = \frac{a_2 + b_2}{2}$; then $h_3 - h_1, h_3 - h_2 \in (a_2, b_2)$ and therefore $h_3 - h_1$ and $h_3 - h_2$ cannot be $W$-translation number of $f$. We proceed and get a sequence $\{h_n\}$ such that no $h_m - h_n$ ($m > n$) is a $W$-translation number of $f$; that is, there exists $t_{mn} \in \mathbb{R}$ with

$$
 f(t_{mn} + h_m - h_n) - f(t_{mn}) \notin W. \quad (8)
$$

Put $s_{mn} = t_{mn} - h_n$. Then (8) becomes:

$$
 f(s_{mn} + h_m) - f(s_{mn} + h_n) \notin W. \quad (9)
$$

Since $H(f)$ is sequentially compact in $C_b(\mathbb{R}, E)$, there exists a subsequence $\{k_n\}$ of $\{h_n\}$ such that $\{f(t + k_n)\}$ is $u$-convergent on $\mathbb{R}$. Then, for $W \in \mathcal{W}$, there exists an $N = \mathcal{N}_W \geq 1$ such that if $m, n > N$ (we may take $m > n$), we have:

$$
 f(t + k_m) - f(t + k_n) \in W \text{ for every } t \in \mathbb{R}. \quad (10)
$$

Taking $t = s_{mn}$ in (10), we get a contradiction to (9). Thus $f$ is almost periodic. \hfill $\Box$

As a converse of the above theorem, we obtain:

**Theorem 11.2.7.** [NGu84, KA11] Let $E$ be an $F$-space and $f : \mathbb{R} \to E$ an almost periodic function. Then the set of translates $H(f) = \{ f_h : h \in \mathbb{R} \}$ is $u$-compact in $C_b(\mathbb{R}, E)$.

*Proof.* Since $E$ is an $F$-space, by Theorem 11.2
.1, $(C_b(\mathbb{R}, E), u)$ is also an F-space. Therefore, it suffices to show that any sequence $\{f_{h_n}\}$ in $H(f)$ has a $u$-Cauchy subsequence.

Proof. Let $S = \{a_n\}$ be a dense sequence in $\mathbb{R}$. Since $f(\mathbb{R})$ is precompact and hence relatively compact in the F-space $E$, we can extract from $\{f_{h_n}(a_1)\}$ a convergent subsequence. Let $\{f_{h_{1,n}}\}$ be the subsequence of $\{f_{h_n}\}$ which converges at $a_1$. We apply the same argument as above to the sequence $\{f_{h_{1,n}}\}$ to choose a subsequence $\{f_{h_{2,n}}\}$ which converges at $a_2$. We continue the process and consider the diagonal sequence $\{f_{h_{n,n}}\}$ which converges at each $a_n$ in $S$. Call this last sequence by $\{f_{h_n}\}$. We claim that this sequence is $u$-Cauchy on $\mathbb{R}$.

Proof. Let $t_o \in \mathbb{R}$, and let $W \in \mathcal{W}$. Choose a balanced $V \in \mathcal{W}$ such that $V + V + V + V + V \subset W$. By almost periodicity of $f$, let $\ell = \ell(V) > 0$ be such that each interval of length $\ell$ in $\mathbb{R}$ contains a point $\tau = \tau_V$ such that

$$f(t + \tau) - f(t) \in V \quad \text{for all } t \in \mathbb{R}. \quad (11)$$

By uniform continuity of $f$ over $\mathbb{R}$, there exists $\delta = \delta_V > 0$ such that

$$f(t) - f(t') \in V \quad \text{for every } t, t' \in \mathbb{R} \text{ with } |t - t'| < \delta. \quad (12)$$

We divide the interval $[0, \ell]$ into subintervals $I_1, ..., I_r$, each of length smaller than $\delta$. Since $S$ is dense in $\mathbb{R}$, for each $1 \leq j \leq r$, $I_j$ contains a point $b_j$ of $S$ and set $S_o = \{b_1, ..., b_r\}$. Since $\{f_{h_n}\}$ is $p$-convergent on $S$, it follows that $\{f_{h_n}\}$ is $u$-convergent on the finite set $S_o$. Then there exists an integer $N = N_V \geq 1$ such that

$$f(b_i + k_n) - f(b_i + k_m) \in V \quad \text{for every } i = 1, ..., J \text{ and } n, m > N. \quad (13)$$

Now, choose $\tau_o \in [-t_o, -t_o + \ell]$ satisfying (11). Then $t_o + \tau \in [0, \ell]$, so choose $b_j \in S_o$ such that $|t_o + \tau - b_j| < \delta$. So, by (12),

$$f(t_o + \tau_o + k_n) - f(b_j + k_n) \in V \quad \text{for every } n \geq 1. \quad (14)$$

Therefore, if $n, m > N$, by applying (11), (13), (14) we get

$$f_{h_n}(t_o) - f_{h_m}(t_o) = f(t_o + k_n) - f(t_o + k_m)$$

$$= [f(t_o + k_n) - f(t_o + k_n + \tau_o)]$$

$$+ [f(t_o + k_n + \tau_o) - f(b_j + k_n)]$$

$$+ [f(b_j + k_n) - f(b_j + k_m)]$$

$$+ [f(b_j + k_m) - f(t_o + \tau_o + k_m)]$$

$$+ [f(t_o + k_m + \tau_o) - f(t_o + k_m)]$$

$$\in V + V + V + V + V \subset W.$$
Therefore the subsequence \( \{f_{h_n}\} \) of \( \{f_{h_n}\} \) is \(-\)Cauchy on \( \mathbb{R} \). Consequently, \( \{f_{h_n}\} \) is \(-\)convergent on \( \mathbb{R} \).

Combining Theorems 11.2.6 and 11.2.7, we have:

**Corollary 11.2.8.** (Bochner’s Criterion) [NGu84, KA11] Let \( E \) be an \( F \)-space. Then a continuous function \( f : \mathbb{R} \to E \) is almost periodic function iff the set \( H(f) = \{f_h : h \in \mathbb{R}\} \) is \(-\)compact in \( C_b(\mathbb{R}, E) \).

**Theorem 11.2.9.** Let \( E \) be an \( F \)-space, and let \( f_1, ... , f_m \in AP(\mathbb{R}, E) \). Define \( F : \mathbb{R} \to E^m \) by

\[
F(t) = (f_1(t), ..., f_m(t)), \quad t \in \mathbb{R}.
\]

Then \( F \in AP(\mathbb{R}, E^m) \). In particular, for any \( W \in W \), \( f_1, ..., f_m \) have common \( W \)-translation numbers.

**Proof.** Let \( \{h_n\} \) be a sequence in \( \mathbb{R} \). Consider the sequence \( \{f_{1,h_n}\} \) of translates of function \( f_1 \) corresponding to \( \{h_n\} \). Since \( f_1 \) is almost periodic, by using Bochner’s criteria, we can extract from \( \{f_{1,h_n}\} \) a uniformly convergent subsequence, denoting again by \( \{f_{1,h_n}\} \). Continuing this process, we extract from \( \{f_{m,h_n}\} \) a uniformly convergent subsequence, denoted also by \( \{f_{m,h_n}\} \). Then the sequence \( \{(f_{1,h_n}, ..., f_{m,h_n})\} \) has a subsequence which is easily seen to be \(-\)convergent on \( \mathbb{R} \). Hence \( F \) is almost periodic. Next, for any \( W \in W \), there exists a number \( \ell = \ell_W > 0 \) such that each interval of length \( \ell \) in \( \mathbb{R} \) contains a point \( \tau_W \) such that

\[
F(t + \tau_W) - F(t) \in W^m \quad \text{for all } t \in \mathbb{R}.
\]

Consequently,

\[
f_i(t + \tau_W) - f_i(t) \in W \quad \text{for all } t \in \mathbb{R} \text{ and } i = 1, ..., m.
\]

**Theorem 11.2.10.** [NGu84, KA11] Let \( E \) be a \( F \)-space, and let \( f, g \in AP(\mathbb{R}, E) \). Then \( f + g \in AP(\mathbb{R}, E) \).

**Proof.** In view of the Bochner’s criteria, we need to show that \( H(f + g) = \{f_h + g_h : h \in \mathbb{R}\} \) is \(-\)compact in \( C_b(\mathbb{R}, E) \). Define \( F : \mathbb{R} \to E^2 \) by

\[
F(t) = (f(t), g(t)), \quad t \in \mathbb{R}.
\]

By Theorem 11.2.8, \( F \in AP(\mathbb{R}, E^2) \) and hence, by Bochner's criteria, \( H(F) = \{(f_h, g_h) : h \in \mathbb{R}\} \) is \(-\)compact in \( C_b(\mathbb{R}, E^2) \). Now define \( S : C_b(\mathbb{R}, E) \times C_b(\mathbb{R}, E) \to C_b(\mathbb{R}, E) \) by

\[
S(u, v) = u + v, \quad u, v \in C_b(\mathbb{R}, E).
\]

This is a continuous function, hence \( S(H(F)) = H(f + g) \) is \(-\)compact in \( C_b(\mathbb{R}, E) \).

We next consider completeness of the space \( (AP(\mathbb{R}, E), u) \).
Theorem 11.2.11. [NGu84, KA11] Let $E$ be an $F$-space. Then the vector space $AP(\mathbb{R}, E)$ is $u$-complete.

Proof. By Theorems 11.2.4. and 11.2.10, $AP(\mathbb{R}, E)$ is a vector space. Further, since each $f \in AP(\mathbb{R}, E)$ is bounded, $AP(\mathbb{R}, E)$ is a vector subspace of $C_b(\mathbb{R}, E)$. Since $E$ is complete, by Theorem 11.2.1, $C_b(\mathbb{R}, E)$ is $u$-complete. So we need only to see that $AP(\mathbb{R}, E)$ is $u$-closed in $C_b(\mathbb{R}, E)$.

Proof. Let $f \in C_b(\mathbb{R}, E)$ with $f \in u-cl[AP(\mathbb{R}, E)]$. Then there exists a sequence $\{f_n\} \subseteq AP(\mathbb{R}, E)$ such that $f_n \xrightarrow{u} f$. Since each $f_n$ is almost periodic and $f_n \xrightarrow{u} f$, by Theorem 11.2.3, $f$ is almost periodic. Hence $f \in AP(\mathbb{R}, E)$. Thus $AP(\mathbb{R}, E)$ is $u$-closed in $C_b(\mathbb{R}, E)$.

Remark. We mention that a continuous functions $f : [a; b] \rightarrow E$ need not be integrable in the Riemann sense, if $E$ is not locally convex ([Rol85], p. 123). However, if $E$ is a locally pseudoconvex $F$-space, then all analytic functions $f : [a; b] \rightarrow E$ are integrable in the Bochner-Lebesgue sense ([Rol85], Theorem 3.5.2). Using a similar approach, almost periodicity of functions with values in $p$-Fréchet spaces, $0 < p < 1$, has been considered in [GaNGu07]. This paper also contains some applications to differential equations and to dynamical systems.
3. Notes and Comments

Section 11.1. A useful reference for the background of the results in this section is [FM91]; see also [Mcl65, Vain64]. These results are given in [Vain64] for $E$ and $F$ normed spaces and in [AS67] for $E$ and $F$ locally convex space. In the general case, similar results are given in [Llo73] without proof. The proofs in the present form are taken from [Kh97]. These include a shorter and direct proof of Theorem 11.5 based on the separation form of the Hahn-Banach theorem (cf. [AS67]). The reader is referred to [PRS01] for more recent versions of the theorem in the TVS setting.

We now indicate further possible generalizations of the mean value theorem.

(I)  


(II)  

For Fuzzy mappings. A useful references in this direction is the paper: M. Ferraro and D.H. Foster, *Differentiation of fuzzy continuous mappings on fuzzy topological vector spaces*, J. Math. Anal. Appl., 121(1987), 589-601. This paper does not contain any version of the Mean Value Theorem, but it seems likely that such a theorem can be formulated.

Section 11.2. The theory of almost periodic functions was mainly created and published during 1924-1926 by the Danish mathematician Harold Bohr; Bohr’s work was preceded by the important investigations of P. Bohl and E. Esclangon (see references in [Cor89, LeZh82]). The theory attracted the interest of a number of researchers because it has been effectively applied to solutions of diverse problems, initially in the theory of harmonic analysis and differential equations. In 1933, S. Bochner [Bo33] published an important article devoted to the extension of the theory of almost periodic functions on the real line $\mathbb{R}$ with values in a Banach space $E$. His results were further developed by several mathematicians (see the monographs by L. Amerio and G. Prouse [AmPr71], C. Corduneanu [Cor89], B.M. Levitan and V.V. Zhikov [LeZh82], and
S. Zaidman [Za85]). The theory of almost periodic functions taking values in a complete metrizable locally convex space $E$ has been initiated by G.M. N’Guérékata [NGu84] and further developed in [BuNgu04, GaNGu07, NGu01]. A survey paper by A.I. Shtern [Sh05] also considers almost periodic functions and representations in locally convex spaces.

In this section, we have considered the concept of almost periodicity of functions having values in a topological vector space $E$, not necessarily locally convex. We are mainly concerned with studying the topological properties of almost periodic functions and demonstrate the validity of several known results, including some from [NGu84, BuNgu04, GaNGu07] to this general setting, as given in [KA11].
CHAPTER 12

Non-Archimedean Function Spaces

Let \( X \) be a Hausdorff topological space. As mentioned earlier, the strict topology was introduced for the first time by Buck in [4] on the space of all bounded continuous functions on a locally compact space \( X \). Several other authors have extended Buck’s results by taking as \( X \) an arbitrary completely regular space and considering spaces of continuous functions on \( X \) which are either real-valued or have values in a classical locally convex space or even in a classical topological vector space. In the non-Archimedean case some authors studied strict topologies on spaces of continuous functions, on a zero-dimensional topological space, with values either in a non-Archimedean valued field \( F \) or in a non-Archimedean locally convex space over \( F \). In this chapter we will consider the case where the functions take their values in a non-Archimedean TVS.

Throughout this chapter, unless it is stated explicitly otherwise, \( X \) will be a non-empty Hausdorff topological space, \( F \) a non-Archimedean non-trivially valued field (with unit element \( e \)) and \( E \) a non-trivial Hausdorff TVS over \( F \) with a base \( W \) of neighborhoods of 0 in \( E \) consisting of balanced and absorbing sets. As before, we denote by \( C(X,E) \) the space of all continuous \( E \)-valued functions on \( X \). By \( C_b(X,E) \) (resp. \( C_o(X,E), C_{oo}(X,E), C_{rc}(X,E) \)), we will denote the space of all \( f \in C(X,E) \) which are bounded (resp. vanish at infinity, vanish outside some compact set, have relatively compact range) in \( E \). In case \( E = F \), we will write \( C(X,F), C_b(X,F), C_o(X,F), C_{oo}(X,F) \) and \( C_{rc}(X,F) \), respectively.
1. Compact-open Topology on $C(X,E)$

In this section, we shall study the $\tau_K$ topology which is the non-Archimedean analogues of the usual compact open topology studied in earlier chapters.

First we recall some terminology (§A.1, A.5). A topological space $(X,\tau)$ is called zero-dimensional if, for each $x \in X$ and each neighborhood $U$ of $x$, there exists a clopen (i.e., both closed and open) set $V$ such that $x \in V$ and $V \subseteq U$. The family of all clopen subsets of $X$ will be denoted by $\text{Clo}(X)$. A valued field $F$ is said to be non-trivially valued if there is an $\alpha \in F$ such that $0 < |\alpha| \neq 1$. By the $F$-characteristic function of a set $A \subseteq X$, we mean the function $\chi_A : X \to F$ defined by $\chi_A(x) = e$ if $x \in A$ and $\chi_A(x) = 0$ if $x \notin X$ ($x \in X$). Note that, for any $\varphi \in C(X,F)$ and $\varepsilon > 0$, the set $\{ t \in X : |\varphi(t)| \geq \varepsilon \}$ is clopen in $X$ since it is the inverse image of the clopen set $F\setminus B(0,\varepsilon)$ under a continuous map. For any non-zero scalar $\lambda \in F$, we shall denote its inverse by $\lambda^{-1}$ (instead of $\frac{1}{\lambda}$). By $S(X,F)$ we will denote the space spanned by the functions $\chi_A, A \in \text{Clo}(X)$, while $S(X,F) \otimes E$ denotes the space spanned by the functions $\chi_A \otimes a, A \in \text{Clo}(X)$ and $a \in E$.

Let $(E,\tau)$ be a topological vector space (in short, a TVS) over a valued field $(F,|\cdot|)$. A subset $S$ of $E$ is called a non-Archimedean set if $S + S \subseteq S$. The pair $(E,\tau)$ is called a non-Archimedean TVS if it has a base of $\tau$neighborhoods of 0 consisting of non-Archimedean sets.

Every non-Archimedean neighborhood of 0 in a TVS is a clopen (i.e. both closed and open) set. Every non-Archimedean valued field $(F,|\cdot|)$ is a non-Archimedean TVS over itself. Further, any indiscrete TVS is a non-Archimedean TVS. In particular, the trivial TVS $E = \{0\}$ is a non-Archimedean TVS.

Recall that a subset $A$ of $X$ is said to be bounding if every continuous real valued function on $X$ is bounded on $A$. If every $f \in C(X,F)$ is bounded on $A$, then $A$ is said to be $F$-bounding. It is easy to see that continuous images of bounding (resp. $F$-bounding ) sets are bounding (resp. $F$-bounding ). The space $X$ is said to be pseudocompact (resp. $F$-pseudocompact), if every continuous real function on $X$ (resp. every $f \in C(X,F)$) is bounded.

**Lemma 12.1.1** If $X$ is zero-dimensional, then a subset $A$ of $X$ is bounding iff it is $F$-bounding.

**Proof.** Assume that $A$ is not bounding and let $g$ be a continuous real function on $X$ which is not bounded on $A$. Then there exist a sequence $(n_k)$ of positive integers and a sequence $(x_k)$ in $A$ such that $n_k < |g(x_k)| < 0$. Therefore, $g$ is unbounded on $A$. This contradicts the fact that $A$ is bounding on $X$. Hence, $A$ is bounding on $X$. Conversely, if $A$ is $F$-bounding, then it is also bounding by definition. Therefore, $A$ is bounding on $X$. This completes the proof. 


n_{k+1} for all k. For each k, there exists a clopen neighborhood \( B_k \) of \( x_k \) contained in \( \{ x : n_k < |g(x)| < n_{k+1} \} \). The set \( B_o = X \setminus \bigcup_{k=1}^\infty B_k \) is open since
\[
B_o = \bigcup_{k=1}^\infty G_k,
\]
where \( G_k = \{ x : |g(x)| < n_k + 1 \} \setminus \bigcup_{i=1}^k B_i \).

Hence the sets \( B_o, B_1, \ldots \) form a clopen partition of \( X \). Choose \( \lambda \in \mathbb{F}, |\lambda| > 1 \), and let \( h = \sum_{k=1}^\infty \lambda^k X_{B_k} \). Then \( h \in C(X, \mathbb{F}) \) and \( h(x_k) = \lambda^k \), and so \( h \) is not bounded on \( A \), which implies that \( A \) is not \( \mathbb{F} \)-bounding. Now the result clearly follows.

Recall that a completely regular Hausdorff space \( Y \) is c-complete or universally complete iff \( Y \) is homeomorphic to a closed subspace of a product of metrizable topological spaces. If \( Y \) is c-complete, then every bounding subset of \( Y \) is relatively compact. The c-completion of \( Y \) is the smallest c-complete subspace of the Stone-Čech compactification \( \beta Y \) of \( Y \), which contains \( Y \), and it is denoted by \( \theta Y \).

**Definition.** A basic sequence in \( E \) is a sequence \( \mathcal{U} = (V_n) \) of balanced absorbing subsets of \( E \) such that
1. \( V_{n+1} + V_{n+1} \subseteq V_n \) for all \( n \).
2. For each \( n \) and each non-zero scalar \( \lambda \), there exists an \( m \) such that \( V_m \subseteq \lambda V_n \).

Every basic sequence \( \mathcal{U} = (V_n) \) induces a pseudometrizable linear topology \( \tau_\mathcal{U} \) on \( E \) for which the sequence \( (V_n) \) is a base at zero. If every \( V_n \in \mathcal{W} \), then the basic \( \mathcal{U} \) is called topological. Given any such \( \mathcal{U} \), the set \( \ker \mathcal{U} = \cap V_n \) is a subspace of \( E \). Let \( E_\mathcal{U} = E/\ker \mathcal{U} \) be the quotient space and \( \pi = \pi_\mathcal{U} \) the quotient map. Then \( \pi(\mathcal{U}) = (\pi(V_n)) \) is a basic sequence in \( E_\mathcal{U} \).

The following Theorem is contained in [BBNW75].

**Theorem 12.1.2.** Every Hausdorff topological vector space \( E \) over \( K \) is topologically isomorphic to a subspace of a product of metrizable topological vector spaces.

**Proof.** We will only sketch the proof. Let \( \mathcal{F} \) be the collection of all topological basic sequences in \( E \) and let \( G = \prod_{\mathcal{U} \in \mathcal{F}} E_\mathcal{U} \). Then the map \( \varphi : E \to G \) given by
\[
\varphi(a)_\mathcal{U} = \pi_\mathcal{U}(a), \quad a \in E,
\]
is a linear homeomorphism between \( E \) and \( \varphi(E) \). \( \square \)

**Remark.** Using the preceding Theorem we get that \( E \) is completely regular.
If \( \hat{E} \) is the completion of \( E \), then the family of all closures \( \overline{W}^E \) in \( \hat{E} \), for closed \( W \in \mathcal{W} \), is a base at 0 in \( \hat{E} \). Using this fact, we get the following

**Lemma 12.1.3.** If \( A \subseteq E \), then \( A \) is totally bounded in \( E \) iff it is totally bounded in \( \hat{E} \).

**Proof.** It is clear that if \( A \) is totally bounded in \( E \), then it is totally bounded in \( \hat{E} \). For the converse, suppose that \( A \) is totally bounded in \( \hat{E} \) and let \( W \in \mathcal{W} \) be a closed. Choose another closed \( W_1 \in \mathcal{W} \) such that \( W_1 + W \subseteq W \). Let \( V \) and \( V_1 \) be the closures of \( W \) and \( W_1 \) in \( \hat{E} \) and let \( S = \{e_1,e_2,\ldots,e_n\} \) be a finite subset of \( \hat{E} \) such that \( A \subseteq S + V_1 \). For \( k = 1,\ldots,n \), choose \( a_k \in E \) such that \( e_k - a_k \in V_1 \). Given \( x \in A \), there exist \( k \) and \( z \in V_1 \) such that

\[
 x = e_k + z + a_k + V_1 \subseteq a_k + V.
\]

Thus \( x - a_k \in V \cap E = W \) and so \( A \subseteq \{a_1,a_2,\ldots,a_n\} + W \). \( \square \)

**Theorem 12.1.4** (1). If \( E \) is complete, then \( E \) is c-complete.

(2). Every bounding subset of \( E \) is totally bounded.

(3). If \( E \) is complete, then for a subset \( A \) of \( E \) the following are equivalent:

(a) \( A \) is bounding.
(b) \( A \) is totally bounded.
(c) \( A \) is relatively compact.

**Proof.** (1). By Theorem 12.1.2, \( E \) is topologically isomorphic to a subspace \( F \) of a product \( \Pi_{i \in I} E_i \) of metrizable topological vector spaces. The space \( F \) is closed in the product since \( E \) is complete. It follows that \( E \) is c-complete.

(2). Let \( A \) be a bounding subset of \( E \). Then \( A \) is a bounding subset of the completion \( \hat{E} \) of \( E \). Since \( \hat{E} \) is c-complete, the closure \( \overline{A}^E \) is compact and hence totally bounded, which implies that \( A \) is totally bounded.

(3). (a) \( \Rightarrow \) (b) follows from (2).

(b) \( \Rightarrow \) (c) follows since \( E \) is complete.

(c) \( \Rightarrow \) (a) Suppose \( A \) is relatively compact. Then \( \overline{A} \) is compact and hence bounding, which implies that \( A \) is bounding. \( \square \)

**Theorem 12.1.5** Assume that \( X \) is zero-dimensional. For a subset \( A \) of \( X \), the following are equivalent:

(1) \( A \) is bounding.
(2) \( A \) is \( \mathcal{F} \)-bounding.
(3) For each \( f \in C(X,E) \), the set \( f(A) \) is totally bounded in \( E \).
(4) For each \( f \in C(X,E) \), the set \( f(A) \) is bounded in \( E \).
Proof. (1)⇔(2) follows from Lemma 12.1.1.

(1)⇒(3) follows since continuous images of bounding sets are bounding and in view of the preceding Theorem 12.1.4.

(3)⇒(4) is clear.

(4)⇒(1) Assume that (4) holds and let \( \varphi \in C(X, F) \). Since \( E \) is Hausdorff and non-trivial, there exists an \( u \in E \) and a closed balanced \( W \in W \) not containing \( u \). Since the function \( f = \varphi \otimes e \) is bounded on \( A \), there exists a non-zero scalar \( \lambda \) such that \( \varphi(A)u \subseteq \lambda W \). If now \( x \in A \), then \( \lambda^{-1}\varphi(x)u \in W \) and so \( |\lambda^{-1}\varphi(x)| < 1 \) since \( W \) is balanced and \( u \notin W \). It follows that \( |\varphi(x)| < |\lambda| \) for all \( x \in A \). Thus \( A \) is \( F \)-bounding. \( \square \)

**Notation.** For \( V \in W \) and \( A \subseteq X \), we denote by

\[
N(A, V) : = N(\chi_A, V) = \{ f \in C(X, E) : f(A) \subseteq V \},
\]

\[
N_b(A, V) : = N_b(\chi_A, V) = N(A, V) \cap C_b(X, E).
\]

Throughout, \( K \) denotes a family of compact subsets of \( X \) which covers \( X \) and it is upwards directed by set inclusion. If \( A \in K \) is compact, then each \( N(A, V) \) is absorbing. Further, if \( V + V \subseteq W \), then \( N(A, V) + N(A, V) \subseteq N(A, W) \). We get now easily the following

**Theorem 12.1.6.** The family

\[
\Omega_{K, B} = \{ N(A, V) : A \in K, V \in W \}
\]

is a base at 0 in \( C(X, E) \) for a Hausdorff linear topology \( \tau_K \). Moreover \( \tau_K \) does not depend on the particular choice of \( B \).

**Note.** (1) Let \( A_o \in K \) and \( V_o \in W \). If \( (V_n) \) is a basic topological sequence in \( E \) such that \( V_1 + V_1 \subseteq V_o \), then \( (N(A_o, V_n)) \) is a basic sequence in \( C(X, E) \) and \( N(A_o, V_1) + N(A_o, V_1) \subseteq N(A_o, V_o) \).

(2) If \( K = K(X) \) is the family of all finite(resp. compact) subsets of \( X \), we will denote \( \tau_K \) by \( \tau_k \) (resp. \( \tau_p \)).

(3) The topology of uniform convergence \( \tau_u \) on \( C_b(X, E) \) has a base of neighborhood of 0 consisting of all sets of the form \( N(X, V), V \in W \).

Recall that a sequence \( (B_n) \), of subsets of of a TVS \( G \), is called a fundamental sequence of bounded sets, if each \( B_n \) is bounded and each bounded subset of \( G \) is contained in some \( B_n \).

**Lemma 12.1.7** If \( (C(X, E), \tau_K) \) has a fundamental sequence \( (G_n) \) of bounded sets, then \( E \) also has the same property.

**Proof.** For each \( A \in K \) and each \( n \) the set \( G_n(A) \) is bounded in \( E \). In particular, since \( K \) covers \( X \), each \( G_n(x), x \in X \), is bounded in \( E \). For \( a \in E \), denote by \( \tilde{a} \) the function which is defined on \( X \) by \( \tilde{a}(x) = a \).
Let $B_n = \{a \in E : \bar{a} \in G_n\}$. For $x \in X$, we have that $B_n \subseteq G_n(x)$ and so $B_n$ is a bounded subset of $E$. Let $B$ be any bounded subset of $E$ and $G = \{\bar{a} : a \in B\}$. Since $G(A) \subseteq B$ for every $A \in \mathcal{K}$, it follows that $G$ is a bounded subset of $(C(X, E), \tau_K)$ and so $G \subseteq G_n$, for some $n$, which implies that $B \subseteq B_n$. Therefore $(B_n)$ is a fundamental sequence of bounded subsets of $E$.

Recall that a Hausdorff zero-dimensional topological space $X$ is a $P$-space if every countable intersection of clopen subsets of $X$ is clopen.

**Theorem 12.1.8.** If $X$ is a zero-dimensional $P$-space and $E$ sequentially complete, then each of the spaces $(C(X, E), \tau_\rho)$ and $(C(X, E), \tau_k)$ is sequentially complete.

**Proof.** (1). Let $\{f_n\}$ be a $\tau_\rho$-Cauchy sequence. Since $E$ is sequentially complete, the limit $f(x) = \lim f_n(x)$ exists in $E$ for each $x$. The function $f$ is continuous. Indeed, let $x \in X$ and let $V \in \mathcal{W}$ be closed. Let $V_1$ be the interior of $V$ and let $m$ be such that $f_n(x) \in f(x) + V_1$ for all $n \geq m$.

For each $n \geq m$, there exists a clopen neighborhood $A_n$ of $x$ such that $f_n(A_n) \subseteq f(x) + V_1$. The set $A = \bigcap_{n \geq m} A_n$ is clopen and $f_n(y) \in f(x) + V$ for all $y \in A$ and all $n \geq m$.

It follows that $f(y) \in f(x) + V$ for all $y \in A$, which proves that $f$ is continuous at $x$.

(2). Suppose that $\{f_n\}$ is $\tau_k$-Cauchy. Then $\{f_n\}$ is $\tau_\rho$-Cauchy and hence $f_n \to f$ pointwise, for some $f \in C(X, E)$. Given a closed $V \in \mathcal{W}$ and a compact subset $Y$ of $X$, there exists $n_o$ such that $(f_n - f_m)(y) \subseteq V$ for all $n, m \geq n_o$.

Since $V$ is closed and $\lim f_m(x) = f(y)$ for all $y$, it follows that $(f_n - f)(Y) \subseteq V$ for all $n \geq n_o$, which proves that $f_n \xrightarrow{\tau_k} f$. □

**Theorem 12.1.9.** Let $K$ be a compact subset of the zero-dimensional topological space $X$ and let $g$ be a continuous $E$-valued function on $K$. Then:

1. Given a $V \in \mathcal{W}$, there exists an $f \in S(X, F) \otimes E$ such that $f(X)$ is a finite subset of $g(K)$ and $(g - f)(K) \subseteq V$.

2. If $E$ is complete and metrizable, then there exists a continuous $E$-valued function $f$ on $X$ which is an extension of $g$ and whose image in $E$ is relatively compact.
Proof. (1). For each \( x \in K \), there exists a clopen neighborhood \( A_x \) of \( x \) in \( K \) such that

\[
g(y) - g(x) \in V \quad \text{for all} \quad y \in A_x.
\]

By the compactness of \( K \), there are \( x_1, x_2, \ldots, x_n \) in \( K \) such that \( K = \cup_{k=1}^n A_{x_k} \). Let

\[
B_1 = A_{x_1}, \quad B_2 = A_{x_2} \setminus B_1, \ldots, \quad B_n = A_{x_n} \setminus \cup_{k=1}^{n-1} B_k.
\]

For each \( k \) there exists a clopen subset \( Z_k \) of \( X \) such that \( Z_k \cap K = B_k \). Replacing \( Z_k \) by \( Z_k \setminus \cup_{j \neq k} Z_j \), we may assume that the sets \( Z_1, \ldots, Z_n \) are pairwise disjoint. Now it suffices to take \( f = \sum_{k=1}^n \chi_{Z_k} g(x_k) \).

(2). Assume that \( E \) is metrizable and complete and let \( \lambda \in \mathbb{F}, \; 0 < |\lambda| < 1 \). There exists a base \( (V_n) \) of closed balanced neighborhoods of \( 0 \) in \( E \) such that \( V_{n+1} + V_{n+1} \subseteq V_n \) and \( V_{n+1} \subseteq \lambda V_n \) for all \( n \). By (1), there exists a function \( f_1 \in S(X, \mathbb{F}) \otimes E \) such that

\[
f_1(X) \subseteq g(K) \quad \text{and} \quad (g - g_1)(K) \subseteq V_2, \quad \text{where} \quad g_1 = f_1|_K.
\]

Applying (1) to the function \( g - g_1 \), we get a function \( f_2 \in S(X, \mathbb{F}) \otimes E \) such that

\[
f_2(X) \subseteq (g - g_1)(K) \quad \text{and} \quad (g - g_1 - g_2)(K) \subseteq V_3, \quad \text{where} \quad g_2 = f_2|_K.
\]

Continuing by induction, we get a sequence \( \{f_n\} \in S(X, \mathbb{F}) \otimes E \) such that, for \( g_k = f_k|_K \), we have that

\[
(g - \sum_{k=1}^n g_k)(K) \subseteq V_{n+1} \quad \text{and} \quad f_n(X) \subseteq (g - \sum_{k=1}^{n-1} g_k)(K) \quad \text{for all} \quad n.
\]

**Claim I** The series \( \sum_{n=1}^\infty f_n \) converges uniformly to a continuous function \( f \) on \( X \). Indeed, let \( h_n = \sum_{k=1}^n f_k \) and let \( V \in \mathcal{W} \) closed

There exists an \( m \) such that \( V_m \subseteq V \). For \( m < n < N \), we have that

\[
h_N - h_n = \sum_{k=n+1}^N f_k.
\]

But, for \( k > n \), we have

\[
f_k(X) \subseteq \left( g - \sum_{i=1}^{k-1} g_i \right)(K) \subseteq V_k
\]

and so

\[
(h_N - h_n)(X) \subseteq V_{n+1} + V_{n+2} + \ldots + V_N \subseteq V.
\]

Thus the sequence \( \{h_n\} \) is \( \tau_w \)-Cauchy in \( C_b(X, E) \) and hence it converges uniformly to a continuous function \( f \in C_b(X, E) \). It is easy to see that \( f(X) \) is totally bounded in \( E \) and hence relatively compact.
Claim II \( f = g \) on \( K \). Indeed, let \( y \in K \) and let \( V \in \mathcal{W} \) be closed and balanced. There exists an \( m \) such that \( V_m \subseteq V \). Let \( n \geq m \). Then
\[
g(y) - \sum_{k=1}^{n} f_k(y) \in V_{n+1} \subseteq V.
\]

Since \( V \) is closed, we get that \( g(y) - f(y) \in V \). This, being true for all closed balanced neighborhoods \( V \) of 0 in \( E \), implies that \( g(y) = f(y) \) since \( E \) is Hausdorff. Hence the result follows. \( \square \)

**Corollary 12.1.10.** If \( X \) is zero-dimensional, then \( S(X, F) \otimes E \) is \( \tau_k \)-dense in \( C(X, E) \).

**Theorem 12.1.11.** Assume that \( X \) is zero-dimensional and let \( \mathcal{K} \) be an upwards directed family of compact subsets of \( X \) covering \( X \). Then \( (C(X, E), \tau_{\mathcal{K}}) \) is complete iff \( E \) is complete and every \( f \in F(X, E) \) such that \( f|_A \) is continuous, for each \( A \in \mathcal{K} \), is continuous on \( X \).

Proof. (\( \Rightarrow \)) Suppose that \((C(X, E), \tau_{\mathcal{K}})\) is complete. Since \( \mathcal{K} \) covers \( X \), it follows easily that \( E \) is complete. Let \( f \in F(X, E) \) be such that \( f|_A \) is continuous, for each \( A \in \mathcal{K} \). Consider the the set \( \mathcal{F} \) of all pairs \((A, V)\), where \( A \in \mathcal{K} \) and \( V \in \mathcal{W} \) closed and balanced. We make \( \mathcal{F} \) into a directed set by defining
\[
(A_1, V_1) \geq (A_2, V_2) \text{ iff } A_2 \subseteq A_1 \text{ and } V_1 \subseteq V_2.
\]

Given \( \alpha = (A, V) \in \mathcal{F} \), there exists (by Theorem 12.1.9) a function \( g_\alpha = g_{A,V} \) in \( C(X, E) \) such that \((f - g_\alpha)(A) \subseteq V\). The net \((g_\alpha)_{\alpha \in \mathcal{F}}\) is Cauchy in \((C(X, E), \tau_{\mathcal{K}})\). Indeed, let \( V \in \mathcal{W} \) be balanced and \( A_0 \in \mathcal{K} \). Choose a closed balanced \( V_0 \in \mathcal{W} \) such that \( V_0 + V_0 \subseteq V \). Let \( \alpha_0 = (A_0, V_0) \) and let \( \alpha_i = (A_i, V_i), i = 1, 2, \alpha_i \geq \alpha_o \). Then \((f - g_{\alpha_0})(A_0) \subseteq V_i \subseteq V_0 \) and so
\[
(g_{\alpha_1} - g_{\alpha_2})(A_0) \subseteq V_0 + V_0 \subseteq V.
\]

This proves that the net \((g_\alpha)_{\alpha \in \mathcal{F}}\) is Cauchy in \((C(X, E), \tau_{\mathcal{K}})\) and hence it converges to some \( g \in C(X, E) \). We will show that \( g = f \). Indeed, let \( x_o \in X \) and let \( V \in \mathcal{W} \). Choose a closed balanced \( V_o \in \mathcal{W} \) such that \( V_o + V_o \subseteq V \). Let \( A_0 \in \mathcal{K} \) containing \( x_o \). There exist \( A_1 \in \mathcal{K} \) containing \( A_o \) and a closed balanced \( V_1 \in \mathcal{W} \) contained in \( V_o \) such that
\[
(g - g_{\alpha})(A_o) \subseteq V_o \text{ for all } \alpha \geq \alpha_1 = (A_1, V_1).
\]

Thus
\[
g(x_o) - g_{\alpha_1}(x_o) \subseteq (g - g_{\alpha_1})(A_o) \subseteq V_o.
\]

Also
\[
f(x_o) - g_{\alpha_1}(x_o) \subseteq (f - g_{\alpha_1})(A_1) \subseteq V_1 \subseteq V_o.
\]
and hence \( g(x_o) - f(x_o) \in V_o - V_o \subseteq V \). It follows that \( f = g \) and so \( f \) is continuous.

\((\Leftarrow)\) Suppose that \( E \) is complete and every \( f \in F(X, E) \) is continuous if \( f \rvert_A \) is continuous for all \( A \in \mathcal{K} \). Let \((f_\alpha)\) be a Cauchy net in \((C(X, E), \tau_K)\). For each \( x \in X \), the net \( \{f_\alpha(x)\} \) is Cauchy in \( E \) and hence convergent. Define \( f(x) = \lim f_\alpha(x) \). If \( V \in \mathcal{W} \) is a closed balanced and \( A_o \in \mathcal{K} \), there exists \( \alpha_o \) such that \((f_\alpha - f)(A_o) \subseteq V\) for all \( \alpha_1, \alpha_2 \geq \alpha_o \). Thus \((f_\alpha - f)(A_o) \subseteq V\) for all \( \alpha \geq \alpha_o \), which proves that \( f_\alpha \to f \) uniformly on \( A_o \). This implies that \( f \rvert_A \) is continuous for each \( A \in \mathcal{K} \) and hence \( f \) is continuous, Moreover, \( f_\alpha \to f \) in \((C(X, E), \tau_K)\). \( \blacksquare \)

**Corollary 12.1.12.** Suppose that \( X \) is zero-dimensional. Then \((C(X, E), \tau_\rho)\) is complete iff \( E \) is complete and \( X \) discrete.

**Proof.** \((\Rightarrow)\) In view of the preceding Theorem, if \( E \) is complete and \( X \) discrete, then \((C(X, E), \tau_\rho)\) is complete.

\((\Leftarrow)\) Conversely, suppose that \((C(X, E), \tau_\rho)\) is complete. Then \( E \) is complete. Let \( A \subseteq X \) and \( a \) a non-zero element of \( E \). Let \( f = \chi_A a \). For every finite subset \( B \) of \( X \), \( f \rvert_B \) is continuous and hence \( f \) is continuous by the Theorem 12.1.11. The set \( A \) is closed. Indeed let \( x \in A \setminus A \) and let \( \{x_\alpha\} \) be a net in \( A \) converging to \( x \). Then \( f(x_\alpha) \to f(x) = 0 \). There exists an open \( V \in \mathcal{W} \) not containing \( a \). Since \( f(x) = 0 \in V \), there exists a \( \alpha \) such that \( a = f(x_\alpha) \in V \), a contradiction. Thus every subset of \( X \) is closed and hence \( X \) is discrete. \( \blacksquare \)
2. Strict Topology on $C_b(X, E)$

In this section, we shall study the $\tau_p$ topology which is the non-Archimedean analogues of the usual strict topology studied in earlier chapters.

For $\varphi \in F(X, F)$ and $V \in \mathcal{W}$, we set

$$N(\varphi, V) = \{ f \in C_b(X, E) : (\varphi f)(X) \subseteq V \}.$$

It is easy to see that:

1. If $\varphi \in B(X, F)$, then $N(\varphi, V)$ is absorbing in $C_b(X, E)$.
2. For $\lambda$ a non-zero scalar, we have $\gamma N(\varphi, V) = N(\lambda^{-1}\varphi, V) = N(\varphi, \lambda V)$.
3. If $V$ is balanced, then $N(\varphi, V)$ is balanced.
4. If $V \subseteq V_1 \cap V_2$ and $|\varphi| \geq \max\{|\varphi_1|, |\varphi_2|\}$ and if $V$ is balanced, then
   $$N(\varphi, V) \subseteq N(\varphi_1, V_1) \cap N(\varphi_2, V_2).$$
5. If $V$ is convex, then $N(\varphi, V)$ is convex.

Let now $\mathcal{P}$ be an upwards directed family of compact subsets of $X$ covering $X$. We denote by $B_{\mathcal{P}}(X, F)$ the collection of all bounded functions $\varphi \in F(X, F)$ which $\mathcal{P}$-vanish at infinity, i.e. for each $\varepsilon > 0$ there exists an $A \in \mathcal{P}$ such that $|\varphi(x)| < \varepsilon$ if $x \notin A$. We denote by $\beta_\mathcal{P}$ the linear topology on $C_b(X, E)$ for which the family of all $N(\varphi, V)$, $\varphi \in B_{\mathcal{P}}(X, F)$ and $V \in \mathcal{W}$, is a base at 0. It follows easily that $\beta_\mathcal{P}$ has as a base at 0 the family of all $N(\varphi, V)$, with $V \in \mathcal{W}$ and $\varphi \in B_{\mathcal{P}}(X, F)$, $\|\varphi\| \leq 1$.

In case $\mathcal{P}$ is the family of all compact (resp. all finite) subsets of $X$, we will denote the corresponding $\beta_\mathcal{P}$ by $\beta_o$ (resp. $\beta_s$). Let $\tau_u$ denote the topology of uniform convergence on $C_b(X, E)$.

**Theorem 12.2.1.**
1. $\tau_p \leq \tau_p \leq \beta_\mathcal{P} \leq \tau_u$ on $C_b(X, E)$.
2. $\tau_u$, $\beta_\mathcal{P}$ and $\beta_o$ have the same bounded sets.
3. $\tau_p$ and $\beta_\mathcal{P}$ coincide on $\tau_u$-bounded sets.
4. A sequence $\{f_n\}$ in $C_b(X, E)$ is $\beta_\mathcal{P}$-convergent to some $f$ iff it is $\tau_u$-bounded and $f_n \xrightarrow{\tau_u} f$.
5. If $(C_b(X, E), \beta_\mathcal{P})$ is bornological, then $\beta_\mathcal{P} = \tau_u$.
6. If $X$ is zero-dimensional, then
   a. $\tau_u = \beta_\mathcal{P}$ iff $X \in \mathcal{P}$.
   b. $\tau_\mathcal{P} = \beta_\mathcal{P}$ iff, for each sequence $(A_n)$ in $\mathcal{P}$, there exists $A$ in $\mathcal{P}$ which contains every $A_n$.
   c. $C_{\omega o}(X, E)$ is $\beta_o$-dense in $C_b(X, E)$ iff $X$ is locally compact.

**Proof.** (1). Since $\mathcal{P}$ covers $X$ and is upwards-directed, it follows easily that $\tau_p$ is coarser than $\tau_\mathcal{P}$. Also $\tau_\mathcal{P}$ is coarser than $\beta_\mathcal{P}$ since the
\(F\)-characteristic function of any \(A \in \mathcal{P}\) belongs to \(B_{\alpha \mathcal{P}}(X, F)\). Finally, let \(V \in \mathcal{W}\) be balanced and \(\varphi \in \mathcal{P}\), \(\|\varphi\| \leq 1\). Then
\[
\{f \in C_b(X, E) : f(X) \subseteq V\} \subseteq N(\varphi, V),
\]
which proves that \(\beta_\mathcal{P}\) is coarser than \(\tau_\alpha\).

(2). Assume that there exists a subset \(G\) of \(C_b(X, E)\) which is \(\beta_s\)-bounded but not \(\tau_\alpha\)-bounded. Then there exists a balanced \(V \in \mathcal{W}\) such that \(G\) is not absorbed by the set \(W = \{f \in C_b(X, E) : f(X) \subseteq V\}\). Let \(\lambda \in F\), \(|\lambda| > 1\). Suppose that we have already chosen \(f_1, f_2, \ldots, f_n\) in \(G\) and distinct \(x_1, x_2, \ldots, x_n\) in \(X\) such that \(f_k(x_k) \notin \lambda^{2k}V\), for \(k = 1, \ldots, n\). The set \(D = \bigcup_{k=1}^n f_k(X)\) is bounded in \(E\). Hence there exists \(|\gamma| > |\lambda|^{2(n+1)}\) such that \(D \subseteq \gamma V\) and \(G \subseteq \gamma \cdot N_b(Z, V)\) where \(Z = \{x_1, x_2, \ldots, x_n\}\) and
\[
N_b(Z, V) = \{f \in C_b(X, E) : f(Z) \subseteq V\}.
\]
Since \(G\) is not absorbed by \(N_b(X, V)\), there exists \(f_{n+1}\) in \(G\) which is not in \(\cap_{\gamma} N_b(X, V)\) and hence \(\gamma^{-1} f_{n+1}(x_{n+1}) \notin V\), for some \(x_{n+1}\). But then \(x_{n+1} \notin Z\). Thus, we get by induction a sequence \(\{f_n\}\) in \(G\) and a sequence \(\{x_n\}\) of distinct elements of \(X\) such that
\[
f_n(x_n) \notin \lambda^{2n}V\quad \text{for all } n.
\]
Now define \(\varphi \in F(X, F)\) by \(\varphi(x_n) = \lambda^{-n}\) and \(\varphi(x) = 0\) if \(x \notin \{x_1, x_2, \ldots\}\). If \(S\) is the family of all finite subsets of \(X\), then \(\varphi \in B_{\alpha S}(X, F)\) and so \(G \subseteq \cap_{\alpha} N(\varphi, V)\) for some non-zero scalar \(\alpha\). Let \(n\) be such that \(|\lambda|^n > |\alpha|\).

Now
\[
\alpha^{-1} \lambda^{-n} f_n(x_n) = \alpha^{-1} f_n(x_n) \varphi(x_n) \in V,
\]
and so \(f_n(x_n) \in \alpha \lambda^n V \subseteq \lambda^{2n}V\), a contradiction. This clearly completes the proof of (2).

(3). Let \(G\) be a \(\tau_\alpha\)-bounded set, \(f_o \in G\), \(\varphi \in B_{\alpha \mathcal{P}}(X, F)\), with \(\|\varphi\| \leq 1\), and \(V \in \mathcal{W}\) be balanced. There exists a balanced \(V_1 \in \mathcal{W}\) with \(V_1 + V_1 \subseteq V\). Since \(G\) is \(\tau_\alpha\)-bounded, there exists a non-zero scalar \(\lambda\) such that \(G(X) \subseteq \lambda V_1\). Choose \(A \in \mathcal{P}\) such that \(|\varphi| < |\lambda|^{-1}\) on \(X \setminus A\). Now
\[
[f_o + N_b(A, V_1)] \cap G \subseteq f_o + N(\varphi, V).
\]
Indeed, let \(f \in G\) be such that \(f - f_o \in N_b(A, V_1)\). If \(x \in A\), then \(f(x) - f_o(x) \in V_1\) and so \(\varphi(x)[f(x) - f_o(x)] \in V_1 \subseteq V\). If \(x \notin A\), then \(|\varphi(x)| < |\lambda|^{-1}\) and hence
\[
\varphi(x)[f(x) - f_o(x)] \in \lambda^{-1} \lambda V_1 - \lambda^{-1} \lambda V_1 \subseteq V.
\]
Thus \(f \in f_o + N(\varphi, V)\), which clearly proves that \(\tau_\mathcal{P} = \beta_\mathcal{P}\) on \(G\).
Choose a balanced \( V \) such a neighborhood exists since \( E \) Thus \( \tau \), \( N \) that \( a \in x \). Now \( A \) and \( P \) sequence in zero-dimensional, there exists a clopen neighborhood \( Z \).

Thus \( \tau_u \) is coarser than \( \beta_p \) and so \( \tau_u = \beta_p \).

(\( \Leftarrow \)) Assume that \( \tau_u = \beta_p \) and let \( V \in W \) be a proper balanced set (such a neighborhood exists since \( E \) is Hausdorff and non-trivial). Choose a balanced \( V_1 \in W \) and \( \varphi \in B_{\beta} (X, E) \), \( 0 < \| \varphi \| \leq 1 \), such that \( N(\varphi, V_1) \subseteq N_b (X, V) \). Let \( a \in E \setminus V \) and choose a non-zero scalar \( \lambda \) such that \( a \in \lambda V_1 \). There exists \( A \in P \) such that \( |\varphi(x)| < |\lambda|^{-1} \) if \( x \notin A \). Now \( A = \lambda \). Indeed, suppose that there exists an \( x \in X \setminus A \). Since \( X \) is zero-dimensional, there exists a clopen neighborhood \( Z \) of \( x \) disjoint from \( A \). If \( f = \chi_Z \), then \( f \in N(\varphi, V_1) \) but \( f \notin N_b (X, V) \), a contradiction.

(b) (\( \Rightarrow \)) Suppose that \( \beta_p = \tau_p \) and let \( 0 < |\lambda| < 1 \). Let \( (A_n) \) be a sequence in \( P \) and \( \varphi = \sum_{n=1}^{\infty} \lambda^n \chi_{A_n} \). Then \( \varphi \in B_{\beta} (X, E) \). Let \( V \in W \) be proper and balanced and \( a \in E \setminus V \). There exists a balanced \( V_1 \in W \) and \( A \in P \) such that \( N_b (A, V_1) \subseteq N(\varphi, V) \). Now assume that there exists an \( x \in (\cup A_n) \setminus A \) and let \( n \) be the smallest integer such that \( x \in A_n \). Then \( |\varphi(x)| = |\lambda|^n \). Choose a clopen neighborhood \( Z \) of \( x \) disjoint from \( A \) and let \( f = \lambda^{-n} \chi_Z \). Clearly \( f \in N_b (A, V_1) \) but \( f \notin N(\varphi, V) \) since \( |\varphi(x)\lambda^{-n}| = 1 \) and \( a \notin V \). This proves that \( \bigcup A_n \subseteq A \).

(c) (\( \Rightarrow \)) Assume that \( X \) is locally compact and let \( V \in W \) be balanced, \( \varphi \in B_{\beta} (X, E) \), with \( \| \varphi \| \leq 1 \). Let \( 0 < |\lambda| < 1 \) and choose, for each positive integer \( n \), an \( A_n \in P \) such that \( |\varphi(x)| < |\lambda|^n \) when \( x \notin A_n \). By our hypothesis, there exists an \( A \in P \) containing each \( A_n \). Now \( N_b (A, V) \subseteq N(\varphi, V) \). In fact, let \( f \in N_b (A, V) \) and \( x \in X \). If \( \varphi(x) \neq 0 \), then \( |\varphi(x)| \geq |\lambda|^n \), for some \( n \), and so \( x \in A_n \subseteq A \), which implies that \( f(x) \in V \) and thus \( \varphi(x)f(x) \in V \).

Thus \( \| \varphi \| \leq 1 \), and \( f \in C_b (X, E) \). There exists a non-zero scalar \( \lambda \) such that \( f(X) \subseteq \lambda V \) and an \( A \in P \) such that \( |\varphi(x)| < |\lambda|^{-1} \) when \( x \notin A \). Every \( y \in A \) has a compact clopen neighborhood. By the compactness of \( A \), there exists a compact clopen set \( Z \) containing \( A \). If \( g = \chi_Z f \), then \( g \in C_{\beta} (X, E) \). Moreover, if \( g(y) \neq f(y) \), then \( g(y) = 0 \) and \( y \notin A \), which implies that \( -\varphi(y)f(y) \in -\lambda \varphi(y)V \subseteq V \), since \( V \) is
balanced. It follows that \( g - f \in N(\varphi, V) \), which proves that \( C_\infty(X, E) \) is \( \beta_p \)-dense in \( C_b(X, E) \).

\( \Leftrightarrow \) Assume that some \( x \in X \) has no compact neighborhood. If \( f \in C_\infty(X, E) \), then the set \( F = \{ y : f(y) \neq 0 \} \) is open. Also \( F \) is contained in a compact set \( B \in \mathcal{P} \). Thus \( x \notin F \) and so \( f(x) = 0 \). Choose any non-zero element \( a \) of \( E \) and let \( V \in \mathcal{W} \) be balanced not containing \( a \). The set \( D = N_\beta(x, V) \) is a \( \tau_p \)-neighborhood of 0 and hence a \( \beta_p \)-neighborhood of zero. Let \( f \in C_b(X, E) \), \( f(y) = a \) for all \( y \). If \( g \in C_\infty(X, E) \), then \( g(x) - f(x) = -f(x) \notin V \) and so \( g - f \notin D \), which implies that \( f \) is not in the \( \beta_p \)-closure of \( C_\infty(X, E) \). Hence the result follows. \( \square \)

**Theorem 12.2.2.** Assume that \( X \) is zero-dimensional and let \( G = (C_b(X, E), \beta_p) \). Then \( G \) is complete iff \( E \) is complete and every bounded \( f \in F(X, E) \) such that \( f|_A \) is continuous, for each \( A \in \mathcal{P} \), is continuous on \( X \).

**Proof.** (\( \Rightarrow \)) Suppose that \( G \) is complete. It is easy to see that \( E \) is complete. Let \( f \in F(X, E) \) be bounded and such that \( f|_A \) is continuous , for each \( A \in \mathcal{P} \). Consider the family \( \Delta \) of all pairs \((A, V)\), where \( A \in \mathcal{P} \) and \( V \in \mathcal{W} \) is balanced. We make \( \Delta \) into a directed set by defining \((A_1, V_1 \geq (A_2, V_2)\) iff \( A_2 \subseteq A_1 \) and \( V_1 \subseteq V_2 \). For each \( \alpha = (A, V) \) in \( \Delta \), there exists a \( g_\alpha \in S(X, E) \otimes E \) such that

\[
  g_\alpha(X) \subseteq f(A) \quad \text{and} \quad (f - g_\alpha)(A) \subseteq V.
\]

The net \( \{g_\alpha\} \) is \( \tau_\alpha \)-bounded. Also the net is \( \tau_\rho \)-Cauchy. Indeed, let \( A_o \in \mathcal{P} \) and \( V \in \mathcal{W} \) be balanced. Choose a closed balanced \( V_o \in \mathcal{W} \) such that \( V_o + V_o \subseteq V \). If \( \alpha_i = (A_i, V_i) \geq \alpha_o = (A_o, V_o), i = 1, 2 \), then

\[
  (f - g_\alpha_i)(A_i) \subseteq (f - g_\alpha_o)(A_i) \subseteq V_i \subseteq V
\]

and so \( (g_{\alpha_1} - g_{\alpha_2})(A_o) \subseteq V_o + V_o \subseteq V \) which proves that \( \{g_\alpha\} \) is \( \tau_\rho \)-Cauchy. Let \( D = \{ g \in C_b(X, E) : g(X) \subseteq f(X) \} \). Then \( F = D - D \) is uniformly bounded. Let now \( \mathcal{N} \) be a \( \beta_p \)-neighborhood of zero. There exists a \( \tau_\rho \)-neighborhood \( \mathcal{N}_1 \) of 0 such that \( \mathcal{N} \cap F = \mathcal{N}_1 \cap F \) since \( \beta_p \) coincides with \( \tau_\rho \) on \( F \). There exists an \( \alpha_o \) such that

\[
  g_o - g_{\alpha_o} \in \mathcal{N}_1 \quad \text{if} \quad \alpha, \alpha' \geq \alpha_o.
\]

For such \( \alpha, \alpha' \geq \alpha_o \) we have that \( g_o - g_{\alpha_o} \in \mathcal{N} \). This proves that the net \( \{g_\alpha\} \) is \( \beta_p \)-Cauchy and it converges to some \( g \in C_b(X, E) \). Now \( g(x) = \lim g_\alpha(x) \) for each \( x \in X \). But \( \lim g_\alpha(x) = f(x) \). Indeed, Let \( V \in \mathcal{W} \) be balanced and choose another one \( V_o \in \mathcal{W} \) which is closed and such that \( V_o + V_o \subseteq V \). Let \( x_o \in X \) and choose \( A_o \in \mathcal{P} \) containing \( x_o \). Since \( \{g_\alpha\} \) converges to \( g \) with respect to the topology \( \tau_\rho \), there exists
a $\alpha_1 = (A_1, V_1)$, where $A_o \subseteq A_1$ and $V_1 \subseteq V_o$, such that, for $\alpha \geq \alpha_1$, we have $(g - g_\alpha)(A_o) \subseteq V_o$. Thus,

$$g(x_o) - g_o(x_o) \in (g - g_\alpha)(A_o) \subseteq V_o.$$  

Also $f(x_o) - g_o(x_o) \in V_o$ and so $g(x_o) - f(x_o) \in V_o + V_o \subseteq V$. Thus $f(x_o) = g(x_o)$ since $E$ is Hausdorff. Thus $f = g$ is continuous.

$(\Leftarrow)$ Suppose that $E$ is complete and the condition is satisfied. Let $(f_\gamma)$ be a Cauchy net in $G$. For each $x \in X$, the net $(f_\gamma(x))$ is Cauchy in $E$ and hence convergent. Define $f(x) = \lim f_\gamma(x)$. For each $\varphi \in B_{o,p}(X, F)$, the function $\varphi f$ is bounded. In fact, let $V \in \mathcal{W}$ and choose another one $V_1 \in \mathcal{W}$, which is closed and balanced, such that $V_1 + V_1 \subseteq V$. There exists a $\gamma_o$ such that

$$\varphi(f_\gamma - f_\gamma')(X) \subseteq V_1 \text{ if } \gamma, \gamma' \geq \gamma_o.$$  

Since $V_1$ is closed, we get that $\varphi(f_{\gamma_o} - f)(X) \subseteq V_1$. Since $\varphi$ and $f_{\gamma_o}$ are bounded, there exists $|\lambda| \geq 1$ such that $(\varphi f_{\gamma_o})(X) \subseteq \lambda V_1$. It follows that

$$(\varphi f)(X) \subseteq V_1 + \lambda V_1 \subseteq \lambda V,$$

which proves that $\varphi f$ is bounded. Next we show that $f$ is bounded. Assume the contrary. Then, there exists a balanced $V \in \mathcal{W}$ which does not absorb $f(X)$. Hence, there are $0 < |\lambda_1| < |\lambda_2| < \ldots$, with $\lim |\lambda_n| \rightarrow \infty$, and a sequence $(x_n)$ such that

$$f(x_n) \in \lambda_n^2 V \setminus \lambda_n V \text{ for all } n.$$  

Define $\varphi$ on $X$ by $\varphi(x_n) = \lambda_n^{-1}$ and $\varphi(x) = 0$ if $x \notin \{x_1, x_2, \ldots\}$. Then $\varphi \in B_{o,p}(X, F)$ but $\varphi f$ is not bounded since $\varphi(x_n) f(x_n) \notin \lambda_n V$. This contradiction proves that $f$ is bounded. Next we prove that $f$ is continuous. In fact, let $V \in \mathcal{W}$ be closed balanced and $A \in \mathcal{P}$. We get easily that there exists $\gamma_o$ such that

$$(f_{\gamma} - f)(A) \subseteq V \text{ for all } \gamma \geq \gamma_o.$$  

Thus $f_{\gamma}(x) \rightarrow f(x)$ uniformly on $A$ and so $f|_A$ is continuous. This, by our hypothesis, implies that $f$ is continuous on $X$. Moreover $f_{\gamma} \rightarrow f$ in $G$.  

**Theorem 12.2.3.** If $X$ is zero-dimensional, then $(C_b(X, E), \beta_P)$ is metrizable iff $E$ is metrizable and $X \in \mathcal{P}$.

Proof. This follows from Theorem 12.2.1 and the fact that a Hausdorff TVS over $F$ is metrizable iff it has a countable base at 0 [BBNW75] (see also Section A.2).
Theorem 12.2.4. \( \beta_P \) has as a base at 0 the family of all sets \( N(\lambda_n, A_n, V) \) of the form

\[
N(\lambda_n, A_n, V) = \bigcap_{n=1}^{\infty} \{ f \in C_b(X, E) : f(A_n) \subseteq \lambda_n V \},
\]

where 0 < |\( \lambda_n | \to \infty, (A_n) \) a sequence in \( P \) and \( V \in W \).

Proof. (\( \Rightarrow \)) Let \( \mathcal{N} = N(\lambda_n, A_n, V) \) be as in the Theorem. We may assume that \( V \) is balanced. Also we may assume that \( (A_n) \) is increasing since \( P \) is upwards directed. As 0 < |\( \lambda_n | \to \infty, for each \( n \) there exists a scalar \( \gamma_n \) such that |\( \gamma_n | = \min\{|\( \lambda_k | : k \geq n \}. \) Now |\( \gamma_n | \to \infty. \) Let \( n_1 < n_2, \ldots \) be such that

\[
|\( \gamma_1 | = \ldots = |\( \gamma_{n_1} | < |\( \gamma_{n_1+1} | = \ldots = |\( \gamma_{n_2} | < |\( \gamma_{n_2+1} | = \ldots
\]

and consider \( \mathcal{N}_1 = N(\gamma_{n_k}, A_{n_k}, V) \). Then \( \mathcal{N}_1 \subseteq \mathcal{N}_n \) [Indeed, let \( f \in N(\gamma_{n_k}, A_{n_k}, V) \) and set \( n_0 = 0 \). For each positive integer \( k \), there exists a \( j \) such that \( n_{j-1} < k \leq n_j \). If \( x \in A_k \), then \( x \in A_{n_j} \) and so \( f(x) \in \gamma_k V \subseteq \lambda_k V \) since \( |\( \gamma_k | \geq |\( \gamma_{n_j} |. \) Thus \( \mathcal{N}_1 \subseteq \mathcal{N}_n \)]. Let \( \alpha_k = \gamma_{n_k} \) and \( B_k = A_{n_k} \). The function \( \varphi = \sum_{k=1}^{\infty} \alpha_k^{-1} \chi_{B_k} \) belongs to \( B_{\bullet P}(X, \mathbb{F}) \). Moreover \( N(\varphi, V) \subseteq W \). Indeed, let \( f \in N(\varphi, V) \) and let \( x \in A_{n_k} = B_k \). Let \( k \) be the smallest positive integer with \( x \in B_k \). Then |\( \varphi(x) | = |\( \alpha_k |^{-1}. \) Now \( \varphi(x)f(x) \in V \). Since |\( \varphi(x) | \geq |\( \alpha_j |^{-1}, \) we have that \( f(x) \in \alpha_j V \). This proves that \( N(\varphi, V) \subseteq N(\gamma_{n_k}, A_{n_k}, V) \subseteq N(\lambda_n, A_n, V) \) and so \( N(\lambda_n, A_n, V) \) is a \( \beta_P \)-neighborhood of zero.

(\( \Leftarrow \)) Let \( \mathcal{N}_n \) be any \( \beta_P \)-neighborhood of zero. There exist a balanced neighborhood \( V \in W \) and \( \varphi \in B_{\bullet P}(X, \mathbb{F}), ||\varphi || < 1 \), such that \( N(\varphi, V) \subseteq W_\varphi \). Let 0 < |\( \lambda | < 1. \) For each \( n \), there exists \( A_n \in P \) such that |\( \varphi(x) | < |\( \lambda |^n if x \notin A_n. \) Set \( \mathcal{N}_2 = N(\lambda^{-n+1}, A_n, V) \). We will finish the proof by showing that \( \mathcal{N}_2 \subseteq N(\varphi, V) \). So let \( f \in \mathcal{N}_2 \). If \( \varphi(x) \neq 0, \) then there exists an \( n \) such that |\( \lambda |^n < |\( \varphi | \leq |\( \lambda |^{n-1}. \) Now \( x \in A_n \) and so \( f(x) \in \lambda^{-n+1} V \), which proves that \( \varphi(x)f(x) \in V \) since |\( \varphi(x) \lambda^{-n+1} | \leq 1. \)

\[ \square \]

Theorem 12.2.5. If \( \mathcal{N} \) is a \( \beta_P \)-neighborhood of zero, then there exists a balanced \( V \in W \) such that, for each non-zero scalar \( \lambda, \) there exist \( A \in P \) and \( \gamma \in K, \gamma \neq 0, \) such that

\[ \{ f \in C_b(X, E) : f(X) \subseteq \lambda V, f(A) \subseteq \gamma V \} \subseteq \mathcal{N}. \]

Proof. By Theorem 12.2.4, there exists a balanced \( V \in W \), an increasing sequence \( (A_n) \) in \( P \) and a sequence \( (\lambda_n) \) in \( \mathbb{F} \), with 0 < |\( \lambda_n | \to \infty, such that \( \mathcal{N}_1 = N(\lambda_n, A_n, V) \subseteq \mathcal{N}. \) Let now \( \lambda \) be a non-zero scalar
and choose \( n \) such that \(|\lambda_k| > |\lambda|\) when \( k > n \). If \( \gamma \in \mathbb{F} \) is such that \(|\gamma| = \min\{|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|\}\), then
\[
\{f \in C_b(Y, E) : f(X) \subseteq \lambda V, f(A_n) \subseteq \gamma V\} \subseteq \mathcal{N}.
\]

**Theorem 12.2.6.** Let \( X, Y \) be Hausdorff topological spaces and let \( h : X \to Y \) be a continuous function. Then the induced linear map
\[
T = T_h : C_b(Y, E) \to C_b(X, E), \quad f \mapsto f \circ h,
\]
is \( \beta_o - \beta_o \) continuous.

*Proof.* Let \( \mathcal{N} \) be a \( \beta_o \)-neighborhood of 0 in \( C_b(X, E) \). In view of Theorem 12.2.4, there exist a sequence \((\lambda_n)\) of scalars, where \( 0 < |\lambda_n| \to \infty \), a sequence \((A_n)\) of compact subsets subsets of \( X \) and \( V \in \mathcal{W} \) such that \( \mathcal{N}_1 = N(\lambda_n, A_n, V) \subseteq \mathcal{N} \). Each \( B_n = h(A_n) \) is compact in \( Y \).

Moreover,
\[
\mathcal{N}_2 = \bigcap_{n=1}^{\infty} \{f \in C_b(Y, E) : f(B_n) \subseteq \lambda_n V\} \subseteq T^{-1}(\mathcal{N}_1)
\]
and so \( T \) is continuous.

**Theorem 12.2.7.** If \( X \) is a zero-dimensional \( P \)-space and \( E \) sequentially complete, then \((C_b(X, E), \beta_o)\) is sequentially complete.

*Proof.* Let \( \{f_n\} \) be a \( \beta_o \)-Cauchy sequence in \( C_b(X, E) \). Then \( \{f_n\} \) is \( \beta_o \)-bounded and hence \( \tau_o \)-bounded. Also, \( \{f_n\} \) is \( \tau_k \)-Cauchy and hence there exists \( f \in C(X, E) \) such that \( f_n \xrightarrow{\tau_k} f \). We show first that \( f \) is bounded. Assume the contrary and let \(|\lambda| > 1\). There exist a closed balanced \( V \in \mathcal{W} \) and a sequence \( \{x_n\} \) of distinct elements of \( X \) such that \( f(x_n) \notin \lambda^n V \) for all \( n \). Choose a closed balanced \( V_1 \in \mathcal{W} \) such that \( V_1 + V_1 \subseteq V \). The function \( \varphi(x_n) = \lambda^{-n} \) and \( \varphi(x) = 0 \) if \( x \notin \{x_n : n = 1, 2, \ldots\} \) belongs to \( B_o(X, \mathbb{F}) \). Hence there exists \( n_o \) such that
\[
[\varphi(f_n - f_m)](X) \subseteq V_1 \quad \text{for all } n, m \geq n_o,
\]
and so
\[
[\varphi(f_n - f)](X) \subseteq V_1 \quad \text{for all } n \geq n_o.
\]
Since \( \{f_n\} \) is uniformly bounded, there exists \( k \geq n_o \) such that \( f_n(X) \subseteq \lambda^k V_1 \) for all \( n \). Now
\[
f(x_k) = (f(x_k) - f_k(x_k)) + f_k(x_k) \in \lambda^k V_1 + \lambda^k V_1 \subseteq \lambda^k V,
\]
a contradiction. Thus \( f \) is bounded. Since \( \tau_k \) coincides with \( \beta_o \) on \( \tau_a \)-bounded sets and since \( f_n \xrightarrow{\tau_k} f \), we get that \( f_n \xrightarrow{\beta_o} f \). \( \square \)
Theorem 12.2.8. Let $X$ be zero-dimensional and let $G$ be an $S(X,F)$-submodule of $C_b(X,E)$. Then $G$ is $\beta_P$-dense in $C_b(X,E)$ iff, for each $x \in X$, the set $G(x) = \{f(x) : f \in G\}$ is dense in $E$.

Proof. ($\Rightarrow$) Assume that $G$ is $\beta_P$-dense and let $x \in X$. Consider the linear map $\widehat{x} : C_b(X,E) \to E$ given by

$$\widehat{x}(f) = f(x), \ f \in C_b(X,E),$$

which is clearly $\beta_P$-continuous. Thus

$$E = \widehat{x}(C_b(X,E)) = \widehat{x}(\overline{G}) \subseteq \overline{\widehat{x}(G)} = \overline{G(x)}$$

and so $G(x)$ is dense in $E$.

($\Leftarrow$) Suppose that the condition is satisfied and let $N = N(\varphi,V)$, where $V \in \mathcal{W}$ is open and balanced and $\varphi \in B_{\sigma P}(X,F)$. Given $f \in C_b(X,E)$, there exists a non-zero scalar $\lambda$ such that $f(X) \subseteq \lambda V$ and $||\varphi|| \leq |\lambda|$. Let $K \in \mathcal{P}$ be such that $|\varphi(x)| < |\lambda|^{-1}$ if $x \notin K$. For each $x \in K$, choose $g_x \in G$ such that $g_x(x) - f(x) \in \lambda^{-1}V$. Since $X$ is zero dimensional, there exists a clopen neighborhood $Z_x$ of $x$ such that

$$g_x(y) - f(y) \in \lambda^{-1}V$$

for all $y \in Z_x$.

By the compactness of $K$, there are $x_1, x_2, \ldots, x_n \in K$ such that $K \subseteq \bigcup_{k=1}^n Z_{x_k}$. Let

$$D_1 = Z_{x_1} \text{ and } D_k = Z_{x_k} \setminus \bigcup_{j=1}^{k-1} D_j \text{ for } k = 2, 3, \ldots, n.$$ 

Each $h_k = \chi_{D_k} g_{x_k}$ belongs to $G$ and so $g = \sum_{k=1}^n h_k \in G$. Now $g - f \in N(\varphi,V)$. Indeed if $x \notin \bigcup_{k=1}^n D_k$, then $g(x) = 0$ and $|\varphi(x)| < |\lambda|^{-1}$ which implies that

$$\varphi(x)[g(x) - f(x)] = -\varphi(x)f(x) \in V$$

because $V$ is balanced. For $x \in D_k$ we have

$$\varphi(x)[g(x) - f(x)] = \varphi(x)[g_{x_k}(x) - f(x)] \in V.$$ 

\[\square\]

Corollary 12.2.9. If $X$ is zero-dimensional, then $S(X,F) \otimes E$ and $C_{rc}(X,E)$ are $\beta_P$-dense in $C_b(X,E)$. 

3. Maximal Ideals in NA Function Algebras

In this section, we give a characterizations of maximal and \( \beta \)-closed ideals in the algebra \( C_b(X, E) \) (\( k \)-closed ideals in the algebra \( C(X, E) \)) when \( E \) is a topological algebra.

Recall that

the strict topology \( \beta \) on \( C_b(X, E) \) is the linear topology which has a base of neighborhoods of 0 in \( C_b(X, E) \) consisting of all sets of the form

\[
N(\varphi, W) = \{ f \in C_b(X, E) : (\varphi f)(X) \subseteq W \},
\]

where \( \varphi \in B_o(X, F) \) and \( W \in W \). It is shown in ([Kat84a], Proposition 2.5) that, in case \( X \) is locally compact and 0-dimensional, the above definition of strict topology is equivalent to the one given in ([Pro77], p. 198-199), where \( \varphi \) is assumed to vary over \( C_o(X, F) \) instead of \( B_o(X, F) \).

We first consider a version of the Stone-Werierstrass theorem for \( C_b(X, F) \)-submodules in \( C_b(X, E) \).

**Theorem 12.3.1.** [KKK11] Assume that either \( X \) or \( E \) is 0-dimensional (in particular, \( E \) is a non-Archimedean TVS), and let \( \mathcal{A} \) be a vector subspace of \( C_b(X, E) \) which is a \( C_b(X, F) \)-module. If, for each \( x \in X \), \( \mathcal{A}(x) \) is dense in \( E \), then \( \mathcal{A} \) is \( \beta \)-dense in \( C_b(X, E) \).

**Proof.** Let \( f \in C_b(X, E) \), and let \( \varphi \in B_o(X, F) \) and \( W \) an open balanced neighborhood of 0 in \( E \). Choose \( \lambda \in F \) with \( ||\varphi|| < |\lambda| \) and \( f(X) \subseteq \lambda W \). There exists a compact set \( K \) of \( X \) such that \( |\varphi(x)| < |\lambda|^{-1} \) for \( x \notin K \). For each \( x \in K \), there is a \( g_x \in \mathcal{A} \) such that

\[
f(x) - g_x(x) \in \lambda^{-1}W.
\]

Let

\[
G(x) = \{ y \in X : f(y) - g_x(y) \in \lambda^{-1}W \} = (f - g_x)^{-1}(\lambda^{-1}W).
\]

If \( E \) is 0-dimensional, we may assume that \( W \) is clopen, and therefore \( G(x) \) is also clopen in \( X \). If \( X \) is 0-dimensional, there exists clopen neighborhood \( N(x) \) of \( x \) such that \( N(X) \subseteq G(x) \). We see that in any case, there exists a clopen neighborhood \( V_x \) of \( x \) such that

\[
V_x \subseteq G(x) = \{ y \in X : f(y) - g_x(y) \in \lambda^{-1}W \}.
\]

By the compactness of \( K \), there are \( x_1, x_2, \ldots, x_n \) in \( Y \) such that \( K \subseteq \bigcup_{k=1}^n V_{x_k} \). Let

\[
D_1 = V_{x_1} \quad \text{and} \quad D_{x_k+1} = V_{x_k+1} \setminus \bigcup_{i=1}^n V_{x_i}.
\]

The sets \( D_1, \ldots, D_n \) are pairwise disjoint and clopen. Let \( \varphi_k \) be the \( F \)-characteristic function of \( D_k \). The function \( g = \sum_{k=1}^n \varphi_k g_{x_k} \) belongs to
\( \mathcal{A} \). Moreover, \([\varphi(f - g)](X) \subseteq W \). Indeed, let \( x \in X \). If \( x \in D_k \), then \( g(x) = g_{x_k}(x) \) and so
\[
\varphi(x)[f(x) - g(x)] \in \varphi(x)\lambda^{-1}W \subseteq W.
\]
If \( x \notin \bigcup_{k=1}^{n}D_k \), then \( g(x) = 0 \) and \( x \notin Y \). Thus
\[
\varphi(x)[f(x) - g(x)] = \varphi(x)f(x) \in \varphi(x)\lambda W \subseteq W.
\]
Thus \( f \in \overline{\mathcal{A}}^\beta \).

A useful equivalent form of the above is the following theorem which characterizes \( \beta \)-closed submodules in \( C_b(X, E) \).

**Theorem 12.3.2.** [KKK11] Assume that either \( X \) or \( E \) is 0-dimensional (in particular, \( E \) is a non-Archimedean TVS), and let \( \mathcal{A} \) be a vector subspace of \( C_b(X, E) \) which is a \( C_b(X, F) \)-module. Then, for any \( f \in C_b(X, E) \), \( f \in \overline{\mathcal{A}}^\beta \) iff \( f(x) \in \overline{\mathcal{A}(x)} \) for all \( x \in X \). In particular, \( \mathcal{A} \) is \( \beta \)-closed iff, for each \( x \in X \), \( f(x) \in \overline{\mathcal{A}(x)} \) for all \( f \in \mathcal{A} \).

**Proof.** Assume that \( f \in \overline{\mathcal{A}}^\beta \), and let \( x \in X \). If \( \varphi \) is the characteristic function of \( \{x\} \), then \( \varphi \in B_o(X, F) \) and so there exists a \( g \in \mathcal{A} \) such that:
\[
f(x) - g(x) \in [\varphi(f - g)](x) \subseteq W,
\]
which implies that \( f(x) \in \overline{\mathcal{A}(x)} \). The converse part follows from Theorem 12.3.1.

Recall that if \( E \) is an associative algebra over a valued field \( F \) and \( \tau \) a topology on \( E \), the pair \((E, \tau)\) is called a topological algebra if \((E, \tau)\) is a TVS and the multiplication \((x, y) \rightarrow xy\) from \( E \times E \) into \( E \) is jointly continuous (i.e. for any \( \tau \)-neighborhood \( W \) of 0 in \( E \), there exists a \( \tau \)-neighborhood \( H \) of 0 in \( E \) such that \( H^2 \subseteq W \), where \( H^2 = \{xy : x, y \in H\} \)). \((E, \tau)\) is called topological simple if \( E \) has no two-sided closed ideals other than \( \{0\} \) and \( E \). It is easy to verify that if \((E, \tau)\) is a topological algebra, then \((C_b(X, E), \beta)\) is also a topological algebra.

**Theorem 12.3.3.** [KKK11] Let \( E \) be a topological algebra with identity over \( F \). Assume that either \( X \) or \( E \) is 0-dimensional (in particular, \( E \) is non-Archimedean) Let \( \Omega \) be the set of all pairs \( \alpha = (N, W) \), where \( N \) is a non-empty closed subset of \( X \) and \( W \) a proper closed two-sided ideal in \( E \). For \( \alpha_i = (N_i, W_i) \), \( i = 1, 2 \), define \( \alpha_1 \leq \alpha_2 \) iff \( N_2 \subseteq N_1 \) and \( W_1 \subseteq W_2 \). Also let \( \mathcal{J}_\alpha = S(N, W) = \{f \in C_b(X, E) : f(N) \subseteq W\} \) for \( \alpha = (N, W) \). Then:

(a) Each \( \mathcal{J}_\alpha \) is a proper two-sided ideal in \( C_b(X, E) \) which is \( \rho \)-closed and hence \( \beta \)-closed.
(b) $J_{\alpha_1} \subseteq J_{\alpha_2}$ iff $\alpha_1 \leq \alpha_2$. Thus the map $\alpha \to J_{\alpha}$ is one-to-one and order preserving.

(c) A $\beta$-closed two-sided ideal $J$ in $C_b(X, E)$ is maximal iff there exist a unique $x_o \in X$ and a unique maximal closed two-sided ideal $W_o$ in $E$ such that $J = S(x_o, W_o)$.

(d) If $E$ is topologically simple and $J$ is a $\beta$-closed two-sided ideal in $C_b(X, E)$, then:

(i) There exists a closed subset $N$ of $X$ such that

$$J = \mathcal{I}(N) = \{f \in C_b(X, E) : f(N) = \{0\}\}.$$  

(ii) $J$ is maximal iff there exists a (unique) $x \in X$ such that $J = \mathcal{I}(x)$.

Proof. (a) It is easy to see that $J_{\alpha}$ is a two-sided ideal in $C_b(X, E)$ which is $a$-closed since $W$ is closed. Moreover $J_{\alpha}$ is a proper subset of $C_b(X, E)$. Indeed let $a \in E \setminus W$. Then the constant function $f(x) = a$ is not in $J_{\alpha}$.

(b) It is clear that $J_{\alpha_1} \subseteq J_{\alpha_2}$ when $\alpha_1 \leq \alpha_2$. Conversely, suppose that $J_{\alpha_1} \subseteq J_{\alpha_2}$. If $x \in N_2 \setminus N_1$, then there exists a clopen neighborhood $V$ of $x$ which is disjoint from $N_1$. Let $a \in E \setminus W_2$ and consider the function $g = \chi_A \otimes a$, where $\chi_A$ is the $F$-characteristic function of $V$. Then $g \in J_{\alpha_1} \setminus J_{\alpha_2}$, a contradiction. Thus $N_2 \subseteq N_1$. Also if $b \in W_1 \setminus W_2$, then the constant function $g(x) = b$ is in $J_{\alpha_1} \setminus J_{\alpha_2}$, a contradiction. Thus $W_1 \subseteq W_2$ and (b) follows.

(c) We first note that $J$ is a $C_b(X, F)$-module. Indeed, let $f \in J$ and $h \in C_b(X, F)$. If $u$ is the identity of $E$, then the function $g = h \otimes u$ belongs to $C_b(X, E)$ and so $g f \in J$. Since $g f = hf$, our claim follows. Now suppose that $J$ is maximal and let $f \in C_b(X, E) \setminus J$. Since $f \notin J = \overline{J}$, there exists, by Theorem

12.3.2, $x_o \in X$ such that $f(x_o) \notin \overline{J(x_o)} = W_o$. Since $W_o$ is a proper ideal and $J \subseteq S(x_o, W_o)$, we have that $J = S(x_o, W_o)$. The ideal $W_o$ is maximal. Indeed, let $W_1$ be a proper closed two-sided ideal which contains $W_o$. Then $J \subseteq S(x_o, W_1) \neq C_b(X, E)$, and so $S(x_o, W_o) = S(x_o, W_1)$, which implies that $W_1 = W_o$. The uniqueness of $x_o$ and $W_o$ follows from (b).

Conversely, suppose that $J = J_{(x_o, W_o)}$, where $W_o$ is a maximal closed ideal in $E$ and let $J_1$ be a proper closed two-sided ideal in $C_b(X, E)$ which contains $J$. Let $f \in C_b(X, E) \setminus J_1$. Since $f \notin \overline{J_1} = \overline{J_1}$, there exists, by Theorem 12.3.2, $y \in X$ such that $f(y) \notin \overline{J_1(y)} = W$. Now $J_{(x_o, W_o)} \subseteq J_1 \subseteq J_{(y, W)}$. But then $W_o \subseteq W$ and $y = x_o$ (by (b)). Since $W_o$
is maximal, we have that \( W_o = W \) and so \( J = J_1 = J_{(y,W)} \), which proves that \( J \) is maximal.

(d) Suppose that \( E \) is topological simple. Then:

(i) Let \( J \) be a \( \beta \)-closed two-sided ideal in \( C_b(X,E) \). Then \( J \) is a \( C_b(X,F) \)-module. Define

\[
N = \{ x \in X : f(x) = 0 \text{ for all } f \in J \}.
\]

Then \( N \) is closed in \( X \). Further, \( J \subseteq I(N) \). We now show that \( I(N) \subseteq J \).

[Let \( g \in I(N) \), but \( g \notin J \).

Since \( J \) is \( \beta \)-closed, by Theorem 12.3.2, \( J(\{0\}) \neq E \).

Since \( E \) is simple and \( J(\{0\}) \) is a two-sided ideal in \( E \), we have \( J(\{0\}) = \{0\} \).

In particular, \( f(x_o) = 0 \) for all \( f \in J \) and so \( x_o \in N \).

But \( g(x_o) \notin J(\{0\}) \) and so \( g(x_o) \neq 0 \) (as \( 0 \in J(\{0\}) \)). This is a contradiction, since \( g \in I(N) \) and \( x_o \in N \) imply that \( g(x_o) = 0 \). Therefore \( I(N) \subseteq J \).

Thus \( J = I(N) \).

(ii) It follows from (c) since the only maximal closed ideal in \( E \) is \( \{0\} \).

\[ \square \]

Remark. Some results related to the above ones for the "compact-open topology" on \( C(X,E) \), due to Prolla ([Pro82], p. 218-222), may be stated without proof as follows:

**Theorem 12.3.4.** [Pro82] Assume that either \( X \) or \( E \) is 0-dimensional (in particular, \( E \) is a non-Archimedean TVS), and let \( A \) be a vector subspace of \( C(X,E) \) which is a \( C(X,F) \)-module. Then, for any \( f \in C(X,E) \), \( f \in \overline{A} \) iff \( f(x) \in \overline{A(x)} \), for all \( x \in X \).

**Theorem 12.3.5.** [Pro82] Let \( E \) be a topological algebra with identity over \( F \). Assume that either \( X \) or \( E \) is 0-dimensional (in particular, \( E \) is non-Archimedean). Then:

(a) For any non-empty closed subset \( N \) of \( X \) and a proper closed two-sided ideal \( W \) in \( E \), \( S(N,W) \) is a proper two-sided ideal in \( C(X,E) \) which is \( k \)-closed.

(b) A \( k \)-closed two-sided ideal \( J \) in \( C(X,E) \) is maximal iff there exist a unique \( x_o \in X \) and a unique maximal closed two-sided ideal \( W_o \) in \( E \) such that

\[
J = S(x_o,W_o) = \{ f \in C(X,E) : f(x_o) \in W_o \}.
\]

(c) If \( E \) is simple and \( J \) is a \( k \)-closed two-sided ideal in \( C(X,E) \), then:
(i) There exists a closed subset $N$ of $X$ such that

$$\mathcal{J} = \mathcal{I}(N) = \{ f \in C(X, E) : f(N) = \{0\} \}.$$

(ii) $\mathcal{J}$ is maximal iff there exists a (unique) $x \in X$ such that $\mathcal{J} = \mathcal{I}(x)$. □
4. NOTES AND COMMENTS

4. Notes and Comments

Section 12.1. In this section, a generalized form of compact-open topology \( \tau_K \) is considered on the non-Archimedean function space \( C(X, E) \). Earlier authors assumed \( X \) a zero-dimensional topological space and \( E \) a non-Archimedean valued field \( \mathbb{F} \) or a non-Archimedean LCS over \( \mathbb{F} \).

In fact, the first result on the Weierstrass type approximation theorem in the non-Archimedean area was proved by Dieudonne [Dieu44] in 1944 for \( C(X, \mathbb{F}) \), where \( \mathbb{F} \) is a \( p \)-adic field. This result was extended in 1950 by I. Kaplansky [Kap50] to the case of \( \mathbb{F} \) a non-Archimedean valued field (i.e. a rank one valued field). These results were completed by Chernoff, Rasala and Waterhouse [CRW68] in 1968 who extended them to any field \( \mathbb{F} \) with a krull valuation (or Archimedean valuations other than the usual absolute value of \( \mathbb{C} \)). When \( E \) is a non-Archimedean normed space, Prolla [Pro77, Pro78] obtained a Stone-Weierstrass theorem which was later extended by Carneiro [Car79] to weighted function spaces \( CV(X, E) \).

In his work, Carneiro considered locally \( \mathbb{F} \)-convex spaces \( E \) over a local field \( (\mathbb{F}, | \cdot |) \), i.e. over a locally compact valued field \( (\mathbb{F}, | \cdot |) \) ). Later, Soares [So80], Prolla [Pro82] and Prolla and Verdoodt [PV97] obtained further generalizations where the field \( (\mathbb{F}, | \cdot |) \) is not assumed to be locally compact.

Here we have considered the case where the functions take their values in a TVS over a non-Archimedean field \( \mathbb{F} \). The main results deal with completeness, completion, extension of function and denseness of subspaces, as given in [Kat11].

Section 12.2. This section makes a parallel study of the strict topology \( \beta_P \) on \( C_b(X, E) \) assuming \( X \) a zero-dimensional topological space and \( E \) a non-Archimedean TVS, similar to that in Section 12.1, as given in [Kat11].

As mentioned earlier, the strict topology was introduced for the first time by Buck in 1958 on the space of all bounded continuous functions on a locally compact space \( X \). Several other authors have extended Buck’s results by taking as \( X \) an arbitrary completely regular space and considering spaces of continuous functions on \( X \) which are either real-valued or have values in a classical locally convex space or even in a classical topological vector space. In the non-Archimedean case some authors studied strict topologies on spaces of continuous functions, on a zero-dimensional topological space, with values either in a non-Archimedean valued field \( \mathbb{F} \) or in a non-Archimedean locally convex space over \( \mathbb{F} \).
this paper we will consider the case where the functions take their values in a non-Archimedean linear topological space.

Section 12.3. In this section, we first consider a Stone-Weierstrass type theorem in the non-Archimedean setting for the spaces \((C_b(X, E), \beta)\) and \((C(X, E), k)\), as given in [KKK11, Pro82]. As applications, we give a characterizations of maximal and closed ideals in the algebra when \(E\) is a topological algebra.
APPENDIX A

Topology and Functional Analysis
1. Topological Spaces

In this section we present some basic definitions and results on topological spaces.

**Definition.** A *topology* on a non-empty set $X$ is a collection $\tau$ of subsets of $X$ having the following properties:

(i) $X$ and the empty set $\emptyset$ belong to $\tau$.

(ii) The union of an arbitrary number of sets in $\tau$ belongs to $\tau$.

(iii) The intersection of a finite number of sets in $\tau$ belongs to $\tau$.

In this case $(X,\tau)$ is called a *topological space* and each set $U$ in $\tau$ is called an *open set*. A subset $A$ of $X$ is said to be *closed set* if its complement $X\setminus A$ is an open set. The *closure* of a set $A$ in $X$, denoted by $\overline{A}$ (or $cl(A)$), is defined as the intersection of all closed set containing $A$ (i.e., $\overline{A}$ is the smallest closed set containing $A$). The *interior* of a set $A$ in $X$, denoted by $A^0$ (or $int(A)$), is defined as the union of all open set contained in $A$ (i.e., $A^0$ is the largest open set contained in $A$). Further, $A$ is open iff $A = A^0$; $A$ is closed iff $A = \overline{A}$. $A$ is said to be *dense* in $X$ if $A = X$.

**Definition.** Let $(X,\tau)$ be a topological space.

(1) For any $x \in X$, a subset $V$ of $X$ is called a *neighborhood* of $x$ if there exists an open set $W$ such that $x \in W \subseteq V$. In particular, every open set $U$ containing $x$ is called an open neighborhood of $x$. The collection $U_x$ of all neighborhoods of $x$ is called the neighborhood system at $x$.

(2) A base of neighborhoods of $x$ (or a local base) at $x$ is a subcollection $B_x$ of $U_x$ such that, for each $U \in U_x$, there exists some $V \in B_x$ such that $V \subseteq U$; in this case, the elements of $B_x$ are called basic neighborhoods of $x$.

**Theorem A.1.1.** Let $(X,\tau)$ be a topological space. For any $x \in X$, let $B_x$ be a base of neighborhoods of $x$. Then:

(NB1) $x \in U$ for all $U \in B_x$.

(NB2) If $U, V \in U_x$, there is some $W \in B_x$ such that $W \subseteq U \cap V$.

(NB3) If $V \in B_x$, there is some $V_o \in B_x$ such that if $y \in V_o$, there is some $W \in B_y$ with $W \subseteq V$ (i.e. $V$ contains a basic neighborhood of each $y \in V_o$).

Furthermore,

(NB4) $G \subseteq X$ is open iff $G$ contains a basic neighborhood of each of its points.
Conversely, given a set $X$ and that for each $x \in X$ is given a non-empty collection $\mathcal{B}_x$ of subsets of $X$ satisfying (NB$_1$)–(NB$_3$) and if we define "open" using (NB$_4$), there exists a unique topology $\mu$ on $X$ such that $\mathcal{B}_x$ is a base of $\mu$ neighborhoods of $x$ for each $x \in X$.

**Definition.** Let $(X, \tau)$ be a topological space.
1. A subcollection $\mathcal{B} \subseteq \tau$ is called a base (or basis) for $\tau$ if, for any $U \in \tau$ and $x \in U$, there exists $V \in \mathcal{B}$ such that $x \in V \subseteq U$ (or equivalently each $U \in \tau$ is a union of sets in $\mathcal{B}$).
2. A subcollection $\mathcal{S} \subseteq \tau$ is called a subbase (or subbasis) for $\tau$ if the finite intersection of sets in $\mathcal{S}$ form a base for $\tau$.

**Remark.** Any collection $\mathcal{B}$ of subsets of a set $X$ is a base for some topology on $X$ iff (i) $X = \bigcup \{B : B \in \mathcal{B}\}$, and (ii) for any $B_1, B_2 \in \mathcal{B}$ and $x \in B_1 \cap B_2$, there is some $B_3 \in \mathcal{B}$ with $x \in B_3 \subseteq B_1 \cap B_2$. On the other hand, any collection of subsets of a set $X$ is a subbase for some topology on $X$. In fact, this is the smallest topology containing the given collection of sets.

**Definition.** (1) If $X$ is a non empty set, then a function $d : X \times X \to \mathbb{R}$ is called a pseudometric on $X$ if, for any $x, y, z \in X$,
- $d(x, y) \geq 0$;
- if $x = y$, then $d(x, y) = 0$;
- $d(x, y) = d(y, x)$ (symmetry);
- $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality).

If, in addition, $d$ satisfies: $d(x, y) = 0 \Rightarrow x = y$, then $d$ is called a metric on $X$. If $d$ is a pseudometric (resp. metric) on a set $X$, then the pair $(X, d)$ is called a pseudometric (resp. metric) space. For any $x \in X$ and $r > 0$, the sets
\[ B(x, r) = \{ y \in X : d(y, x) < r \} \quad \text{and} \quad B[x, r] = \{ y \in X : d(y, x) \leq r \} \]
are called open ball and closed ball, respectively, with center $x$ and radius $r$.

(2) A subset $U$ of a pseudometric space $(X, d)$ is called a $d$-open set if, for each $x \in U$, there exists $r > 0$ such that $B(x, r) \subseteq U$. The collection $t_d$ of all $d$-open sets is a topology on $X$, called the pseudometric topology induced by $d$. Clearly, the collection $\mathcal{B} = \{ B(x, r) : x \in X, r > 0 \}$ is a base for the topology $t_d$.

(3) A topological space $(X, \tau)$ is said to be pseudometrizable (resp. metrizable) if there exists a pseudometric (resp. metric) $d$ on $X$ such that $\tau = t_d$. 
Definition. Let \((X,d)\) be a metric space.

1. A sequence \(\{x_n : n \in \mathbb{N}\}\) is said to be **convergent** to \(x \in X\) if, given any \(\varepsilon > 0\), there exists an integer \(n_o\) such that 
   \[d(x_n, x) < \varepsilon\] for all \(n \geq n_o\); in this case, we write \(x_n \to x\).

2. \(\{x_n\}\) is said to be a **Cauchy sequence** if, given any \(\varepsilon > 0\), there exists an integer \(n_o\) such that 
   \[d(x_n, x_m) < \varepsilon\] for all \(n, m \geq n_o\).

Clearly, every convergent sequence is a Cauchy sequence.

Definition. Let \((X,d)\) be a metric space, and let \(A \subseteq X\). Then \(A\) is called **complete** if every Cauchy sequence in \(A\) converges to a point in \(A\).

Definition. Let \((X,\tau)\) be a topological space and \(A \subseteq X\).

1. A point \(x\) in \(A\) is called an **interior point** of \(A\) if there exists an open set \(U\) in \(X\) such that \(x \in U \subseteq A\).

2. A point \(x \in A\) is called an **isolated point** of \(A\) if it has a neighborhood which does not contain any other point of \(A\).

3. A point \(x \in X\) is called a **limit point** of \(A\) if each neighborhood of \(x\) contains a point of \(A\) other than \(x\). The set of all limit points of \(A\) is denoted by \(A_d\).

Note that, for any \(A \subseteq X\), its closure \(\overline{A} = A \cup A_d\); so \(A\) is closed iff every limit point of \(A\) is in \(A\). Further, every \(x \in \overline{A}\) is either an isolated point of \(A\) or a limit point of \(A\).

Definition. (1) A relation \(\geq\) on a set \(D\) is called a direction (or a directed relation) if:

   i. \(\alpha \geq \alpha\), for all \(\alpha \in D\) (reflexive),
   ii. if \(\alpha \geq \beta, \beta \geq \gamma\), then \(\alpha \geq \gamma\) (transitive);
   iii. if \(\alpha, \beta \in D\), there exists a \(\gamma \in D\) such that \(\gamma \geq \alpha, \gamma \geq \beta\) (directive).

   In this case, \((D, \geq)\) is called a directed set.

2. A net in a set \(X\) is a mapping \(f\) of a directed set \(D\) into \(X\). We denote this net by \(\{f(\alpha) : \alpha \in D\}\) or \(\{x_\alpha : \alpha \in D\}\), where \(x_\alpha = f(\alpha) \in X\). If \(B\) is a subset of a directed set \((D, \geq)\), with the induced ordering directed, then \((B, \geq)\) is called a directed subset of \((D, \geq)\). A subnet \(\{x_\beta : \beta \in B\}\) in \(X\) is a directed subset of a net \(\{x_\alpha : \alpha \in D\}\).

3. A net \(\{x_\alpha : \alpha \in D\}\) in a topological space \((X,\tau)\) is said to be **convergent** to \(x \in X\) if, given any neighborhood \(U\) of \(x\), there exists
\[ \alpha_o \in D \text{ such that } x_\alpha \in U \text{ for all } \alpha \geq \alpha_o. \]

(i.e., \( x_\alpha \in U \) eventually). In this case, \( x \) is called the \textit{limit of the net} and we write \( x_\alpha \to x \).

**Theorem A.1.2.** Let \( X \) be a topological space and \( A \subseteq X \). Then:

(a) For any \( x \in X, x \in A \) iff there exists a net \( \{x_\alpha\} \subseteq A \) such that \( x_\alpha \to x \).

(b) \( A \) is closed iff, for any net \( \{x_\alpha\} \subseteq A \) with \( x_\alpha \to x \in X \), we have \( x \in A \).

**Definition.** Let \( X \) and \( Y \) be topological spaces.

(1) A function \( f : X \to Y \) is said to be \textit{continuous at} \( x_o \in X \) if, for each neighborhood \( V_o \) of \( f(x_o) \) in \( Y \), there exists a neighborhood \( U_o \) of \( x_o \) in \( X \) such that \( f(U_o) \subseteq V_o \). \( f \) is said to be \textit{continuous on} \( X \) if it is continuous at each \( x \in X \).

(2) A function \( f : X \to Y \) is called a \textit{homeomorphism} if \( f \) is continuous, one-one, onto and \( f^{-1} : Y \to X \) is continuous.

It is often convenient to use the following equivalent conditions for continuity.

**Theorem A.1.3.** Let \( X \) and \( Y \) be topological spaces and \( f : X \to Y \) a function.

(a) The following conditions are equivalent:

(i) \( f : X \to Y \) is continuous on \( X \).

(ii) For each open set \( V \subseteq Y, f^{-1}(V) \) is open in \( X \).

(iii) For each closed set \( W \subseteq Y, f^{-1}(W) \) is closed in \( X \).

(iv) For any net \( \{x_\alpha\} \subseteq X \) with \( x_\alpha \to x \) in \( X \), \( f(x_\alpha) \to f(x) \) in \( Y \).

(b) If \( X \) and \( Y \) are metric spaces, then \( f : X \to Y \) is continuous on \( X \) iff, for any sequence \( \{x_n\} \subseteq X \) with \( x_n \to x \) in \( X \), \( f(x_n) \to f(x) \) in \( Y \).

(c) If \( \tau_1 \) and \( \tau_2 \) are two topologies on a set \( X \), then the identity map \( i : (X, \tau_1) \to (X, \tau_2) \) is continuous iff \( \tau_2 \subseteq \tau_1 \).

**Remark.** Clearly, if \( X \) and \( Y \) are topological spaces, then any function \( f : X \to Y \) is continuous on \( X \) in each of the following cases: (1) \( X \) has the discrete topology \( \tau = P(X) \), the power set of \( X \); (2) \( Y \) has the indiscrete topology \( \tau = \{X, \emptyset\} \).

**Definition.** A function \( f : X \to Y \) is called an \textit{open function} (resp. a \textit{closed function}) if, for any open set (resp. closed set) \( U \) in \( X \), \( f(U) \) is open (resp. closed) in \( Y \).
In general, \( f \) is continuous \( \iff \) \( f \) is open \( \iff \) \( f \) is closed. However, if \( f : X \rightarrow Y \) is one-one and onto, then \( f \) is open iff \( f \) is closed iff \( f^{-1} \) is continuous; hence \( f \) is a homeomorphism iff \( f \) is continuous and open iff \( f \) is continuous and closed.

**Theorem A.1.4.** (Map Gluing theorem) Let \( X \) and \( Y \) be topological spaces, and let \( X = A \cap B \), where \( A \) and \( B \) are closed subsets of \( X \). Let \( f : A \rightarrow Y \) and \( g : B \rightarrow Y \) be continuous functions such that \( f = g \) on \( A \cap B \). Then the function \( h : X \rightarrow Y \) defined by

\[
h(x) = \begin{cases} f(x) & \text{if } x \in A, \\ g(x) & \text{if } x \in B, \end{cases}
\]

is continuous on \( X \).

**Proof.** Let \( W \) be a closed set in \( Y \). Then \( f^{-1}(W) \) is closed in \( A \) and hence also in \( X \), since \( A \) is closed in \( X \). Similarly, \( f^{-1}(W) \) is closed both in \( B \) and \( X \). Hence \( h^{-1}(W) = f^{-1}(W) \cap g^{-1}(W) \), and thus \( h^{-1}(W) \) is closed in \( X \). This implies that, \( h \) is continuous on \( X \). \( \square \)

**Definition.** A mapping \( \varphi : (X, d) \rightarrow (Y, \rho) \) is called an isometry of \((X, d)\) into \((Y, \rho)\) if

\[
\rho(\varphi(x_1), \varphi(x_2)) = d(x_1, x_2) \quad \text{for all } x_1, x_2 \in X.
\]

( i.e. \( \varphi \) preserves distances). In this case, \( X \) and \( \varphi(X) \) are said to be isometric. Clearly, every isometry is one-one, continuous and an open mapping. Hence an isometry \( \varphi : (X, d) \rightarrow (Y, \rho) \) is a homeomorphism \( \iff \) \( \varphi \) is onto. However, a homeomorphism \( \varphi : (X, d) \rightarrow (Y, \rho) \) need not be an isometry.

**Definition.** If \( X, Y, Z \) are topological spaces, then a function \( f : X \times Y \rightarrow Z \) is called

(i) separately continuous on \( X \times Y \) if, given any \( (x, y) \in X \times Y \) and any neighborhood \( G \) of \( f(x, y) \) in \( Z \), there exist neighborhoods \( H \) of \( x \) in \( X \) and \( J \) of \( y \) in \( Y \) such that \( f(H \times \{y\}) \subseteq G \) and \( f(\{x\} \times J) \subseteq G \).

(ii) jointly continuous on \( X \times Y \) if, given any \( (x, y) \in X \times Y \) and any neighborhood \( G \) of \( f(x, y) \) in \( Z \), there exist neighborhoods \( H \) of \( x \) in \( X \) and \( J \) of \( y \) in \( Y \) such that \( f(H \times J) \subseteq G \).

Clearly, jointly continuity \( \Rightarrow \) separately continuity; the converse need not hold.

**Definition.** A topological space \((X, \tau)\) is called:

(1) a \( T_1 \)-space if, for any two points \( x \neq y \) in \( X \), there exist two open sets \( U, V \subseteq X \) such that \( x \in U, y \in V \) and \( x \notin V, y \notin U \);
(2) **Hausdorff** if, for any two points \( x \neq y \) in \( X \), there exist two open sets \( U, V \subseteq X \) such that \( x \in U \), \( y \in V \) and \( U \cap V = \emptyset \);

(3) **regular** if, for any closed set \( A \) in \( X \) and \( x \in X \) with \( x \notin A \), there exist two open sets \( U, V \subseteq X \) such that \( x \in U \), \( A \subseteq V \) and \( U \cap V = \emptyset \);

(4) **normal** if for any two closed sets \( A, B \subseteq X \) with \( A \cap B = \emptyset \), there exist two open sets \( U, V \subseteq X \) such that \( A \subseteq U \), \( B \subseteq V \) and \( U \cap V = \emptyset \);

(5) **completely regular** if, for any closed set \( A \subseteq X \) and \( x \in X \) with \( x \notin A \), there exists a continuous function \( f : X \to [0, 1] \) such that \( f(x) = 0 \) and \( f(y) = 1 \) for all \( y \in A \).

**Definition.** Let \( X \) be a topological space, and let \( A \subseteq X \).

1. A collection \( \mathcal{U} = \{ G_\alpha : \alpha \in I \} \) of open subsets of \( X \) is said to be an **open cover** of \( A \) if \( A \subseteq \bigcup_{\alpha \in I} G_\alpha \). If \( \mathcal{U} \) and \( \mathcal{V} \) are two covers of \( X \), then \( \mathcal{V} \) is said to be a **refinement** of \( \mathcal{U} \) if each \( V \in \mathcal{V} \) is contained in some \( U \in \mathcal{U} \).

2. \( A \) is called **compact** if every open cover of \( A \) has a finite subcover.

3. \( A \) is called **countably compact** if every countable open cover of \( A \) has a finite subcover.

4. \( A \) is called **sequentially compact** if every sequence in \( A \) has a convergent subsequence with limit in \( A \).

5. \( A \) is called **relatively compact** if its closure \( \overline{A} \) is compact.

6. If \( A \) is a subset of a pseudometric space \((X, d)\), then \( A \) is called **precompact** (or **totally bounded**) if, for each \( \varepsilon > 0 \), there exists a finite subset \( D = \{ x_1, ..., x_n \} \) of \( X \) such that

\[
A \subseteq \bigcup_{i=1}^{n} B(x_i, \varepsilon).
\]

**Definition.** A topological space \((X, \tau)\) is called:

1. **locally compact** if each \( x \in X \) has a relative compact neighborhood;

2. **pseudocompact** if every continuous real-valued function on \( X \) is bounded;

3. \( \sigma\)-**compact** if it can be written as a countable union of compact sets;

4. **Lindelöf** if every open cover of \( A \) has a countable subcover;

5. **hemicompact** if \( X \) can be expressed as a countable union of compact sets \( K_n \) such that each compact subset of \( X \) is contained in some \( K_n \).
(6) paracompact if every open cover $\mathcal{U}$ of $X$ has refinement $\mathcal{V}$ which is locally finite (i.e. each $x \in X$ has a neighborhood which intersects only a finite number of members of $\mathcal{V}$);

(7) a $k$-space if, for any $U \subseteq X$, $U$ is closed in $X$ whenever $U \cap K$ is closed in $K$ for each compact $K \subseteq X$;

(8) a $k_\mathbb{R}$-space if a function $f : X \to \mathbb{R}$ is continuous on $X$ whenever $f \mid K$ is continuous for every compact subset $K$ of $X$;

(9) separable if it has a countable dense subset;

(10) zero-dimensional (or 0-dimensional) if, for each $x \in X$ and each neighborhood $U$ of $x$, there exists a clopen (i.e., both closed and open) set $V$ such that $x \in V$ and $V \subseteq U$;

(11) totally disconnected if, for any $x \in X$, $C_x = \{x\}$, where $C_x$ is the component of $x$ (i.e., the maximal connected subset of $X$ containing $x$).

(12) $X$ is called extremely disconnected (or Stonian) if the closure of every open set is open.

We now summarize some properties of the above spaces in the following theorem.

**Theorem A.1.5.** (a) A topological space $X$ is Hausdorff iff every convergent net in $X$ has a unique limit.

(b) A closed subset $A$ of a compact space $X$ is compact.

(c) A compact subset $A$ of a Hausdorff space $X$ is closed.

(d) Every compact Hausdorff space is normal.

(e) A subset $A$ of a topological space $X$ is compact iff every net in $A$ has a convergent subnet with limit in $A$.

(f) Every compact space and every sequentially compact space is countably compact and hence pseudocompact.

(g) A pseudometric space is compact iff it is sequentially compact iff it is countably compact iff it is precompact and complete.

(h) Every locally compact space and every first countable (in particular, metric) space is a $k$-space.

(i) The continuous image of a compact set is compact.

(j) If $X$ is a $k$-space and $Y$ any topological space, then a function $f : X \to Y$ is continuous on $X$ iff $f \mid K$ is continuous for every compact subset $K$ of $X$; hence every $k$-space is a $k_\mathbb{R}$-space.

(k) Every locally compact $\sigma$-compact Hausdorff space is hemicompact.

(l) Every compact metric space is separable.

(m) Every open subspace of a separable space is separable; an arbitrary subspace of a separable metric space is also separable.

(n) The continuous image of a separable space is separable.
(o) Every zero-dimensional $T_1$-space is totally disconnected and completely regular.

(p) A locally compact Hausdorff is 0-dimensional iff it is totally disconnected.

(q) Continuous image of a totally disconnected space need not be totally disconnected.

Examples. (1) $\mathbb{Q}$, $\mathbb{P}$ and discrete spaces are zero-dimensional and totally disconnected.

(2) $\mathbb{Q}$ is not extremely disconnected.

Note. Clearly, a finite subset $A$ of a topological space $X$ is always compact; but need not be closed. So it need not be relatively compact (e.g. if $X$ is neither $T_1$ nor regular).

Example. Let $X = \mathbb{N}$ with the topology
$$\tau = \{\mathbb{N}\} \cup \{A_n : n \in \mathbb{N}\},$$
where $A_n = \{1, 2, \ldots n-1\}$ for every $n \in \mathbb{N}$. This is $T_0$ but not $T_1$ (hence also not Hausdorff). In this case, the only closed compact subsets of $X$ and $\emptyset$, so the only relatively compact set is $\emptyset$.

Theorem A.1.6. Let $X$ be any topological space and $Y$ a Hausdorff topological space, and let $f, g : X \to Y$ be continuous. Then:

(a) The set $A = \{x \in X : f(x) = g(x)\}$ is closed in $X$.

(b) If $D \subseteq X$ and $f = g$ on $D$, then $f = g$ on $\overline{D}$. In particular, if $D$ is dense in $X$ and $f = g$ on $D$, then $f = g$ on $X$.

Notation. For any function $\varphi : X \to \mathbb{R}$, $A \subseteq X$, and $t \in \mathbb{R}$, we shall often write $\varphi(A) = t$ to mean that $\varphi(A) = \{t\}$, that is, $\varphi(x) = t$ for all $x \in A$.

Theorem A.1.7. (Urysohn lemma)

Let $X$ be a normal space, and let $A$ and $B$ be closed subsets of $X$ with $A \cap B = \emptyset$. Then there exists a continuous function $\varphi : X \to [0, 1]$ such that $\varphi(A) = 0$ and $\varphi(B) = 1$.

Theorem A.1.8. (a) Let $X$ be a completely regular space, $K$ a compact subset of $X$, and $B$ a closed of $X$ with $K \cap B = \emptyset$. Then there exists a continuous function $\varphi : X \to \mathbb{R}$ such that $\varphi(K) = 0$ and $\varphi(B) = 1$.

(b) Let $X$ be a locally compact Hausdorff space, $K$ a compact subset of $X$, and $U$ a neighborhood of $K$. Then there exists a continuous function $\varphi : X \to [0, 1]$ such that $\varphi(K) = 1$ and the support of $\varphi$ is
compact and contained in $U$. (The support of $\varphi$ is the closure of the set \{x \in X : \varphi(x) \neq 0\}).

In particular, every normal $T_1$-space and every locally compact Hausdorff space is completely regular. Further, every 0-dimensional $T_1$-space is completely regular.

**Definition.** A topological space $X$ is called a Baire space if whenever $X = \bigcup_{n=1}^{\infty} A_n$ with each $A_n$ closed, then, for at least one $m$, int$(A_m) \neq \emptyset$.

**Theorem A.1.9.** (Baire) (a) A topological space $X$ is a Baire space iff, for any countable family \{${U_n}$\} of open dense subsets in $X$, $\cap_{n=1}^{\infty} U_n \neq \emptyset$.

(b) Every locally compact Hausdorff space is a Baire space.

(c) Every complete metric space is a Baire space.

**Theorem A.1.10.** (Tietze extension theorem) Let $X$ be a normal space, and let $A$ be a closed subset of $X$. Then, for any continuous function $f : A \rightarrow [0, 1]$ (or $\mathbb{R}$), there exists a continuous function $g : X \rightarrow [0, 1]$ (or $\mathbb{R}$) such that $g = f$ on $A$. (Here $g$ is called a continuous extension of $f$ from $A$ to $X$).

**Definition.** Let \{(X_{\alpha}, \tau_{\alpha}) : \alpha \in I\} be an indexed family of topological spaces. The product topology on the Cartesian product $\prod_{\alpha \in I} X_{\alpha}$ is defined as the topology $\tau_{\Pi}$ obtained by taking as its base the collection of all sets of the form $\prod_{\alpha \in I} U_{\alpha}$, where each $U_{\alpha}$ is $\tau_{\alpha}$-open in $X_{\alpha}$ and $U_{\alpha} = X_{\alpha}$ except for a finite number of indices $\alpha$'s.

**Remark.** If \{${X_{\alpha}}$\} is a family of compact spaces, then $\prod_{\alpha \in I} X_{\alpha}$ with the product topology $\tau_{\Pi}$ is a compact space (Tychonoff theorem).

**Definition.** If $(X, \tau)$ is a completely regular Hausdorff space, let $C = C_b(X, \mathbb{R})$, the set of all continuous and bounded functions $f : X \rightarrow \mathbb{R}$. For each $f \in C$, define a closed bounded (hence compact) interval in $\mathbb{R}$ by

$$I_f = [\inf_{x \in X} f(x), \sup_{x \in X} f(x)].$$

Define a map $e : X \rightarrow \prod_{f \in C} I_f$ (with the product topology) by

$$e(x) = (f(x))_{f \in C}, \; x \in X.$$  

Then $\beta X = \overline{e(X)}$ is a compact space, called the Stone-Čech compactification of $X$.
Theorem A.1.11. (Stone-Čech) Let $X$ be a completely regular Hausdorff space and $\beta X$ its Stone-Čech compactification. Let $Y$ be a compact Hausdorff space or $Y = \mathbb{R}$. Then every continuous function $f : X \to Y$ has a unique continuous extension $\hat{f} : \beta X \to Y$.

Definition. (1) A subset $A$ of a topological space $X$ is called a zero set [GJ60] if there exists a continuous function $\varphi : X \to \mathbb{R}$ and that
\[ A = \varphi^{-1}(0) = \{ x \in X : \varphi(x) = 0 \}. \]
In this case, we write $A = Z(\varphi)$. The complement of a zero set is called a cozero set.

(2) A subset $D$ of $X$ is called a $G_\delta$-set if it is a countable intersection of open sets; $D$ is called an $F_\sigma$-set if it is a countable union of closed sets.

(3) $X$ is called a $P$-space ([GJ60], p. 62-63) if every zero set (or equivalently, every $G_\delta$-set) in $X$ is open.

Clearly, the complement of a $G_\delta$-set is an $F_\sigma$-set, and vice versa. For any function $f : X \to \mathbb{R}$, the set $D$ of all points of discontinuity of $f$ is an $F_\sigma$-set and the set of all points of continuity of $f$ is a $G_\delta$-set. It follows from ([GJ60], 4K(3)) that every compact subset of a $P$-space is finite.

Definition. [GJ60, Nag65] A subset $A$ of a topological space $X$ is said to have finite covering dimension if there exists a non-negative integer $n$ such that for each finite open cover $\mathcal{U}$ of $A$ there exists an open refinement $\mathcal{V}$ of $\mathcal{U}$ which is a cover of $A$ and such that any point of $A$ belongs to a most $n + 1$ member of $\mathcal{V}$. The least such $n$ is called the covering dimension of $A$. If no such finite $n$ exists, then we say that $A$ has an infinite covering dimension.

Theorem A.1.12. (a) For each $n \geq 1$, the covering dimensions of $\mathbb{R}^n$ is $n$.

(b) Every compact subset of a $P$-space has finite covering dimension.

(c) If a normal space $X$ has finite covering dimensions, then so has its Stone-Čech compactification $\beta X$.

Recall that, if $X$ is a topological space, then a function $\varphi : X \to \mathbb{R}$ is continuous at $x_o \in X$ if, for any $\varepsilon > 0$, there exists an open neighborhood $N(x_o)$ of $x_o$ such that
\[ |\varphi(y) - \varphi(x_o)| < \varepsilon \text{ for all } y \in N(x_o), \]
or equivalently,
\[ \varphi(x_o) - \varepsilon < \varphi(y) < \varphi(x_o) + \varepsilon \text{ for all } y \in N(x_o). \]
\text{(*)}
Definition. If $X$ is a topological space, then a function $\varphi : X \to \mathbb{R}$ is said to be

(a) upper semicontinuous at $x_o \in X$ if, for any $r \in \mathbb{R}$ with $\varphi(x_o) < r$, there exists an open neighborhood $N(x_o)$ of $x_o$ such that

$$\varphi(y) < r \text{ for all } y \in N(x_o);$$

(b) lower semicontinuous at $x_o \in X$ if, for any $r \in \mathbb{R}$ with $\varphi(x_o) > r$, there exists an open neighborhood $N(x_o)$ of $x_o$ such that

$$\varphi(y) > r \text{ for all } y \in N(x_o);$$

(c) upper (resp. lower) semicontinuous on $X$ if it is upper (resp. lower) semicontinuous at each point of $X$, or equivalently, if, for each $r \in \mathbb{R}$, the set $\{ x \in X : \varphi(x) < r \}$ (resp. $\{ x \in X : \varphi(x) > r \}$) is open ([Wil70], p. 49).

Clearly, by ($\ast$), a function $\varphi : X \to \mathbb{R}$ is continuous at $x_o \in X$ iff it is both upper and lower semicontinuous at $x_o \in X$.

Theorem A.1.13. (a) A function $\varphi : X \to \mathbb{R}$ is continuous iff it is both upper and lower semicontinuous.

(b) The characteristic function $\chi_A$ of a subset $A$ of $X$ is upper (resp. lower) semicontinuous at each point of $X$, or equivalently, if, for each $r \in \mathbb{R}$, the set $\{ x \in X : \varphi(x) < r \}$ (resp. $\{ x \in X : \varphi(x) > r \}$) is open ([Wil70], p. 49).

(c) Every upper (resp. lower) semicontinuous function assumes its supremum (resp. infimum) on a compact set. In particular, every non-negative upper semicontinuous function on a compact set is bounded.

Definition: Let $X$ and $Y$ be topological spaces and $T : X \to Y$ a mapping. Then:

(a) The set $G(T) = \{(x,Tx) : x \in X\}$ is called the graph of $T$ in $X \times Y$.

(b) $T$ is said to have a closed graph if $G(T)$ is a closed subset of $X \times Y$.

(Here $X \times Y$ has the product topology.)

The following theorem gives equivalent condition for a mapping $T : X \to Y$ to have a closed graph.

Theorem A.1.14. Let $X$ and $Y$ be topological spaces and $T : X \times Y$ a mapping. Then the following are equivalent

(i) $T$ has a closed graph.

(ii) If $\{x_\alpha\}$ is a net in $X$ with $x_\alpha \to x \in X$ and $Tx_\alpha \to y \in Y$, then $y = T(x)$

(iii) For each $x \in X$ and $y \neq T(x)$ in $Y$, there exist neighborhoods $U$ of $x$ in $X$ and $V$ of $y$ in $Y$ such that

$$T(U) \cap V = \emptyset.$$
Proof. (i) \(\Rightarrow\) (ii). Let \(\{x_{\alpha} = \alpha \in I\} \subseteq X\) with \(x_{\alpha} \to x \in X\) and \(Tx_{\alpha} \to y \in Y\). Then clearly \((x_{\alpha}, Tx_{\alpha}) \to (x, y)\) in \(X \times Y\) (in the product topology). Since \(\{(x_{\alpha}, Tx_{\alpha}) = \alpha \in I\} \subseteq G(T)\) and, by (i), \(G(T)\) is closed, we have \((x, y) \in G(T)\). Hence \(y = Tx\).

(ii) \(\Rightarrow\) (i). Suppose (ii) holds. To show that \(G(T) \subseteq G(T)\), let 
\((x, y) \in X \times Y\) with \((x, y) \in G(T)\). Then there exists a net \(\{(x_{\alpha}, Tx_{\alpha}) : \alpha \in I\} \subseteq G(T)\) such that \((x_{\alpha}, Tx_{\alpha}) \to (x, y)\). This implies that \(x_{\alpha} \to x\) and \(Tx_{\alpha} \to y\). Hence, by hypothesis (ii), \(y = Tx\). So \((x, y) = (x, Tx) \in G(T)\). Thus \(G(T) = G(T)\) and so \(T\) has a closed graph. \(\square\)

The next result is concerned with the questions: Under what conditions continuity of \(T\) implies that \(T\) has a closed graph? Conversely, under what conditions, \(T\) has a closed graph implies that \(T\) is continuous?

**Theorem A.1.15.** Let \(X\) and \(Y\) be topological spaces with \(Y\) Hausdorff, and let \(T : X \to Y\) be a mapping. If \(T\) is continuous, then \(T\) has a closed graph.

Proof. Suppose \(T\) is continuous. To show that \(T\) has a closed graph, let \((x, y) \in X \times Y\) with \((x, y) \in G(T)\). Then there exists a net \(\{(x_{\alpha}, Tx_{\alpha}) : \alpha \in I\} \subseteq G(T)\) such that \((x_{\alpha}, Tx_{\alpha}) \to (x, y)\). This implies that \(x_{\alpha} \to x\) and \(Tx_{\alpha} \to y\). Since \(T\) is continuous and \(x_{\alpha} \to x\), we have \(Tx_{\alpha} \to Tx\) in \(Y\). Since \(Y\) is Hausdorff, \(\{Tx_{\alpha}\}\) has a unique limit and so \(y = Tx\). Hence \((x, y) = (x, Tx) \in G(T)\). Thus \(G(T) = G(T)\); i.e. \(T\) has a closed graph. \(\square\)

The converse of the above theorem is known as the "closed graph theorem" and will be considered in Section A.3.

**Definition:** Let \(E\) be a vector space over the field \(\mathbb{K}\) (= \(\mathbb{R}\) or \(\mathbb{C}\)). Then a function \(\|\cdot\| : E \to \mathbb{R}\) is called a norm on \(E\) if

\[
\begin{align*}
(N_1) \quad &\|x\| \geq 0 \text{ for all } x \in E; \\
(N_2) \quad &\|x\| = 0 \iff x = 0; \\
(N_3) \quad &\|\lambda x\| = |\lambda| \|x\| \text{ for all } x \in E \text{ and } \lambda \in \mathbb{K}; \\
(N_4) \quad &\|x + y\| \leq \|x\| + \|y\| \text{ for all } x, y \in E.
\end{align*}
\]

**Definition.** (1) A vector space \(E\) with a norm \(\|\cdot\|\) is called a normed vector space or, simply, a normed space. Clearly, every normed space \((E, \|\cdot\|)\) is a metric space with metric given by:

\[d(x, y) = \|x - y\|, \quad x, y \in E.\]

(2) A normed space \(E\) is called a Banach space if it is complete with respect to the metric induced by the norm.
Definition. Let $A$ be an algebra over $\mathbb{K}$ (= $\mathbb{R}$ or $\mathbb{C}$) which is also a normed space with norm $\| \cdot \|$. Then $(A, \| \cdot \|)$ is called
(a) a normed algebra if the norm $\| \cdot \|$ is submultiplicative, i.e.
\[
\|xy\| \leq \|x\|\|y\| \text{ for all } x, y \in A;
\]
(b) a Banach algebra if $A$ is a complete normed algebra.
2. Topological Vector Spaces

In the sequel, we consider all vector spaces over the field $\mathbb{K} (=\mathbb{R} \text{ or } \mathbb{C})$ with the usual absolute value.

**Notations.** Let $E$ be a vector space and $A, B \subseteq E$, $x \in E$, $\alpha, \beta \in \mathbb{K}$. We denote

$$x \pm A = \{x + a : a \in A\}; \quad A \pm B = \{a + b : a \in A, b \in B\};$$

$$\alpha A = \{\alpha a : a \in A\}; \quad \alpha A + \beta B = \{\alpha a + \beta b : a \in A, b \in B\}.$$

Clearly $(\alpha + \beta)A \subseteq \alpha A + \beta A$, but $\alpha A + \beta A \subseteq (\alpha + \beta)A$ need not hold, in general.

**Definition.** If $E$ is a vector space and $\tau$ is a topology on $E$, then the pair $(E, \tau)$ is called a topological vector space (TVS, in short) if the following conditions hold:

- (TVS$_1$) The operation of addition $(x, y) \to x + y$ of $E \times E \to E$ is jointly continuous; i.e., given any $x, y \in E$ and any $\tau$-neighborhood $U$ of $x + y$ in $E$, there exist $\tau$-neighborhoods $V$ of $x$ and $W$ of $y$ in $E$ such that $V + W \subseteq U$.

- (TVS$_2$) The operation of scalar multiplication $(\lambda, x) \to \lambda x$ of $\mathbb{K} \times E \to E$ is jointly continuous; i.e., given any $x \in E$ and $\lambda \in \mathbb{K}$ and any $\tau$-neighborhood $U$ of $\lambda x$ in $E$, there exist a $\tau$-neighborhood $V$ of $x$ in $E$ and a neighborhood $D$ of $\lambda$ in $\mathbb{K}$ such that $DV \subseteq U$.

**Examples:** (1)

Any normed space $(E, \|\|)$ is a TVS since the conditions (TVS$_1$) and (TVS$_2$) follow, respectively, from

- (i) If $x, y \in E$ and $x_n \to x$, $y_n \to y$, then $x_n + y_n \to x + y$ in $E$.
- (ii) If $x \in E$, $\lambda \in \mathbb{K}$ and $\lambda_n \to \lambda$, $x_n \to x$, then $\lambda_n x_n \to \lambda x$ in $E$.

(2) Any vector space $E$ with the indiscrete topology $\tau_I = \{E, \emptyset\}$ is a TVS.

(3) A vector space $E$ with the discrete topology $\tau_D = P(E)$ is not a TVS unless $E = \{0\}$ (since the scalar multiplication need not be continuous).

**Definition:** Let $E$ be a vector space over the field $\mathbb{K} (=\mathbb{R} \text{ or } \mathbb{C})$. A subset $A$ of a vector space $E$ is called:

- (i) **absorbing** if, for each $x \in E$, there exists a number $r = r(x) > 0$ such that
  $$x \in \lambda A \text{ for all } \lambda \in \mathbb{K} \text{ with } |\lambda| \geq r;$$
  (or equivalently, there exists a number $t = t(x) > 0$ such that $\lambda x \in A$ for all $\lambda \in \mathbb{K}$ with $|\lambda| \leq t$);
(ii) balanced if \( \lambda x \in A \) for all \( x \in A \) and \( \lambda \in \mathbb{K} \) with \( |\lambda| \leq 1 \);
(iii) convex if \( tx + (1 - t)y \in A \) for all \( x, y \in A \) and \( 0 \leq t \leq 1 \).

**Examples.** (1) Any vector subspace \( A \) of \( E \) is balanced and convex, but it need not be absorbing.
(2) If \((E, ||\cdot||)\) is a normed vector space, then, for each \( r > 0 \), both the open and closed balls
\[
B(0, r) = \{ x \in E : ||x|| < r \}, \quad B[0, r] = \{ x \in E : ||x|| \leq r \}
\]
are absorbing, balanced and convex subset of \( E \).

**Lemma A.2.1.** Let \((E, \tau)\) be a TVS.
(a) For any fixed \( a \in E \), the translation map \( f_a : E \to E \) given by
\[
f_a(x) = x + a = a + x, \quad x \in E,
\]
is a homeomorphism of \( E \) into itself.
(b) For any fixed \( \lambda \neq 0 \in \mathbb{K} \), the scalar multiplication map \( g_{\lambda} : E \to E \) given by
\[
g_{\lambda}(x) = \lambda x, \quad x \in E,
\]
is also a homeomorphism of \( E \) into itself.

**Proof.** It is easy to see that \( f_{-a}^{-1} = f_{-a} \) and \( g_{\lambda}^{-1} = g_{1/\lambda} \).
(a) Clearly \( f_{a} \)
is one-one and onto. Since \( f_{a}^{-1} = f_{-a} \), by (TVS1), both \( f_{a} \) and \( f_{a}^{-1} \) are continuous. Hence \( f_{a} \) is a homeomorphism of \( E \) into itself.
(b) Since \( \lambda \neq 0 \), it easily follows that \( g_{\lambda} \) is one-one and onto. Since \( g_{\lambda}^{-1} = g_{1/\lambda} \), by (TVS2), both \( g_{\lambda} \) and \( g_{\lambda}^{-1} \) are continuous. Hence \( g_{\lambda} \) is also a homeomorphism of \( E \) into itself.

**Lemma A.2.2.** Let \((E, \tau)\) be a TVS, and let \( a \in E \) and \( \lambda \neq 0 \in \mathbb{K} \).
(a) If \( U \) is an open (respectively, a closed) set in \( E \), then \( a + U \) and \( \lambda U \) are also open (respectively, closed) in \( E \).
(b) \( U \) is a neighborhood of \( 0 \) iff \( a + U \) is a neighborhood of \( a \).
(c) \( U \) is a neighborhood of \( 0 \) iff \( \lambda U \) is a neighborhood of \( 0 \).
(d) A collection \( \mathcal{W} \) of subsets of \( E \) is a base of neighborhoods of \( 0 \) in \( E \) iff the collection
\[
a + \mathcal{W} = \{ a + U : U \in \mathcal{W} \}
\]
is a base of neighborhoods of \( a \) in \( E \).

**Proof.** (a) By above Lemma, the translation map \( f_{a} : E \to E \) is a homeomorphism; in particular, \( f_{a} \) is both an open and closed map. Hence, if \( U \) is an open (resp. a closed) set in \( E \), then \( f_{a}(U) = a + U \) is also open (resp. closed) in \( E \). Next, the map \( g_{\lambda} : E \to E \) is a homeomorphism;
hence, if $U$ is an open (resp. a closed) set in $E$, then $g_\lambda(U) = \lambda U$ is also open (resp. closed) in $E$.

(b) & (c). These follow immediately from (a).

(c) By (b), $U$ is a neighborhood of 0 iff $a + U$ is a neighborhood of $a$ in $E$. Hence $\mathcal{W}$ is a base of neighborhoods of 0 in $E$ iff $a + \mathcal{W}$ is a base of neighborhoods of $a$ in $E$. $\square$

The following theorem shows that every TVS has a base $\mathcal{W}$ of neighborhoods of 0 consisting of closed, balanced and absorbing sets.

**Theorem A.2.3.** Let $(E, \tau)$ be a TVS, and let $\mathcal{W}$ be a base of neighborhoods of 0 in a TVS $E$. Then:

(A$_1$) For every $U, V \in \mathcal{W}$, there exists $W \in \mathcal{W}$ such that $W \subseteq U \cap V$.

(A$_2$) For each $U \in \mathcal{W}$, there exists $V \in \mathcal{W}$ such that $V + V \subseteq U$.

(A$_3$) For each $U \in \mathcal{W}$ and $\lambda(\neq 0) \in \mathbb{K}$, $\lambda U \in \mathcal{W}$.

(A$_4$) Each $U \in \mathcal{W}$ is absorbing.

(A$_5$) For each $U \in \mathcal{W}$, there exists a balanced $W \in \mathcal{W}$ such that $W \subseteq U$.

(A$_6$) For each $U \in \mathcal{W}$, there exists a closed neighborhood $W \in \mathcal{W}$ such that $W \subseteq U$.

Conversely, if $\mathcal{W}$ is a non-empty collection of subsets of $E$ satisfy

(A$_1$) – (A$_5$) then there exist a topology $\tau$ on $E$ making $(E, \tau)$ a TVS with a base $\mathcal{W}$ of $\tau$-neighborhoods of 0.

**Proof.** (A$_1$) This follows from the property (NB$_2$) of a local base of neighborhoods of a topological space.

(A$_2$) Let $U \in \mathcal{W}$. By (TVS$_1$), the mapping $f : (x, y) \mapsto x + y$ is continuous at $(x, y) = (0, 0) \in E \times E$. Since $f(0, 0) = 0 + 0 = 0$ and $U$ is a neighborhood of 0, there exist neighborhoods $V_1$ and $V_2$ of 0 such that $V_1 + V_2 \subseteq U$. Put $V = V_1 \cap V_2$. Then $V \in \mathcal{W}$ with $V + V \subseteq V_1 + V_2 \subseteq U$.

(A$_3$) Let $U \in \mathcal{W}$ and $\lambda(\neq 0) \in \mathbb{K}$. Choose an open neighborhood $V$ of 0 such that $V \subseteq U$. By (TVS$_2$), the function $g_\lambda : E \rightarrow E$ defined by $g_\lambda(x) = \lambda x$ $(x \in E)$ is a homeomorphism and hence an open map. Hence $g_\lambda(V) = \lambda V$ is an open set. Now $\lambda V \subseteq \lambda U$, and so $\lambda U$ is a neighborhood of 0. Thus $\lambda U \in \mathcal{W}$.

(A$_4$) Let $U \in \mathcal{W}$, and let $x_o \in E$. By (TVS$_2$), the mapping $h_{x_o} : \mathbb{K} \rightarrow E$, defined by $h_{x_o}(\lambda) = \lambda x_o$ $(\lambda \in \mathbb{K})$ is continuous at $\lambda = 0$. So there exist $t > 0$ such that $h_{x_o}(B[0, t]) \subseteq U$, where $B[0, t] = \{\mu \in \mathbb{K} : |\mu| \leq t\}$. Hence $\mu x_o \in U$ for all $|\mu| \leq t$, and so $U$ is absorbing.

(A$_5$) Let $U \in \mathcal{W}$. The function $g : (\lambda, x) \mapsto \lambda x$ is continuous at $(\lambda, x) = (0, 0)$ by (TVS$_2$). So there exist neighborhoods $V$ of 0 in $E$ and $B[0, r]$ of 0 in $\mathbb{K}$ such that $W = B[0, r] V \subseteq U$. Clearly, $W$ is a balanced. Further, $W$ is a neighborhood of 0 (since $r V \subseteq B[0, r] V = W$ and $r V$ is
a neighborhood of 0 by (A₃)). Thus, we have a balanced \( W \subseteq \mathcal{W} \) such that \( W \subseteq U \).

(A₆) Let \( U \in \mathcal{W} \). By (A₂) and (A₅), we can choose a balanced \( V \in \mathcal{W} \) such that \( V + V \subseteq U \). We now show that \( \overline{V} \subseteq U \). [Let \( x \in \overline{V} \). Then, since \( x + V \) is a neighborhood of \( x \), \( (x + V) \cap V \neq \emptyset \). Choose \( y \in (x + V) \cap V \).

Then \( y = x + v_1 = v_2 \), where \( v_1, v_2 \in V \). So

\[
x = v_2 - v_1 \in V - V = V + V \subseteq U.
\]

This shows that \( \overline{V} \subseteq U \). Thus \( W = \overline{V} \in \mathcal{W} \) is closed with \( W \subseteq U \).

Conversely, let \( \mathcal{V} = \{ V \subseteq E : V \text{ contains some } U \in \mathcal{W} \} \). Clearly, \( \mathcal{W} \subseteq \mathcal{V} \). We first show that, for each \( x \in E \), \( B_x = x + \mathcal{V} \) satisfies \((NB₁) - (NB₃)\).

\((NB₁)\) By (A₄), each \( U \in \mathcal{W} \) is absorbing and so \( 0 \in U \); hence \( 0 \in V \) for all \( V \in \mathcal{V} \). Therefore \( x = x + 0 \in x + V \) for all \( V \in \mathcal{V} \).

\((NB₂)\) Let \( x + V_1 \), \( x + V_2 \in B_x \), where \( V_1, V_2 \in \mathcal{V} \). We show that \( (x + V_1) \cap (x + V_2) \subseteq x + \mathcal{V} \). Now

\[
(x + V_1) \cap (x + V_2) = x + V_1 \cap V_2.
\]

Choose \( U_1, U_2 \in \mathcal{W} \) such that \( U_1 \subseteq V_1 \), \( U_2 \subseteq V_2 \). By (A₄), there exists a \( U_3 \in \mathcal{W} \) such that \( U_3 \subseteq U_1 \cap U_2 \). Hence \( x + U_3 \subseteq x + V_1 \cap V_2 \).

\((NB₃)\) Let \( x + V \in B_x \), where \( V \in \mathcal{V} \). Choose \( U \in \mathcal{W} \) with \( U \subseteq V \). By (A₂), there exists a \( W \in \mathcal{W} \) such that \( W + W \subseteq U \). Now, if \( y \in x + W \), then \( y + W \in B_y = y + \mathcal{V} \) and

\[
y + W \subseteq x + W + W \subseteq x + V.
\]

Now \((NB₁)-(NB₃)\) are satisfied and so, by a general result on base of neighborhoods in Topology (Theorem A.1.1), there exists a unique topology \( \tau \) on \( E \) such that \( B_x = x + \mathcal{V} \) forms a base of \( \tau \)neighborhoods of \( x \) for each \( x \in E \). Clearly, for any \( x \in E \) and \( U \in \mathcal{W} \), \( x + U \) is a \( \tau \)neighborhood of \( x \) (since \( U \in \mathcal{W} \subseteq \mathcal{V} \)). Hence \( x + \mathcal{W} \) is a base of \( \tau \)neighborhoods of \( x \) for each \( x \in E \). In particular, \( \mathcal{W} \) is a base of \( \tau \)neighborhoods of 0. It remains to verify (TVS₁) and (TVS₂).

(TVS₁) By (A₂), it follows that the map \( (x, y) \to x + y \) is continuous at \((x, y) = (0, 0)\). Let \( x_o, y_o \in E \), and let \( U \in \mathcal{W} \). Then \( x_o + y_o + U \) is a \( \tau \)neighborhood of \( x_o + y_o \). By (A₂), there exists a \( V \in \mathcal{W} \) such that \( V + V \subseteq U \). Then \( x_o + V \) is a \( \tau \)neighborhood of \( x_o, y_o + V \) is a \( \tau \)neighborhood of \( y_o \), and

\[
(x_o + V) + (x_o + V) = x_o + y_o + V + V \subseteq x_o + y_o + U.
\]

Hence the map \((x, y) \to x + y\) is continuous at \((x, y) = (x_o, y_o) \in E \times E\).
Let \( \lambda_0 \in \mathbb{K} \) and \( x_0 \in E \). We show that the map \( g : (\lambda, x) \to \lambda x \) is continuous at \((\lambda, x) = (\lambda_0, x_0)\). Let \( W \in \mathcal{W} \), so that \( \lambda_0 x_0 + W \) is a \( \tau \)-neighborhood of \( \lambda_0 x_0 \). Choose a balanced \( V \in \mathcal{W} \) such that \( V + V \subseteq W \) (by \((A_2)\) and \((A_5)\)). Since \( V \) is absorbing (by \((A_4)\)), there exists a \( r > 0 \) such that \( x_0 \in rV \). By \((A_3)\), \( U = \frac{r}{1 + |\lambda_0|} V \in \mathcal{W} \). We now show that \( B(\lambda_0, 1/r)(x_0 + U) \subseteq \lambda_0 x_0 + W \). Let \( \lambda \in B(\lambda_0, 1/r) \) and \( x \in x_0 + U \). Then, since \( V \) is balanced and

\[
|\lambda| \leq |\lambda - \lambda_0| + |\lambda_0| < \frac{1}{r} + |\lambda_0| = \frac{1}{r} \frac{|\lambda_0|}{r},
\]

so \( \lambda x - \lambda_0 x_0 = \lambda(x - x_0) + (\lambda - \lambda_0)x_0 \)

\[
\subseteq \lambda \cdot \frac{r}{r|\lambda_0| + 1} V + (\lambda - \lambda_0)rV 
\subseteq V + V \subseteq W.
\]

Consequently, the map \( g : (\lambda, x) \to \lambda x \) is continuous at \((\lambda, x) = (\lambda_0, x_0)\). Thus \((E, \tau)\) is a TVS.

We now summarize topological properties of a TVS.

**Theorem A.2.4.** Let \((E, \tau)\) be a TVS. Then:

(a) \((E, \tau)\) is regular.

(b) \((E, \tau)\) is Hausdorff iff \( \bigcap_{U \in \mathcal{W}} U = \{0\} \), where \(\mathcal{W}\) is a base of \(\tau\)-neighborhoods of 0.

(c) \((E, \tau)\) is a \(T_1\)-space iff \((E, \tau)\) is Hausdorff iff \((E, \tau)\) is completely regular.

(d) \((E, \tau)\) is locally compact iff \((E, \tau)\) has a compact \(\tau\)-neighborhood of 0 iff \((E, \tau)\) is homeomorphic to \(\mathbb{K}^n\) for some \(n \geq 1\) (i.e. \((E, \tau)\) is finite dimensional).

(e) \((E, \tau)\) is compact iff \(E = \{0\}\).

(f) \((E, \tau)\) is both connected and locally connected.

If a vector space \(E\) is non-trivial (i.e. \(E \neq \{0\}\)) with the indiscrete topology \(\tau_I = \{E, \emptyset\}\), then \((E, \tau_I)\) is a TVS which is not a \(T_1\)-space, hence also not a Hausdorff space. But it is regular and normal.

**Definition.** A subset \(A\) of a TVS \(E\) is called:

(a) bounded if, for any neighborhood \(W\) of 0 in \(E\), there exists \(r > 0\) such that

\[A \subseteq \lambda W\text{ for all }\lambda \in \mathbb{K}\text{ with }|\lambda| \geq r,\]

or, equivalently, \(A \subseteq rW\) if \(W\) is balanced.
(b) precompact (or totally bounded) if, for each neighborhood $W$ of 0, there exists a finite subset $D = \{x_1, ..., x_n\}$ of $E$ such that

$$A \subseteq D + W = \bigcup_{i=1}^{n}(x_i + W).$$

Clearly, compactness $\Rightarrow$ precompactness $\Rightarrow$ boundedness; the reverse implications need not be true.

**Definition.** A net $\{x_\alpha : \alpha \in I\}$ in a TVS $E$ is said to be convergent to $x \in E$ if, given any neighborhood $U$ of 0 in $E$, there exists an $\alpha_0 \in I$ such that $x_\alpha - x \in U$ for all $\alpha \geq \alpha_0$; in this case, we write $x_\alpha \to x$. A net $\{x_\alpha\}$ is said to be a Cauchy net if, given any neighborhood $U$ of 0 in $E$, there exists an $\alpha_0 \in I$ such that $x_\alpha - x_\beta \in U$ for all $\alpha, \beta \geq \alpha_0$.

**Definition.** Let $E$ be a TVS, and let $A \subseteq E$. Then $A$ is called:

(i) complete if every Cauchy net in $A$ converges to a point in $A$;
(ii) sequentially complete if every Cauchy sequence in $A$ converges to a point in $A$;
(iii) quasi-complete (or boundedly complete) if every bounded Cauchy net in $A$ converges to a point of $E$ (i.e. if every bounded closed subset of $E$ is complete).

Clearly, completeness $\Rightarrow$ quasi-completeness $\Rightarrow$ sequential completeness; the reverse implications need not hold.

**Theorem A.2.5.** A subset $A$ of a TVS $E$ is compact iff it is complete and precompact.

**Definition.** Let $E$ be a TVS. Then a complete TVS $\hat{E}$ is called a completion of $E$ if there exists a topological isomorphism $\varphi : E \to \hat{E}$ such that $\varphi(E)$ is dense in $\hat{E}$. Every TVS has a unique Hausdorff completion ([KN63], p. 63-64).

**Definition.** A TVS $(E, \tau)$ is called:

(i) a locally convex space (or a locally convex TVS) if it has a base of convex neighborhoods of 0;
(ii) metrizable if $\tau$ is induced by a metric $d$ on $E$;
(iii) an $F$-space if it is a complete metrizable TVS.

In general, $E$ is locally convex $\Leftrightarrow$ $E$ is metrizable.
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Examples. (1) (i) ([Rud91], p. 86) For $0 < p < 1$, the sequence space

$$\ell_p(\mathbb{N}) = \{ x = \{ x_n \} \subseteq \mathbb{K} : \sum_{n=1}^{\infty} |x_n|^p < \infty \},$$

the set of all $p$-summable sequences in $\mathbb{K}$, is a metrizable TVS (in fact, an $F$-space) with metric given by

$$d_p(x, y) = \sum_{n=1}^{\infty} |x_n - y_n|^p, \, x = \{ x_n \}, y = \{ y_n \} \in \ell_p(\mathbb{N}).$$

but it is not locally convex.

(ii) ([Rud91], p. 36-37) For $0 < p < 1$, the space

$$L_p[a, b] = \{ f : [a, b] \rightarrow \mathbb{K} : \int_a^b |f(t)|^p \, dt < \infty \},$$

the set of all Lebesgue $p$-integrable functions, is also an $F$-space but not locally convex.

(2) For $X$ a topological space, let $E = C(X)$ with the compact-open topology $k$, i.e. the topology which has a base $\mathcal{W}$ of $k$-neighborhoods of $0$ consisting of all sets of the form

$$U(K, \varepsilon) = \{ f \in C(X) : \sup_{x \in K} |f(x)| \leq \varepsilon \},$$

where $K \subseteq X$ is compact and $\varepsilon > 0$. Then each $U(K, \varepsilon)$ is absorbing, balanced and convex, and so $(C(X), k)$ is locally convex. However, $(C(X), k)$ is not metrizable. In fact, if $X$ is a locally compact Hausdorff space, then $(C(X), k)$ is metrizable if $X$ is hemicompact (i.e. $X$ can be expressed as a countable union of compact sets $K_n$ such that each compact subset of $X$ is contained in some $K_n$).

Definition. Let $E$ be a vector space over $\mathbb{K}$.

(1) A function $q : E \rightarrow \mathbb{R}$ is called a seminorm on $E$ if it satisfies

- $(S_1)$ $q(x) \geq 0$ for all $x \in E$;
- $(S_2)$ $q(x) = 0$ if $x = 0$;
- $(S_3)$ $q(\lambda x) = |\lambda| q(x)$ for all $x \in E$ and $\lambda \in \mathbb{K}$ (absolutely homogeneous);
- $(S_4)$ $q(x + y) \leq q(x) + q(y)$ for all $x, y \in E$ (subadditive).

(2) A function $q : E \rightarrow \mathbb{R}$ is called a $k$-norm on $E$, where $0 < k \leq 1$, if it satisfies

- $(K_1)$ $q(x) \geq 0$ for all $x \in E$;
- $(K_2)$ $q(x) = 0$ if and only if $x = 0$;
- $(K_3)$ $q(\lambda x) = |\lambda|^k q(x)$ for all $x \in E$ and $\lambda \in \mathbb{K}$.
(K4) \( q(x + y) \leq q(x) + q(y) \) for all \( x, y \in E \).

(3) A function \( q : E \to \mathbb{R} \) is called an \( F \)-seminorm on \( E \) if it satisfies
\[
(F_1) \quad q(x) \geq 0 \quad \text{for all} \quad x \in E;
\]
\[
(F_2) \quad q(x) = 0 \quad \text{if} \quad x = 0;
\]
\[
(F_3) \quad q(\lambda x) \leq |\lambda| q(x) \quad \text{for all} \quad x \in E \quad \text{and} \quad \lambda \in \mathbb{K} \quad \text{with} \quad |\lambda| \leq 1;
\]
\[
(F_4) \quad q(x + y) \leq q(x) + q(y) \quad \text{for all} \quad x, y \in E \quad \text{(subadditive)};
\]
\[
(F_5) \quad \text{if} \quad \lambda_n \to 0, \text{ then} \quad q(\lambda_n x) \to 0 \quad \text{for all} \quad x \in E;
\]
\[
(F_6) \quad \text{if} \quad q(n) \to 0, \text{ then} \quad q(\lambda x_n) \to 0 \quad \text{for all} \quad \lambda \in \mathbb{K}.
\]

(4) If \( q : E \to \mathbb{R} \) satisfies
\[
q(\lambda x) = \lambda q(x) \quad \text{for all} \quad \lambda \geq 0,
\]
then \( q \) is called positively homogeneous.

(5) If a seminorm (resp. \( F \)-seminorm) \( q \) satisfies \( q(x) = 0 \) implies \( x = 0 \), then \( q \) is called a norm (resp. \( F \)-norm) on \( E \).

Remarks. (1) If \( q \) is a seminorm or \( F \)-seminorm on a vector space \( E \), it may happen that \( q(x) = 0 \) for some \( x \neq 0 \). Hence, if \( q \) is a seminorm or \( F \)-seminorm on a vector space \( E \), then it defines a pseudometric \( d_q \) on \( E \) given by
\[
d_q(x, y) = p(x - y), \quad x, y \in E.
\]

(2) It is easy to verify that if \( q \) is a seminorm on a vector space \( E \), then, for each \( r > 0 \), the \( q \)-balls
\[
B_q(0, r) = \{ x \in E : q(x) < r \}, \quad B_q[0, r] = \{ x \in E : q(x) \leq r \}
\]
are absorbing, balanced and convex subset of \( E \).

Definition: Let \( A \) be an absorbing subset of a vector space \( E \). Then the function \( \rho_A : E \to \mathbb{R} \) defined by
\[
\rho_A(x) = \inf \{ r > 0, \quad x \in rA \}, \quad x \in E,
\]
is called the Minkowski functional of \( A \).

Remark. It is easy to see that if, in addition, \( A \) is either balanced or convex, then
\[
(\text{i}) \quad \text{for any} \quad x \in E \quad \text{and} \quad \varepsilon > 0, \quad x \in (\rho_A(x) + \varepsilon)A.
\]
\[
(\text{ii}) \quad \{ x \in E : \rho_A(x) < 1 \} \subseteq A \subseteq \{ x \in E : \rho_A(x) \leq 1 \}.
\]

Lemma A.2.6. Let \( A \) be an absorbing subset of a vector space \( E \). Then:
\[
(\text{a}) \quad 0 \leq \rho_A(x) < \infty \quad \text{with} \quad \rho_A(0) = 0.
\]
\[
(\text{b}) \quad (\text{i}) \quad \rho_A(tx) = t \rho_A(x) \quad \text{for all} \quad x \in E \quad \text{and} \quad t \geq 0 \quad \text{(positively homogeneous)};
\]
\[
(\text{ii}) \quad A \text{ is balanced iff} \quad \rho_A(\lambda x) = |\lambda| \rho_A(x) \quad \text{for all} \quad x \in E \quad \text{and} \quad \lambda \in \mathbb{K} \quad \text{(absolutely homogeneous)}.
\]
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(c) A is convex iff
\[ \rho_A(x + y) \leq \rho_A(x) + \rho_A(y) \] for all \( x, y \in E \) (subadditive).

Hence, if \( A \) is an absorbing, balanced and convex subset of a vector space \( E \), then \( \rho_A \) is a seminorm on \( E \).

Note. If \( A \) is not convex, then \( \rho_A \) need not be subadditive. However, we have:

Lemma A.2.7. Let \( A \) be an absorbing subset of a vector space \( E \). If \( H \) is a balanced subset of \( E \) with \( H + H \subseteq A \), then
\[ \rho_A(x + y) \leq \rho_H(x) + \rho_H(y) \] for all \( x, y \in E \).

Proof. Let \( x, y \in E \) and let \( \varepsilon > 0 \). Then there exist \( r, s > 0 \) such that
\[ \rho_H(x) < r < \rho_H(x) + \frac{\varepsilon}{2} \] and \( x \in rH \),
\[ \rho_H(y) < s < \rho_H(y) + \frac{\varepsilon}{2} \] and \( y \in sH \).

so
\[ x + y \in rH + sH = (r + s)[rH + sH] \subseteq (r + s)H \] (since \( H \) is balanced)
\[ \subseteq (r + s)A. \]

Hence
\[ \rho_A(x + y) \leq r + s \leq \rho_H(x) + \frac{\varepsilon}{2} + \rho_H(y) + \frac{\varepsilon}{2} = \rho_H(x) + \rho_H(y) + \varepsilon. \]

Since \( \varepsilon \) is arbitrary,
\[ \rho_A(x + y) \leq \rho_H(x) + \rho_H(y). \] □

Lemma A.2.8. Let \( E \) be a vector space and \( p, q : E \to R^+ \) be positively homogeneous (e.g. \( p, q \) be seminorms on \( E \) or \( p = \rho_G, q = \rho_H \) be Minkowski functionals of neighborhoods \( G \) and \( H \) of \( 0 \) in a TVS \( E \)). Then
\[ p(x) \leq q(x) \forall x \in E \iff q(x) \leq 1 \text{ implies } p(x) \leq 1 \forall x \in E. \]

Proof. (\( \Rightarrow \)) Suppose \( p(x) \leq q(x) \) \( \forall \) \( x \in E \). If \( q(x) \leq 1 \), then
\[ p(x) \leq q(x) \leq 1. \]

(\( \Leftarrow \)) Suppose \( q(x) \leq 1 \) implies \( p(x) \leq 1 \) for all \( x \in E \). Fix \( x \in E \), we consider two cases:
Case I. Suppose \( q(x) \neq 0 \). Then 
\[
q \left( \frac{x}{q(x)} \right) = 1; \quad \text{and so, by hypothesis,}
\]
\[
p \left( \frac{x}{q(x)} \right) \leq 1, \quad \text{or} \quad \frac{p(x)}{q(x)} \leq 1, \quad \text{or} \quad p(x) \leq q(x).
\]

Case II. Suppose \( q(x) = 0 \). Then 
\[
q \left( \frac{x}{\varepsilon} \right) < \varepsilon \quad \text{for every} \quad \varepsilon > 0,
\]
\[
q \left( \frac{x}{\varepsilon} \right) \leq 1 \quad \text{for every} \quad \varepsilon > 0.
\]

Hence, by hypothesis, for any \( \varepsilon > 0 \), 
\[
p \left( \frac{x}{\varepsilon} \right) < 1, \quad \text{or} \quad p(x) < \varepsilon
\]

Therefore, \( p(x) = 0 \). So, in this case \( p(x) = 0 = q(x) \).

Therefore, \( p(x) = 0 \). So, in this case \( p(x) = 0 = q(x) \).

Thus, in each case, \( p(x) \leq q(x) \) for all \( x \in E \).

**Lemma A.2.9.** Let \( A \) be an absorbing and balanced (or convex) subset of a TVS \((E, \tau)\). Then:

(a) \( A^0 \subseteq \{ x \in E : \rho_A(x) < 1 \} \subseteq A \subseteq \{ x \in E : \rho_A(x) \leq 1 \} \subseteq \bar{A} \).

(b) \( \rho_A \) is \( \tau \)-continuous iff \( A \) is a \( \tau \)neighborhood of \( 0 \).

(c) If \( A \) is a \( \tau \)neighborhood of \( 0 \), then:

(i) \( A^0 = \{ x \in E : \rho_A(x) < 1 \} \).

(ii) \( \bar{A} = \{ x \in E : \rho_A(x) \leq 1 \} \).

We shall later see that if \( A \) is a “shrinkable” (not necessarily convex) \( \tau \)neighborhood of \( 0 \), then \( \rho_A \) is \( \tau \)-continuous on \( E \). From the above lemmas, we obtain:

**Theorem A.2.10.** If \((E, \tau)\) is a locally convex space, then its topology \( \tau \) can be defined by a family of seminorms, namely, the Minkowski functionals of balanced convex \( \tau \)neighborhoods of \( 0 \). Conversely, if \( \mathcal{P} \) is a family of seminorms on a vector space \( E \) over \( \mathbb{K} \), then \( \mathcal{P} \) defines a locally convex topology \( \tau \) (say) on \( E \) and a base of \( \tau \)neighborhoods of \( 0 \) consists of all sets of the form

\[
\{ x \in E : \max_{1 \leq j \leq n} p_j(x) \leq \varepsilon \}, \quad \text{where} \quad \varepsilon > 0 \quad \text{and} \quad p_j \in \mathcal{P}, \quad n \in \mathbb{N}.
\]

For general TVSs, we have:

**Theorem A.2.11.** If \((E, \tau)\) is TVS, then its topology \( \tau \) can be defined by a family of \( F \)-seminorms. If \( \tau \) is generated by a countable family of \( F \)-seminorms, then \( \tau \) can be generated by a single \( F \)-norm and so it is metrizable.

**Proof.** ([Wae71], p. 2-3) Let \( V \in \mathcal{W} \) be any balanced \( \tau \)neighborhood of \( 0 \) in \( E \). Define \( V_1, ..., V_k, ... \) inductively in such a way that each \( V_k \) is balanced, \( V \supseteq V_i + V_i \), and \( V_k \supseteq V_{k+1} + V_{k+1} \). Let \( D = \{ p/2^n : p \) is
rational, $n \in \mathbb{N}$, the set of dyadic rationals. Fix any $t = \frac{q}{2^n} \in D$, $0 \leq t < 1$. Then $t = \sum_{k=1}^{N} \frac{\varepsilon_k}{2^n}$ with $\varepsilon_k = 0$ or 1 for all $k$. Put

$$W_t = \sum_{k=1}^{N} \varepsilon_k V_k; \text{ that is, } W_t = \sum_{\varepsilon_k=1}^{N} V_k.$$  

Then $W_t$ is balanced, and clearly $W_{t_1} + W_{t_2} \subseteq W_{t_1 + t_2}$ if $t_1 + t_2 < 1$. We put $W_t = E$ for $t \geq 1$, and define

$$q_V(x) = \inf \{ t : x \in W_t \}, x \in E.$$  

Then $q_V$ is an $F$-seminorm on $E$. Thus the family $\{q_V : V \in \mathcal{W}\}$ generates the topology $\tau$. If $\tau$ is generated by a countable family $\{q_n : n \in \mathbb{N}\}$ of $F$-seminorms, then clearly $\tau$ can be generated by a single $F$-norm $q$ given by

$$q(x) = \sum_{n=1}^{\infty} 2^{-n} \frac{q_n(x)}{1 + q_n(x)}, x \in E.$$  

Thus $(E, \tau)$ is metrizable. \hfill $\square$  

The general criterion for the metrizability of a TVS may be stated as:

**Theorem A.2.12.** (Metrization Theorem)

(\cite{KN63}, p. 48-49) (a) A TVS $(E, \tau)$ is pseudo-metrizable iff it has a countable base of $\tau$-neighborhoods of 0. In this case, $\tau$ may be defined by a pseudo-metric $d$ which is translation invariant and each ball about 0 is balanced; i.e.,

(i) $d(x + z, y + z) = d(x, y) = d(x - y, 0)$ for all $x, y, z \in E$;

(ii) $d(\lambda x, 0) \leq d(x, 0)$ for all $x \in E$ and $\lambda \in \mathbb{K}$ with $|\lambda| \leq 1$;

(b) A TVS $(E, \tau)$ is metrizable iff it is Hausdorff and has a countable base of $\tau$-neighborhoods of 0. In this case, $\tau$ may be given by an $F$-norm $q$, where $q(x) = d(x, 0)$ for all $x \in E$.

**Definition.** A TVS $E$ is said called locally bounded if $E$ has a bounded neighborhood of 0. In this case, if $V$ is a bounded neighborhood of 0 in $E$, then clearly, $\left\{ \frac{1}{n} V : n \geq 1 \right\}$ is a countable base of neighborhoods of 0 in $E$. Then, by above theorem, every Hausdorff locally bounded TVS is metrizable.

**Theorem A.2.13.** (Kolmogoroff) (i) The topology of a Hausdorff locally bounded TVS $E$ can always be generated by a $k$-norm for some $k$, $0 < k \leq 1$.  


(ii) A Hausdorff TVS \((E, \tau)\) is normable iff it has a bounded convex neighborhood of 0. (Hence every Hausdorff locally bounded locally convex space is normable.)

**Theorem A.2.14.** Let \(E\) and \(F\) be TVSs. Then:
(a) A linear mapping \(T : E \to F\) is continuous on \(E\) iff \(T\) is continuous at \(0 \in E\) iff \(T\) is uniformly continuous on \(E\).

(b) If \(f : E \to K\) a non-zero linear functional, then \(f\) is continuous on \(E\) iff \(f^{-1}(0)\) is closed in \(E\) iff \(f\) is bounded on some neighborhood \(U\) of 0 in \(E\).

**Definition:** The set of all linear functionals on a vector space (or a TVS) \(E\) is a vector space, called the algebraic dual of \(E\) and is denoted by \(E'\). If \(E\) is a TVS, the set of all continuous linear functionals on \(E\) is also a vector space, called the topological dual of \(E\) and is denoted by \(E^*\). Clearly, \(E'\) is a vector space over \(K\). Further, \(E^*\) is a vector subspace of \(E'\).

**Definition.** Let \((E, \tau)\) be a TVS and \(M\) a vector subspace of \(E\).
1. The quotient space \(E/M\) is defined by
   \[E/M = \{x + M : x \in E\},\]
which is a vector space under the operations
   \[(x + M) + (y + M) = (x + y) + M,\quad \lambda(x + M) = \lambda x + M,\]
where \(x, y \in E\) and \(\lambda \in K\).
2. The canonical map (or quotient map) \(\pi : E \to E/M\) is defined by
   \[\pi(x) = x + M, \quad x \in E.\]
3. The quotient topology \(\tau_M\) on \(E/M\) is defined by
   \[\tau_M = \{A \subseteq E/M : \pi^{-1}(A) \in \tau\}.\]

**Theorem A.2.15.** ([Scha71], p. 20, 31) Let \((E, \tau)\) be a TVS and \(M\) a vector subspace of \(E\). Then:
(a) \((E/M, \tau_M)\) is a TVS.
(b) \(\pi : (E, \tau) \to (E/M, \tau_M)\) is a linear, continuous and open map.
(c) \(\tau_M\) is Hausdorff iff \(M\) is closed.
(d) If \((E, \tau)\) is locally convex whose topology is given by a family \(s_E\) of seminorms, then \((E/M, \tau_M)\) is also a locally convex space and its topology is given by the family \(\{\bar{p} : p \in s_E\}\) of seminorms, where \(\bar{p} : E/M \to \mathbb{R}\) is given by
   \[\bar{p}(x + M) = \inf\{p(x + m) : m \in M\}, x + M \in E/M.\]
(e) If \((E, \tau)\) is metrizable (resp. normable) and \(M\) is closed, then \((E/M, \tau_M)\) is also metrizable (resp. normable).

(f) If \((E, \tau)\) is complete and metrizable (resp. a Banach space) and \(M\) is closed, then \((E/M, \tau_M)\) is complete and metrizable (resp. a Banach space).

Remarks. (1) If \(M\) is a closed vector subspace of a TVS \(E\), then the seminorm \(\tilde{p}: E/M \to \mathbb{R}\) defined above in (d) becomes a norm on \(E/M\), since
\[
\tilde{p}(x + M) = 0 \Rightarrow x \in \overline{M} = M \Rightarrow x + M = M, \text{ the zero of } E/M.
\]

(2) ([Rud91], p. 33) Let \(p\) be a seminorm on a vector space \(E\) and \(N_p = \ker(p) = p^{-1}(0)\). Then
(i) \(\inf\{ p(x + m) : m \in N_p \} = p(x)\).
(ii) If \(\tilde{p}: E/N_p \to \mathbb{R}\) is defined by
\[
\tilde{p}(x + N_p) = p(x), \quad x + N_p \in E/N_p,
\]
then \(\tilde{p}\) is a norm on \(E/N_p\).

Definition. (1) If \(M\) be a proper maximal vector subspace of a vector space \(E\) and \(x_0 \in E\), then the set \(H = x_0 + M\) is called a hyperplane through \(x_0\); in particular, \(M\) is a hyperplane through 0.

(2) Two vector subspaces \(M\) and \(N\) of \(E\) are said to be complementary if \(E = M \oplus N\) (i.e. if \(E = M + N\) with \(M \cap N = \{0\}\)).

(3) The co-dimension of vector subspace \(M\) of \(E\) is defined as the dimension of its complementary subspace \(N\) (or equivalently, the dimension of the quotient space \(E/M\)).

Theorem A.2.16. Let \(E\) be a vector space and \(H \subseteq E\). Then:
(a) \(H\) is a hyperplane iff \(H = \varphi^{-1}(\alpha) = \{x \in E : \varphi(x) = \alpha\}\) for some non-zero linear functional \(\varphi\) on \(E\) and \(\alpha \in \mathbb{K}\).
(b) \(H\) is a hyperplane through 0 iff \(M\) is of co-dimension one iff \(M = \varphi^{-1}(0)\) for some non-zero linear functional \(\varphi\) on \(E\).
(c) If \(E\) is a TVS, then a hyperplane \(H = \varphi^{-1}(\alpha)\) in \(E\) is either closed or dense in \(E\). Further, \(H = \varphi^{-1}(\alpha)\) is closed iff \(\varphi\) is continuous on \(E\).

Theorem A.2.17. (Hahn-Banach) Let \((E, \tau)\) be a TVS and \(M\) a vector subspace of \(E\).
(a) If \(f\) is a continuous linear functional on \(M\) and \(p\) a continuous seminorm on \(E\) with
\[
|f(x)| \leq p(x) \text{ for all } x \in M,
\]
then there exists a $g \in E^*$ such that 
$$g = f \text{ on } M \text{ and } |g(x)| \leq p(x) \text{ for all } x \in E.$$ 

(b) If $E$ is locally convex and $f$ a continuous linear functional on $M$, then there exist a $g \in E^*$ and a continuous seminorm $p$ on $E$ such that 
$$g = f \text{ on } M \text{ and } |g(x)| \leq p(x) \text{ for all } x \in E.$$ 

**Corollary A.2.18.** Let $E$ be a Hausdorff locally convex space (in particular, a normed space) over $\mathbb{K}$. Then $E^*$ separates points of $E$ (i.e. for any $x_o \neq 0$ in $E$, there exist a $g \in E^*$ such that $g(x_o) \neq 0$).

Note that if $E$ is not Hausdorff or not locally convex, then $E^*$ need not separate the points of $E$.

**Examples.** For $0 < p < 1$, the spaces $\ell_p(\mathbb{N})$ and $L_p[a,b]$ are $F$-spaces but not locally convex spaces (as seen before). However, $(\ell_p(\mathbb{N}))^*$ separates points of $\ell_p(\mathbb{N})$ and $(L_p[a,b])^*$ does not separate points of $L_p[a,b]$ (as $(L_p[a,b])^* = \{0\}$).

**Remark.** In fact, if $E$ is a TVS, then $E^* \neq \{0\}$ iff $E$ contains a non-empty open convex subset $U \neq E$ ([Scha71], p. 47). For more material on non-locally convex spaces, see the monographs by Waelbroeck ([Wae71, Wae73]), Adasch, Ernst and Keim [AEK78], Khaleelulla [Khal81], Kalton, Peck and Roberts [KPR84], Rolewicz [Rol85]; see also the papers by Klee ([Kle60a, Kle60b]) and Kakol ([Kako85, Kako87, Kako90, Kako92]).

**Definition.** Let $(E, \tau)$ be a TVS. For any finite (or singleton) set $A \subseteq E^*$, the mapping $p_A : E \to \mathbb{R}$ defined by 
$$p_A(x) = \sup\{|f(x)| : f \in A\}, \quad x \in E,$$ 
is a seminorm $p_A$ on $E$. If $\mathcal{A}$ is the collection of all finite subsets of $E^*$, then the collection $\{p_A : A \in \mathcal{A}\}$ of seminorms defines a locally convex topology on $E$, called the **weak topology** and denoted by $w = w(E, E^*)$ or $\sigma(E, E^*)$. Similarly, for any finite (or singleton) set $B \subseteq E^*$, the mapping $q_B : E^* \to \mathbb{R}$ defined by 
$$q_B(f) = \sup\{|f(x)| : x \in B\}, \quad f \in E^*.$$ 
is a seminorm $q_B$ on $E^*$. If $\mathcal{B}$ is the collection of all finite subsets of $E$, then the collection $\{q_A : B \in \mathcal{B}\}$ of seminorms defines a locally convex topology on $E^*$, called the **weak$^*$ topology** on $E$ and denoted by $w^* = w(E^*, E)$ or $\sigma(E^*, E)$.

**Theorem A.2.19.** ([Hor66], p. 185-209) (a) $w(E, E^*) \leq \tau$. 

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(b) \((E, w(E, E^*))\) is Hausdorff iff \(E^*\) separates points of \(E\); in particular, if \((E, \tau)\) is a Hausdorff locally convex space, then \((E, w(E, E^*))\) is Hausdorff.

(c) \((E, w(E, E^*))^* = (E, \tau)^*\).

(d) If \((E, \tau)\) a Hausdorff locally convex space, then:

(i) (Mazur) For any convex subset \(M\) of \(E\), \(M\) is \(\tau\)-closed iff \(M\) is \(w(E, E^*)\)-closed; in particular, \(M^w = M^\tau\).

(ii) (Mackey) For any subset \(M\) of \(E\), \(M\) is \(\tau\)-bounded iff \(M\) is \(w(E, E^*)\)-bounded.

(e) If \((E, \tau)\) is finite-dimensional, then \(\tau = w(E, E^*)\).

We next consider strong and strong* topologies on \(E\) and \(E^*\), respectively. We first note that, for any set \(B \subseteq E\), the mapping \(q_B : E \to \mathbb{R} \) defined by

\[ q_B(f) = \sup \{|f(x)| : x \in B\}, \quad f \in E^*, \]

is a seminorm on \(E^*\) iff \(B\) is \(w(E^*, E)\)-bounded in \(E^*\) (see [RR64], p.45; [Hor66], p. 195).

**Definition.** Let \((E, \tau)\) be a TVS. For any \(w(E^*, E)\)-bounded set \(A \subseteq E^*\), the mapping \(p_A : E \to \mathbb{R}\) defined by

\[ p_A(x) = \sup \{|f(x)| : f \in A\}, \quad x \in E, \]

is a seminorm \(p_A\) on \(E\). If \(A\) is the collection of all \(w(E^*, E)\)-bounded subsets of \(E^*\), then the collection \(\{p_A : A \in \mathcal{A}\}\) of seminorms defines a locally convex topology on \(E\), called the strong topology and denoted by \(\beta = \beta(E, E^*)\). Clearly, \(w(E, E^*) \leq \beta(E, E^*)\). Similarly, for any \(\tau\)-bounded (or equivalently, \(w(E^*, E)\)-bounded) set \(B \subseteq E\), the mapping \(q_B : E^* \to \mathbb{R}\) defined by

\[ q_B(f) = \sup \{|f(x)| : x \in B\}, \quad f \in E^*. \]

is a seminorm \(q_B\) on \(E^*\). If \(B\) is the collection of all \(\tau\)-bounded subsets of \(E\), then the collection \(\{q_B : B \in \mathcal{B}\}\) of seminorms defines a locally convex topology on \(E^*\), called the strong* topology and denoted by \(\beta^* = \beta(E^*, E)\). Clearly, \(w(E^*, E) \leq \beta(E^*, E)\).

**Remarks.** (1) ([Wila78], p. 119) If \((E, ||\cdot||)\) is a normed space, then the strong* topology \(\beta(E^*, E)\) on \(E^*\) is the norm topology given by

\[ ||f|| = \sup\{|f(x)| : ||x|| \leq 1, \quad f \in E^*. \]

However, in this case, the strong topology \(\beta(E, E^*)\) on \(E\) need not the norm topology for \(E\); in general, \(||\cdot|| \leq \beta(E, E^*)\). If \(E\) is a Banach space, then \(\beta(E, E^*) = ||\cdot||\).
The weak and strong topologies are often useful in giving counterexamples. For instance, the Banach space $\ell_1 = \ell_1(\mathbb{N})$ with the weak topology $w = w(\ell_1, \ell_\infty)$ is sequentially complete but not complete; $\ell_1$ with the weak* topology $w^* = w(\ell_1, c_0)$ is quasi-complete but not complete. (Recall that $c_0^* \cong \ell_1$ and $\ell_1^* = \ell_\infty$ as normed spaces.) Hilbert space is weakly quasi-complete but not weakly complete.

**Definition:** Let $(E, \tau)$ be a TVS. A subset $H$ of $E^*$ is said to be equicontinuous if, given any $\varepsilon > 0$, there exists a neighborhood $U$ of 0 in $E$ such that

$$|f(x)| < \varepsilon \text{ for all } x \in U \text{ and } f \in H.$$ 

**Definition.** Let $(E, \tau)$ is a TVS such that $E^*$ separates points of $E$. For each set $U \subseteq E$, the set

$$U^p = \{ f \in E^* : q_U(f) = \sup \{|f(x)| : x \in U\} \leq 1 \}$$

is called the polar of $U$ in $E^*$.

**Theorem A.2.20.** ([Hor66], p. 190, 195) Let $(E, \tau)$ be a TVS and $U \subseteq E$. Then:

(i) $U^p$ is convex, balanced and $w(E^*, E)$-closed in $E^*$.

(ii) $U^p$ is absorbing iff $U$ is $w(E^*, E)$-bounded iff $q_U$ is a seminorm on $E^*$.

**Theorem A.2.21.** ([Hor66], p. 200) Let $(E, \tau)$ be a TVS. Then

(i) [Hor66, p. 200] A subset $H$ of $E^*$ is equicontinuous iff there exists a $\tau$-neighborhood $U$ of 0 in $E$ such that $H \subseteq U^p$. In particular, for any $\tau$-neighborhood $U$ of 0 in $E$, $U^p$ is an equicontinuous subset of $E^*$.

(ii) Every equicontinuous subset $H$ of $E^*$ is $w(E^*, E)$-bounded.

**Theorem 1.2.22.** (Bourbaki-Alaoglu) ([Hor66], p. 201) Let $(E, \tau)$ be a TVS. Then any $\tau$-equicontinuous subset $H$ of $E^*$ is relatively $w(E^*, E)$-compact. In particular, for any $\tau$-neighborhood $U$ of 0 in $E$, $U^p$ is $w(E^*, E)$-compact.

**Corollary A.2.23.** (Banach-Alaoglu) Let $E$ be a normed space. Then the norm closed unit ball

$$B^*[0, 1] = \{ f \in E^* : \|f\| \leq 1 \}$$

of $E^*$ is $w(E^*, E)$-compact.

**Note.** ([Ko69], p. 282) A norm closed unit ball of a normed space $E$ need not be $w(E, E^*)$-compact in $E$; take, for example, $E = \ell^1$. 

Theorem A.2.24. ([Sim63], p. 234]). Let \((E, ||\cdot||)\) is a Banach space, and let \(S^* = (B_1^*, w^*)\), a compact Hausdorff space. For any \(x \in X\), define a continuous map \(\hat{x} : S^* \to \mathbb{K}\) by

\[\hat{x}(f) = f(x), \ f \in S^*.\]

Then the map \(\pi : x \to \hat{x}\) is an isometric isomorphism of \(E\) onto a closed vector subspace of the Banach space \(C(S^*)\).

Remarks. (1) By above theorem, any general Banach space \((E, ||\cdot||)\) can be considered as a closed vector subspace of the function space \((C(X), ||\cdot||_{\infty})\), where \(X\) is a compact Hausdorff space.

(2) [Jam74, p. 273] It is fair to ask whether \(C[0, 1]\) is typical of many other spaces \(C(X)\). In this regard, we mention a famous result of A.A. Milyutin (1966) which states that: if \(X\) is an uncountable compact metric space, then \(C(X)\) is isomorphic to \(C(I)\).

Definition. A sequence \(\{x_n\}\) in a TVS \(E\) is called a topological basis of \(E\) if, for each \(x \in E\), there is a unique sequence \(\{\lambda_n\}\) of scalars such that

\[x = \lim_{k \to \infty} \sum_{n=1}^{k} \lambda_n x_n\] (in the topology of \(E\));

that is, for any neighborhood \(U\) of 0 in \(E\), there exists an integer \(N \geq 1\) such that

\[\sum_{i=1}^{k} \lambda_i x_i - x \in U\] for all \(k \geq N\).

In this case we write \(x = \sum_{i=1}^{\infty} \lambda_i x_i\). For each \(m = 1, 2, ...,\) the linear functionals \(f_m : E \to \mathbb{K}\) defined by

\[f_m(x) = \lambda_m\] for \(x = \sum_{i=1}^{\infty} \lambda_i x_i \in E,\]

are called the associated coefficient functionals of the basis \(\{x_n\}\). Note that \(f_m(x_n) = \delta_{mn}\). If all \(f_m\) are continuous, the basis \(\{x_n\}\) is called a Schauder basis.

Definition. (1) A TVS \(E\) is said to have the approximation property if the identity map on \(E\) can be approximated uniformly on precompact sets by continuous and linear maps of finite rank (i.e. with range contained in finite dimensional subspaces of \(E\)).
(2) A TVS $E$ is said to be *admissible* if the identity map on $E$ can be approximated uniformly on compact sets by continuous maps of finite rank.

Every locally convex space and every $F$-space with a basis (e.g. $\ell_p$, $0 < p < 1$) is admissible ([Kl60a, Kl60b, Shu72]).
3. Spaces of Continuous Linear Mappings

**Definition.** Let $E$ and $F$ be TVSs over the same field $\mathbb{K}$. Then a linear mapping $T : E \to F$ is called *bounded* if it maps bounded sets of $E$ into bounded sets of $F$.

**Lemma A.3.1.** Every continuous linear mapping $T : E \to F$ is bounded.

*Proof.* Let $S$ be a bounded set in $E$, and let $W$ be a neighborhood of 0 in $F$. By continuity of $T$, $T^{-1}(W)$ is a neighborhood of 0 in $E$. Since $S$ is a bounded set in $E$, there exists $r > 0$ such that $S \subseteq \lambda T^{-1}(W)$ for all $|\lambda| \geq r$. Then $T(S) \subseteq \lambda W$ for all $|\lambda| \geq r$. Hence $T(S)$ is a bounded set in $F$.

In general, a bounded linear map $T : E \to F$ need not be continuous. We shall see below that, under some additional hypothesis, every bounded linear mapping $T : E \to F$ is continuous.

**Theorem A.3.2.** Let $E$ be a metrizable TVS and $F$ any TVS. Then any bounded linear map $T : E \to F$ is continuous (i.e. $T$ is continuous iff $T$ is bounded).

*Proof.* First suppose that $E$ is a metrizable TVS and $F$ any TVS, and let $T : E \to F$ be a bounded linear map. In this case, there exists a countable base $\{U_1, U_2, \ldots\}$ of neighborhood of 0 in $E$. Let

$$V_1 = U_1, \; V_2 = U_1 \cap U_2, \ldots, \; V_n = U_1 \cap U_2 \cap \ldots \cap U_n, \ldots$$

Then $\{V_1, V_2, \ldots\}$ is also a base of neighborhood of 0 with $V_1 \supseteq V_2 \supseteq \ldots$. If $T$ is not continuous, there is some neighborhood $W$ of 0 in $F$ such that $T^{-1}(W)$ is not a neighborhood of 0 in $E$. Hence, for each $n \geq 1$, $V_n \nsubseteq nT^{-1}(W)$ and so $V_n \setminus nT^{-1}(W) \neq \emptyset$. For each $n \geq 1$, choose $x_n \in V_n \setminus nT^{-1}(W)$. Then $\{x_n\}$ is a bounded set in $E$. [Indeed, let $V$ be any neighborhood of 0 in $E$. There exists a $V_m$ with $V_m \subseteq V$. Now, for any $n \geq m$, $x_n \in V_n \subseteq V_m$. Choose $r > 1$ such that $\{x_1, \ldots, x_{m-1}\} \subseteq rV_m$. Hence $x_n \in rV_m \subseteq rV$ for all $n \geq 1$.] However, $\{Tx_n\}$ is not bounded in $F$ since, for all $n \geq 1$, $x_n \notin nT^{-1}(W)$ and hence $T(x_n) \notin nW$. This contradicts the fact that $T$ is bounded. □

**Notations.** The set of all bounded (resp. continuous) linear mappings $T : E \to F$ is denoted by $BL(E, F)$ (resp. $CL(E, F)$). Clearly, $BL(E, F)$ and $CL(E, F)$ are vector spaces with the usual pointwise operations and $CL(E, F) \subseteq BL(E, F)$. Note that $E' = L(E, \mathbb{K})$ and $E^* = CL(E, \mathbb{K})$. Further, if $F = E$, $BL(E) = BL(E, E)$ and $CL(E) =$
$CL(E,E)$ are algebras under composition (i.e. $(ST)(x) = S(T(x)))$ and have the identity $I : E \to E$ given by $I(x) = x$ ($x \in E$).

**Definition.** (1) A TVS $E$ is called ultrabornological [Iya68] if every bounded linear map from $E$ into any TVS $F$ is continuous. In this case, $CL(E,F) = BL(E,F)$. By above theorem, every metrizable TVS is ultrabornological.

(2) A TVS $(E,\tau)$ is called ultrabarrelled ([RobW58, Edw65, Hus65]) if, any linear topology $\tau'$ on $E$, having a base of neighborhoods of 0 formed of $\tau$-closed sets, is weaker than $\tau$. Equivalently, a TVS $E$ is ultrabarrelled if every closed linear map from $E$ into any complete metrizable TVS is continuous [Iya68].

(3) $E$ is called quasi-ultrabarrelled if every bounded linear map from $E$ into any complete metrizable TVS is continuous.

Every Baire TVS (in particular, $F$-space) is ultrabarrelled ([Edw65], p. 428). Further, every ultrabornological and every ultrabarrelled TVS is quasi-ultrabarrelled.

We next consider some linear topologies on $CL(E,F)$, known as the $G$-topologies.

**Definition.** Let $E$ and $F$ be TVSs, and let $\mathcal{G}$ be a collection of subsets of $E$ and $W_F$ a base of closed balanced neighborhoods of 0 in $F$. We topologize $CL(E,F)$, as follows. For any $D \in \mathcal{G}$ and $W \in W_F$, let

$$U(D,W) = \{T \in CL(E,F) : T(D) \subseteq W\}.$$  

Then the family $\{U(D,W) : D \in \mathcal{G}, W \in W_F\}$ forms a subbase for a topology on $CL(E,F)$, called the topology of uniform convergence on members of $\mathcal{G}$, or briefly, the $\mathcal{G}$-topology and is denoted by $t_\mathcal{G}$.

Note that, for any $D \in \mathcal{G}$ and $W \in W_F$,

(i) $U(D,\lambda W) = \lambda U(D, W)$ ($\lambda \in \mathbb{K}$);

(ii) $U(D,W)$ is balanced (resp. convex) if $W$ is so;

(ii) $U(D,W)$ is absorbing iff $T(D)$ is absorbed by $W$ for each $T \in CL(E,F)$.

**Theorem A.3.4.** ([Scha71], p. 79-80) (a) $(CL(E,F),t_\mathcal{G})$ is a TVS iff $T(D)$ is bounded in $F$ for each $D \in \mathcal{G}$ and $T \in CL(E,F)$. In particular, if $\mathcal{G}$ is the collection of all bounded subsets of $E$, then $(CL(E,F),t_\mathcal{G})$ is a TVS.

(b) In addition, if $E = \bigcup\{D : D \in \mathcal{G}\}$, then $(CL(E,F),t_\mathcal{G})$ is Hausdorff; if $F$ is locally convex, so is $(CL(E,F),t_\mathcal{G})$.

The most important cases of the $\mathcal{G}$-topology are as follows:
Definition. (1) If \( G \) is the collection of all bounded subsets of \( E \), then the \( G \)-topology on \( CL(E,F) \) is called the topology of uniform convergence on bounded sets (or the uniform topology) and is denoted by \( t_u \).

(2) If \( G \) is the collection of all precompact subsets of \( E \), then the \( G \)-topology on \( CL(E,F) \) is called the topology of precompact convergence and is denoted by \( t_{pc} \).

(3) If \( G \) is the collection of all compact subsets of \( E \), then the \( G \)-topology on \( CL(E,F) \) is called the topology of compact convergence and is denoted by \( t_c \).

(4) If \( G \) is the collection of all finite subsets of \( E \), then the \( G \)-topology on \( CL(E,F) \) is called the topology of pointwise convergence and is denoted by \( t_p \).

Clearly, \( t_p \leq t_c \leq t_{pc} \leq t_u \).

Since each \( T \in CL(E,F) \) maps bounded sets of \( E \) into bounded sets in \( F \), it follows from the above theorem that \( CL(E,F) \) with each of these topologies is a TVS. We also mention that, if \( F = K \), we have \( CL(E,F) = CL(E,K) = E^* \) and so the weak topology \( w(E^*,E) \) and the strong topology \( \beta(E^*,E) \) on \( E^* \) are particular cases of the \( G \)-topologies.

Theorem A.3.5. ([Edw65], p. 87; [Sch71], p. 85) Let \( E \) and \( F \) TVSs.

(a) If \( E \) is ultrabornological (in particular, metrizable) and \( F \) is complete (resp. quasi-complete), then \( (CL(E,F),t_u) \) is complete (resp. quasi-complete).

(b) If \( E \) is a Baire space (in particular, an \( F \)-space) and \( F \) is quasi-complete, then \( (CL(E,F),t_u) \) is quasi-complete.

Proof. We only prove the first part of (a). Let \( \{T_{\alpha} : \alpha \in I \} \) be a Cauchy net in \( (CL(E,F),t_u) \). Then, for each \( x \in E, \{T_{\alpha}(x) : \alpha \in I \} \) is a Cauchy net in \( F \), as follows. [Let \( x \in E \), and consider any \( W \in W_F \). Then \( U(\{x\},W) \) is a \( t_u \)-neighborhood of 0 in \( CL(E,F) \). Since \( \{T_{\alpha}\} \) is a Cauchy net in \( (CL(E,F),t_u) \), there exists \( \alpha_0 \in I \) such that \( T_{\alpha}(x) - T_{\beta}(x) \in W \) for all \( \alpha, \beta \geq \alpha_0 \).]

Hence \( \{T_{\alpha}(x)\} \) is a Cauchy net in \( F \).] Since \( F \) is complete, the mapping \( T : E \to F \) given by

\[
T(x) = \lim_\alpha T_{\alpha}(x), \ x \in E,
\]

is well-defined. Clearly \( T \) is linear. To show that \( T_{\alpha} \overset{t_u}{\to} T \), let \( D \) be a bounded set in \( E \) and \( W \in W_F \). Since \( \{T_{\alpha}\} \) is \( t_u \)-Cauchy, there exists \( \alpha_0 \in I \) such that

\[
T_{\alpha}(x) - T_{\beta}(x) \in W \text{ for all } x \in D \text{ and } \alpha, \beta \geq \alpha_0.
\]
Since $W$ is closed, fixing $\alpha \geq \alpha_o$ and taking $\lim_\beta$, we have
\[ T_\alpha(x) - T(x) \in W \text{ for all } x \in D \text{ and } \alpha \geq \alpha_o. \]
Hence
\[ T_\alpha - T \in U(D, W) \text{ for all } \alpha \geq \alpha_o. \]
Therefore $T_\alpha \to T$.

We next show that $T$ is continuous. Since $E$ is ultrabornological, by definition, $CL(E, F) = BL(E, F)$ and so it suffices to show that $T$ is bounded (i.e. $T$ maps bounded sets into bounded sets). Let $D$ be any bounded set in $E$. To show that $T(D)$ is bounded in $F$, let $W \in W_F$. Choose a closed and balanced $V \in W_F$ such that $V + V \subseteq W$. Since $T_\alpha \to T$, there exists $\alpha_o \in I$ such that
\[ T_\alpha(x) - T(x) \in V \text{ for all } x \in D \text{ and } \alpha \geq \alpha_o. \]
Since $T_\alpha$, being continuous, is bounded, there exists $r \geq 1$ such that $T_\alpha(D) \subseteq rV$. Now, for any $x \in D$,
\[ T(x) = (T(x) - T_{\alpha_o}(x)) + T_{\alpha_o}(x) \in V + rV \subseteq r(V + V) \subseteq rW; \]
that is, $T(D) \subseteq rW$. Hence $T$ is bounded or, equivalently, $T$ is continuous. Thus $T \in CL(E, F)$, and so $(CL(E, F), t_u)$ is complete. □

Remark. In general, $(CL(E, F), t_p)$ need not be complete. In fact, completeness results in the other $G$-topologies are mostly of negative character. If $(E, \tau)$ is a locally convex space, then $(E^*, w(E^*, E))$ is complete iff $E$ is finite dimensional; hence, if $E$ is an infinite dimensional metrizable locally convex space, then $(E^*, w(E^*, E))$ is never complete. However, if $E$ is complete, then $(E^*, w(E^*, E))$ is quasi-complete.

Definition: Let $E$ and $F$ be TVSs. Then a subset $\mathcal{F}$ of $CL(E, F)$ is said to be
(i) equicontinuous if for each neighborhood $V$ of 0 in $F$, there exists a neighborhood $U$ of 0 in $E$ such that $T(U) \subseteq V$ for all $T \in \mathcal{F}$;
(ii) uniformly bounded if, each bounded set $B$ in $E$,
\[ \cup\{T(B) : T \in \mathcal{F}\} \text{ is a bounded set in } F; \]
(iii) pointwise bounded if, each $x \in E$,
\[ \{T(x) : T \in \mathcal{F}\} \text{ is a bounded set in } F. \]

Clearly, if $\mathcal{F} \subseteq CL(E, F)$ is uniformly bounded, then it is pointwise bounded. Further, we obtain:

Lemma A.3.6. ([Scha71], p. 83, 85) Let $E$ and $F$ be TVSs, and let $\mathcal{F} \subseteq CL(E, F)$ be equicontinuous. Then:
3. SPACES OF CONTINUOUS LINEAR MAPPINGS

(a) $\mathcal{F}$ is uniformly bounded (i.e. $t_u$-bounded).

(b) The topologies $t_{pc}$ and $t_p$ coincide on $\mathcal{F}$.

Proof. (a) Let $D$ be a bounded set in $E$ and $W$ a neighborhood of 0 in $F$. Since $\mathcal{F}$ is equicontinuous, there exists a neighborhood $G$ of 0 in $E$ such that $T(G) \subseteq W$ for all $T \in \mathcal{F}$.

Since $D$ is a bounded set in $E$, there exists $r > 0$ such that $D \subseteq rG$. Then $T(D) \subseteq rT(G) \subseteq rW$ for all $T \in \mathcal{F}$.

Thus $\bigcup \{T(D) : T \in \mathcal{F}\}$ is a bounded set in $F$.

(b) Since $t_{pc} \subseteq t_p$, we need to prove that $t_{pc} \subseteq t_p$ on $\mathcal{F}$. Let $T_0 \in \mathcal{F}$, and let $K$ be a precompact subset of $E$ and $W \in \mathcal{W}_F$. We show that there exists a finite set $D \subseteq K$ and $V \in \mathcal{W}_F$ such that

$$[T_0 + U(D, V)] \cap \mathcal{F} \subseteq T_0 + U(K, W).$$

Choose a balanced $V \in \mathcal{W}$ with $V + V + V \subseteq W$. Since $\mathcal{F}$ is equicontinuous at 0, there exists a balanced neighborhood $G$ of 0 in $E$ such that $T(G) \subseteq V$ for all $T \in \mathcal{F}$.

Now $K$ is precompact, there exists a finite set $D = \{y_i : 1 \leq i \leq m\} \subseteq K$ such that $K \subseteq D + G$.

Now, let $T \in [T_0 + U(D, V)] \cap \mathcal{F}$. Then $T = T_0 + S$, where $S \in U(D, V)$ \cap $\mathcal{F}$. By (2) and (3),

$$S(K) \subseteq S(D + G) \subseteq S(D) + S(G) \subseteq V + (T - T_0)(G) \subseteq V + V + V \subseteq W.$$

Thus $T = T_0 + S \in T_0 + U(K, W)$. This proves (1). □

**Theorem A.3.7.** (Principle of uniform boundedness) [RobW58] Let $(E, \tau)$ be an ultrabarrelled TVS and $F$ any TVS. Let $\mathcal{F} \subseteq CL(E, F)$ be a collection such that, for each $x \in E$, the set $\mathcal{F}(x) = \{T(x) : T \in \mathcal{F}\}$ is bounded in $F$. Then $\mathcal{F}$ is equicontinuous; hence, for any bounded set $D$ in $E$, $\bigcup \{T(D) : T \in \mathcal{F}\}$ is a bounded set in $F$.

Proof. ([Edw65], p. 465) Let $\mathcal{W}_E$ and $\mathcal{W}_F$ denote a base of closed neighborhoods of 0 in $E$ and $F$, respectively. (Let $V$ be a closed neighborhood of 0 in $F$. We show that there exists a neighborhood $V^*$ of 0 in $E$ such that $T(V^*) \subseteq V$ for all $T \in \mathcal{F}$.)

For each $V \in \mathcal{W}_F$, let $V^* = \cap \{T^{-1}(V) : T \in \mathcal{F}\}$. 


Then $V^*$ is $\tau$-closed and absorbing in $E$, as follows. Since the inverse image of a closed set under a continuous mapping is a closed set and also the intersection of any number of closed sets is closed, it follows that $V^*$ is $\tau$-closed in $E$. Since $F$ is pointwise bounded, for each $x \in E$, there exists $r_x \geq 1$ such that

$$T(x) \in r_x V \text{ for all } T \in F. \tag{2}$$

i.e. $x \in r_x (\cap \{T^{-1}(V) : T \in F\}) = r_x V^*$. This implies that $V^*$ is absorbing. Then the collection

$$U = \{V^* : V \in \mathcal{W}_F\}$$

may be taken as a base of neighborhoods of 0 for a vector topology $\tau^*$ on $E$. Since $(E, \tau)$ is ultrabarrelled, $\tau^* \leq \tau$ and so each $V^*$ is a neighborhood of 0 in $E$. Thus $T(V^*) \subseteq V$ for all $T \in F$, and so $F$ is equicontinuous.

**Theorem A.3.8.** (Banach-Steinhaus) Let $E$ be an ultrabarrelled TVS and $F$ is a quasi-complete TVS. Let $\{T_\alpha : \alpha \in I\}$ be a net in $CL(E, F)$ such that, for each $x \in E$,

$$T(x) = \lim_\alpha T_\alpha(x)$$

exists. Then $T$ is continuous on $E$ and $T_\alpha \to T$ uniformly on precompact subsets of $E$. Further, if the net $\{T_\alpha : \alpha \in I\}$ is replaced by a sequence $\{T_n\}$, it is enough that $F$ be sequentially complete.

**Proof.** (see [Edw65], p. 465) It is easy to verify that $T$ is linear. Since $\{T_\alpha : \alpha \in J\}$ is pointwise bounded, by the principle of uniform boundedness, $\{T_\alpha : \alpha \in J\}$ is equicontinuous. We now show that $T$ is continuous. Let $W$ be a closed neighborhood of 0 in $F$. Since $\{T_\alpha : \alpha \in J\}$ is equicontinuous, there exists a neighborhood $U$ of 0 in $E$ such that $T_\alpha(U) \subseteq W$ for all $\alpha \in J$. Since $W$ is closed, for each $x \in U$, $\lim_\alpha T_\alpha x \in W$; that is, for each $x \in U$, $Tx \in W$. So $T(U) \subseteq W$ and hence $T$ is continuous. Next, by Lemma A.3.6(b), the topologies $t_{pc}$ and $t_p$ coincide on $A$; hence $T_\alpha \to T$ uniformly on precompact subsets of $E$. □

We next state the open mapping and the closed graph theorems.

**Definition.** Let $E$ and $F$ be TVSs. A mapping $T : E \to F$ is called open if $T(U)$ is open in $F$ for every open set $U$ in $E$. $T$ is called open at $x \in E$ if $T(U)$ contains a neighborhood of $T(x)$ whenever $U$ is a neighborhood of $x$ in $E$. It is clear that $T$ is open iff $T$ is open at every point of $E$. Because of the invariance of vector topologies, it follows that a linear mapping $T : E \to F$ is open iff it is open at $0 \in E$. 


Theorem A.3.9. (Open mapping theorem) [RobW58] Let $E$ be an $F$-space and $F$ an ultrabarrelled TVS, and let $T : E \to F$ be a continuous and onto linear mapping. Then $T$ is an open mapping. If, in addition $T$ is one-one, then $T^{-1} : F \to E$ is also continuous.

Proof. See ([Edw65], p. 437; [Hus65, p. 53]).

As before, a mapping $T : E \to F$ is said to have a closed graph if its graph $G(T) = \{(x, Tx) : x \in E\}$ is closed in $E \times F$ or, equivalently, for any net $\{x_\alpha\}$ in $E$ with $x_\alpha \to x \in E$ and $T(x_\alpha) \to y \in F$, we have $y = T(x)$. Every continuous map $T : E \to F$ has a closed graph. The converse holds under some additional hypotheses in the following form.

Theorem A.3.10. (Closed graph theorem) [RobW58] Let $E$ be an ultrabarrelled TVS and $F$ an $F$-space. If $T : E \to F$ is linear and has closed graph, then $T$ is continuous.

Proof. See ([Edw65], p. 437; [Hus65], p. 53).
4. Shrinkable Neighborhoods in a TVS

In this section, we study the notion of a shrinkable neighborhood of 0 in a TVS $E$, as given by Klee [Kl60a, Kl60b].

**Definition.** A neighborhood $W$ of 0 in a TVS $E$ is said to be shrinkable if $tW \subseteq W^0$ for all $0 \leq t < 1$.

**Lemma A.4.1.** Any convex neighborhood $V$ of 0 in a TVS $E$ is shrinkable.

**Proof.** Let $\rho_V$ be the Minkowski functional of $V$. Then, since $\rho_V$ is continuous, we have

$$V^0 = \{x \in E : \rho_V(x) < 1\}, V = \{x \in E : \rho_V(x) \leq 1\}.$$

Let $y \in V$ and $0 \leq t < 1$. Now

$$1 - \rho_V(ty) \geq \rho(y) - t\rho_V(y) = (1 - t)\rho_V(y) > 0.$$

Hence $\rho_V(ty) < 1$ and so $ty \in V^0$. Thus $tV \subseteq V^0$. □

Important facts about shrinkable neighborhoods are:

(A) Every Hausdorff TVS $E$ has a base of shrinkable neighborhoods of 0.

(B) If $W$ is a shrinkable neighborhood of 0 in a TVS $E$, then its Minkowski functional $\rho_W$ is continuous and positively homogeneous.

The proofs of these important results, due to Klee [Kl60a], may not be found in the text books and therefore we include them here. We first need to prove the following three results.

**Theorem A.4.2.** Every Hausdorff TVS is linearly homeomorphic with a linear subspace of a product of metric linear spaces.

**Proof.** Let $E$ be a Hausdorff TVS and $W$ the class of all open balanced neighborhoods $W$ of 0 in $E$. For each $W \in W$, let $W_0, W_1, W_2, ...$ be a sequence of members of $W$ such that $W_o = W$, and $W_n + W_n \subseteq W_{n-1}$ for $n = 1, 2, ...$.

Let $M_W = \bigcap_{n=0}^{\infty} W_n$. Then clearly

$$M_W = [-1, 1]M_W \text{ and } M_W + M_W \subseteq M_W,$$

and so $M_W$ is a linear subspace of $E$. Let $\pi_W$ be the quotient map of $E$ onto $E/M_W$. Let $E/M_W$ be topologized by taking $\{\pi_W(W_n) : n \geq 1\}$ as a base of neighborhoods of 0. It is easily verified that $E/M_W$ is a TVS and $\pi_W$ is continuous. Further,

$$\pi_W^{-1}(\pi_W(W_n)) = W_n + M_W \subseteq W_n + W_n \subseteq W_{n-1}.$$
and so it follows that \( \bigcap_{n=0}^{\infty} W_n = \{ M_W \} \). Thus the topology of \( E/M_W \) is Hausdorff; further it is metrizable since it has a countable base of neighborhoods of 0. Let \( \mathcal{P} = \prod_{W \in \mathcal{W}} E/M_W \) (the product space), let \( \pi : E \to \mathcal{P} \) be defined by
\[
(\pi(x))_W = \pi_W(x), \quad x \in E, \quad W \in \mathcal{W}.
\]
Then \( \pi(E) \) is a linear subspace of \( \mathcal{P} \) and \( \pi \) is a linear homeomorphism. \( \Box \)

This next two lemmas are easy to verify.

**Lemma A.4.3.** Suppose \( E \) is a Hausdorff TVS, \( M \) a vector subspace of \( E \), and \( W \) a shrinkable neighborhood of 0 in \( E \). Then \( W \cap M \) is a shrinkable neighborhood of 0 in \( M \).

**Lemma A.4.4.** Suppose \( \{ E_\alpha : \alpha \in I \} \) is a family of Hausdorff TVSs, \( W_\alpha \) is a shrinkable neighborhood of 0 in \( E_\alpha \) for each \( \alpha \in I \), and \( W_\alpha = E_\alpha \) for all but finitely many \( \alpha \in I \). Then the set \( \prod_{\alpha \in I} W_\alpha \) is a shrinkable neighborhood of 0 in the product space \( \prod_{\alpha \in I} E_\alpha \).

**Theorem A.4.5.** In a Hausdorff TVS \( E \), every neighborhood of 0 contains a shrinkable neighborhood of 0.

**Proof.** In view of the above results, we may assume that \( E \) is a metrizable TVS. According to a classical result of Edelheit and Mazur (see [DS58], p.91), the topology of \( E \) can be generated by an invariant metric \( d \) which is strictly monotonic in the sense that
\[
d(tx,0) < d(sx,0) \quad \text{for} \quad 0 \leq t < s \quad \text{and} \quad x \neq 0.
\]
We need to show that, for any \( \varepsilon > 0 \), \( B(0,\varepsilon) = \{ x \in E : d(x,0) < \varepsilon \} \) is shrinkable. [Let \( y \in B(0,\varepsilon) \) and 0 \( \leq t < 1 \). To show that \( ty \in [B(0,\varepsilon)]^0 \), we show that there exists a \( \delta > 0 \) such that \( B(ty,\delta) \subseteq B(0,\varepsilon) \). Since \( t < 1 \), \( d(ty,0) < d(y,0) \) and so \( \delta = d(y,0) - d(ty,0) > 0 \). Let \( z \in B(ty,\delta) \). Then
\[
d(z,0) \leq d(z,ty) + d(ty,0) - \delta + d(ty,0) = d(y,0) < \varepsilon.
\]
Hence \( z \in B(ty,\delta) \subseteq B(0,\varepsilon) \). Thus \( ty \in [B(0,\varepsilon)]^0 \) and so \( B(0,\varepsilon) \) is shrinkable.] \( \Box \)

**Theorem A.4.6.** Let \( W \) be a balanced neighborhood of 0 in a Hausdorff TVS \( E \) and let \( \rho = \rho_W \) be the Minkowski functional of \( W \). Then:
(a) \( \rho \) is upper semicontinuous iff \( W^0 = \{ x \in E : \rho(x) < 1 \} \);
(b) \( \rho \) is lower semicontinuous iff \( \overline{W} = \{ x \in E : \rho(x) \leq 1 \} \).
(c) \( \rho \) is continuous iff \( W \) is shrinkable.
Proof. We first note that
\[ W_0 \subseteq \{ x \in E : \rho(x) < 1 \} \subseteq W \subseteq \{ x \in E : \rho(x) \leq 1 \} \subseteq \overline W. \] (1)

(a) From (1) it follows that \( W_0 = \{ x \in E : \rho(x) < 1 \} \) iff the latter set is open. Since \( \rho \) is positively homogeneous, openness of the set \( \{ x \in E : \rho(x) < 1 \} \) is equivalent to that of the set \( \{ x \in E : \rho(x) < t \} \) for all \( t > 0 \), and that in turn is equivalent to the upper semicontinuity of \( \rho \).

(b) Again by (1) we note that \( W = \{ x \in E : \rho(x) \leq 1 \} \) iff the latter set is closed. Since \( \rho \) is positively homogeneous, closedness of the set \( \{ x \in E : \rho(x) \leq 1 \} \) is equivalent to that of the set \( \{ x \in E : \rho(x) \leq r \} \) for all \( r > 0 \), and that is equivalent to the lower semicontinuity of \( \rho \).

(c) Suppose \( \rho \) is continuous. To show that \( W \) is shrinkable, let \( 0 \leq t < 1 \). Then
\[
y \in t \overline W = t \{ x : \rho(x) \leq 1 \} \Rightarrow y = tx, \rho(x) \leq 1 \Rightarrow \rho(y) = t \rho(x) < 1 \Rightarrow y \in \{ z \in E : \rho(z) < 1 \} = W_0,
\]
as required.

Conversely, suppose that \( W \) is shrinkable. We first show that \( \rho \) is lower semicontinuous or that \( A = \{ x \in E : \rho(x) < 1 \} \) is open. Let \( y \in A \). Let \( t = \max \{ \frac{1}{r}, \rho(y) \} < 1 \). Then
\[
\rho(\frac{1}{t} y) = \frac{1}{t} \rho(y) \leq 1, \text{ and so } y \in t \overline W \subseteq W_0 \subseteq A.
\]
Hence \( A \) is open. To establish upper semicontinuity of \( \rho \), we need to show that \( \{ x \in E : \rho(x) > 1 \} \) is open or that \( B = \{ x \in E : \rho(x) \leq 1 \} \) is closed. We claim that \( B = \overline W \). Clearly, \( B \subseteq \overline W \). Suppose \( \overline W \not\subseteq B \), and let \( z \in \overline W \setminus B \). Then \( r = \rho(z) > 1 \), and so \( \rho(\frac{1}{r^{1/2}} z) = (r)^{1/2} > 1 \). Now, since \( \frac{1}{r^{1/2}} < 1 \) and \( W \) is shrinkable,
\[
\frac{1}{r^{1/2}} z \in \frac{1}{r^{1/2}} \overline W \subseteq W_0 \subseteq \{ x \in E : \rho(x) < 1 \},
\]
and so \( \rho(\frac{1}{r^{1/2}} z) < 1 \), a contradiction. Hence \( B = \overline W \), and so \( B \) is closed. Thus \( \rho \) is both lower and upper semicontinuous, and hence it is continuous. \( \square \)

Theorem A.4.7. Let \( W \) be a closed shrinkable neighborhood of 0 in a TVS \( E \), \( \rho = \rho_W \) the Minkowski functional of \( W \), and \( h : E \to E \) is given by
\[
h(x) = \begin{cases} 
x & \text{if } x \in W, \\
\frac{x}{\rho(x)} & \text{if } x \in E \setminus W.
\end{cases}
\]
Then $h$ is continuous.

Proof. Let $A = W = \{x \in E : \rho(x) \leq 1\}$ and $B = \{x \in E : \rho(x) \geq 1\}$. Then $A$ and $B$ are closed with $E = A \cup B$. Define $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ by

$$f(x) = x, \quad x \in A,$$
$$g(x) = \frac{x}{\rho(x)}, \quad x \in B.$$

Hence, by the Map Gluing Theorem, $h$ is continuous on $E$. $\square$
5. Non-Archimedean Functional Analysis

In this section, we consider the notions of non-Archimedean TVSs, locally \( F \)-convex spaces and almost non-Archimedean TVSs.

**Definition.** Let \( F \) be any field with unit element \( e \). Then a function \( |·|: F \to \mathbb{R} \) is called a valuation on \( F \) if, for any \( a, b \in F \),

- (i) \( |a| \geq 0 \), with \( |a| = 0 \) iff \( a = 0 \);
- (ii) \( |ab| = |a||b| \);
- (iii) \( |a+b| \leq |a|+|b| \).

In this case, the pair \((F, |·|)\) is called a valued field. Clearly, \( |e| = 1 \).

**Definition.** A valued field \((F, |·|)\) is called a non-Archimedean field (in short, an NA field) if

\[
|a + b| \leq \max\{|a|, |b|\} \quad \text{for all } a, b \in F \quad \text{(Strong Triangular inequality).}
\]

(ST)

**Remark.** Recall that an important property of the field \( \mathbb{R} \) is that it has the Archimedean Property in the sense that, given any \( x > 0 \) in \( \mathbb{R} \), there exists an \( n \in \mathbb{N} \) such that \( x \leq n \); equivalently, \( \mathbb{N} \) is not bounded in \( \mathbb{R} \). The proof of this fact is based on the supremum (or completeness) axiom of \( \mathbb{R} \). However, if \((F, |·|)\) is an NA valued field, then, using (ST) inequality, for any integers \( n \in \mathbb{N} \),

\[
|ne| = |e + e + .... + e| \leq \max\{|e|, |e|, ..., |e|\} = |e| = 1;
\]

hence the set \( Ne \) is bounded in \((F, |·|)\). Thus \((F, |·|)\) does not have the Archimedean property. This justifies the use of the term non-Archimedean for such fields. The formula (ST) is essential to the entire non-Archimedean theory.

**Examples**

(1) Clearly, \( \mathbb{R} \) and \( \mathbb{C} \) with the usual absolute value are valued fields but not non-Archimedean.

(2) Any field \( F \) with the trivial valuation

\[
|a| = \begin{cases} 
1 & \text{if } a \neq 0 \\
0 & \text{if } a = 0 
\end{cases}
\]

is an NA field. All other valuations on \( F \) are called non-trivial valuations.

(3) Let \( F = \mathbb{Q} \), the field of rationals, and let \( p \) be any fixed prime number. Every \( x \in \mathbb{Q}, x \neq 0 \), can be written uniquely as \( x = p^k \cdot \frac{a}{b} \), where \( k \in \mathbb{Z} = \{0, \pm1, \pm2, \ldots\} \) and \( a, b \) cannot be divided by \( p \). Define \( x \to |x|_p \) by

\[
|x|_p = \begin{cases} 
p^{-k} & \text{if } x \neq 0, \\
0 & \text{if } x = 0.
\end{cases}
\]
Then \( | \cdot |_p \) is an NA valuation on \( \mathbb{Q} \), called the \( p \)-adic valuation on \( \mathbb{Q} \). Note that \( (\mathbb{Q}, | x |_p) \), being a metric space, has a completion, denoted by \( \mathbb{Q}_p \). Then \( \mathbb{Q}_p \) is called the field of \( p \)-adic numbers.

**Remark.** Let \( E \) be a vector space over a valued field \( (\mathbb{F}, | \cdot |) \). Then the definition of a TVS \( E \) over \( \mathbb{F} \) is identical with that of a TVS \( E \) over \( \mathbb{K} \). However, the result on the characterization of a base of neighborhoods of 0 with the desired properties needs some modification. In fact, in most cases, we require that the valued field \( (\mathbb{F}, | \cdot |) \) is non-trivial. The reason is that, for any real \( r > 0 \), we can choose \( a, b \in \mathbb{F} \) such that \( 0 < |a| \leq r \) and \( |b| \geq r \). [Choose \( c \in \mathbb{F} \) such that \( |c| \neq 0 \) and \( |c| \neq 1 \). Clearly, if \( 0 < |c| \leq r \), then \( |c^{-1}| \geq r \); if \( |c| \geq r \), then \( 0 < |c^{-1}| \leq r \).]

**Definition.** A subset \( S \) of vector space \( E \) over \( \mathbb{F} \) is called a non-Archimedean set (in short, an NA set) if \( S + S \subseteq S \). By induction, for any \( k \in \mathbb{N} \), the \( k \)-fold sum

\[
S + S + \ldots + S \subseteq S.
\]

A TVS \( (E, \tau) \) over a valued field \( (\mathbb{F}, | \cdot |) \) is called a non-Archimedean TVS (in short, an NA TVS) if it has a base of \( \tau \)neighborhoods of 0 consisting of NA sets.

**Examples.** (1). Every NA valued field \( (\mathbb{F}, | \cdot |) \) is an NA TVS over itself.

(2). Any indiscrete TVS is an NA TVS.

(3). The trivial TVS \( E = \{0\} \) is an NA TVS.

**Theorem**

**A.5.3.** Let \( (E, \tau) \) be a TVS over a valued field \( (\mathbb{F}, | \cdot |) \). If \( V \) is any NA neighborhood of 0 in \( E \), then \( V \) is clopen (i.e., both closed and open).

**Proof.** \( V \) is open, as follows. Let \( y \in V \). Then \( y + V \subseteq V + V \subseteq V \) (since \( V \) is NA), so \( y \) has a neighborhood \( y + V \) with \( y + V \subseteq V \). Hence \( V \) is open.

\( V \) is closed, as follows. We need to show that \( E \setminus V \) is open. Let \( x \in E \setminus V \). Then \( V \cap (x - V) = \emptyset \). [Suppose \( V \cap (x - V) \neq \emptyset \), and let \( z \in V \cap (x - V) \). Then \( z = y_1, z = x - y_2 \) with \( y_1, y_2 \in V \), and so \( y_1 = x - y_2 \). Hence

\[
x = y_1 + y_2 \in V + V \subseteq V,
\]

a contradiction since \( x \in E \setminus V \).] Therefore, \( x - V \subseteq E \setminus V \). Since \( x - V \) is a neighborhood of \( x \), \( E \setminus V \) is open. Thus \( V \) is closed. \( \square \)

Recall that a topological space \( X \) (\( \neq \emptyset \)) is said to have dimension 0 at a point \( x \in X \) if \( x \) has a local base \( B_x \) of neighborhood consisting of
sets which are both open and closed; $X$ is said to be 0-dimensional if $X$
has a dimension 0 at each point of $X$.

**Theorem A.5.4.** *Every NA TVS $(E, \tau)$ is a 0-dimensional topological space.*

*Proof.* Since $(E, \tau)$ is an NA TVS, it has a base $\mathcal{W}$ of $\tau$neighborhoods
of 0 consisting of NA sets. Then, by above theorem, each $V \in \mathcal{W}$ is both
open and closed. Hence, for each $x \in E$, $B_x = \{x + V : V \in \mathcal{W}\}$ consists
of both open and closed neighborhoods of $x$. $\square$

**Theorem**

**A.5.5.** *Let $(E, \tau)$ be an NA TVS over a non-trivially valued field
$(F, | \cdot |)$. Then there exist a base of neighborhoods of 0 consisting of NA
(hence clopen) 'balanced' sets.*
6. Topological Algebras and Modules

**Definition.** Let $A$ be an algebra over $\mathbb{K}$ (= $\mathbb{R}$ or $\mathbb{C}$) and $\tau$ a topology on $A$. Then the (algebra) multiplication $(x, y) \mapsto xy$ from $A \times A \to A$ is called:

(i) **separately continuous** if, for any $\tau$-neighborhood $W$ of 0 in $A$ and any $x \in A$, there exists a $\tau$-neighborhood $V$ of 0 in $A$ such that $xV \subseteq W$ and $Vx \subseteq W$.

(ii) **hypocontinuous** if, for any $\tau$-neighborhood $W$ of 0 in $A$ and any $\tau$-bounded set $D$ in $A$, there exists a $\tau$-neighborhood $V$ of 0 in $A$ such that $DV \subseteq W$ and $VD \subseteq W$.

(iii) **jointly continuous** if, for any $\tau$-neighborhood $W$ of 0 in $A$, there exists a $\tau$-neighborhood $U$ of 0 in $A$ such that $U^2 \subseteq W$, where $U^2 = \{xy : x, y \in U\}$.

Clearly, joint continuity $\Rightarrow$ hypocontinuity $\Rightarrow$ separate continuity.

**Definition:** Let $A$ be an algebra over $\mathbb{K}$ (= $\mathbb{R}$ or $\mathbb{C}$) and $\tau$ a topology on $A$ such that $(A, \tau)$ is a TVS. Then the pair $(A, \tau)$ is called a **topological algebra** if it has a separately continuous multiplication. A complete metrizable topological algebra is called an $F$-**algebra**.

Recall that a topological space $X$ is called a **Baire space** if whenever $X = \bigcup_{n=1}^{\infty} A_n$ with each $A_n$ closed, then, for at least one $m$, $\text{int}(A_m) \neq \emptyset$.

**Theorem A.6.1.** (Arens [Are47])

*Every Baire metrizable topological algebra $(A, \tau)$ has jointly continuous multiplication. In particular, every $F$-algebra $(A, \tau)$ has jointly continuous multiplication.*

**Proof.** ([Hus83], p. 5-6) Since $(A, \tau)$ is metrizable, there exists a metric $d$ on $A$ such that $\tau = \tau_d$. To show that the map $(x, y) \mapsto xy$ is jointly continuous at $(0, 0) \in A \times A$, let $W$ be any $\tau_d$-neighborhood of 0 in $A$. We shall find a $\tau_d$-neighborhood $U$ of 0 in $A$ such that $U^2 \subseteq W$.

For each $n \geq 1$, let $B(0, \frac{1}{n}) = \{x \in A : d(x, 0) < \frac{1}{n}\}$. Then $\{B(0, \frac{1}{n})\}$ is a countable base of $\tau_d$-neighborhoods of 0. Choose a closed balanced $\tau$-neighborhood $V$ of 0 in $A$ such that $V + V \subseteq W$. Let $A_n = \{x \in A : xB(0, \frac{1}{n}) \subseteq V\}$. Clearly $A_nB(0, \frac{1}{n}) \subseteq V$. We now show that (i) each $A_n$ is closed; (ii) $A = \bigcup_{n=1}^{\infty} A_n$.

(i) $A_n$ is closed. [Let $x \in \overline{A_n}$. Then there exists $\{x_k\} \subseteq A_n$ such that $x_k \xrightarrow{\tau_d} x$. Then $x_kB(0, \frac{1}{n}) \subseteq V$ for all $k \geq 1$. By separate continuity of multiplication, for any $y \in B(0, \frac{1}{n})$, $x_ky \xrightarrow{\tau_d} xy \in \overline{V}$. Since $V$ is closed, $\overline{V} = V$, and so $xy \in V$. Therefore $xB(0, \frac{1}{n}) \subseteq V$, and so $x \in A_n$. Hence $A_n$ is closed.]
(ii) \( A = \bigcup_{n=1}^{\infty} A_n \). [Clearly \( \bigcup_{n=1}^{\infty} A_n \subseteq A \). To show \( A \subseteq \bigcup_{n=1}^{\infty} A_n \), let \( x \in A \). Since \( \{B(0, \frac{1}{n})\} \) is a base of \( \tau_d \)-neighborhoods of 0, by separate continuity of \( (x, y) \rightarrow xy \) at \( (x, 0) \), there exists a \( B(0, \frac{1}{N}) \in \{B(0, \frac{1}{n})\} \) such that \( xB(0, \frac{1}{N}) \subseteq V \). Hence \( x \in A_N \subseteq \bigcup_{n=1}^{\infty} A_n \). This proves that \( A \subseteq \bigcup_{n=1}^{\infty} A_n \).]

Since \( A \) is a Baire space and \( A = \bigcup_{n=1}^{\infty} A_n \) with each \( A_n \) closed, by the Baire Category theorem, \( \text{int}(A_m) \neq \emptyset \) for some \( m \geq 1 \). Let \( x_o \in \text{int}(A_m) \). Then there exists \( \varepsilon > 0 \) such that \( B(x_o, \varepsilon) \subseteq A_m \). (1)

Now \( U = B(0, \varepsilon) \cap B(0, \frac{1}{m}) \) is a \( \tau_d \)-neighborhood of 0 in \( A \). We now show that \( U \subseteq W \). Clearly, \( U \subseteq B(0, \varepsilon)B(0, \frac{1}{m}) \). We need to show that \( B(0, \varepsilon)B(0, \frac{1}{m}) \subseteq W \). [For any \( x \in B(0, \varepsilon) \), \( d(x + x_o, x_o) = d(x, 0) < \varepsilon \) and so, by (1),

\[
x + x_o \in B(x_o, \varepsilon) \subseteq A_m;
\]

hence, by definition of \( A_m \),

\[
xB(0, \frac{1}{m}) = [(x + x_o) - x_o]B(0, \frac{1}{m}) \subseteq (x + x_o)B(0, \frac{1}{m}) - x_oB(0, \frac{1}{m}) \subseteq V - V \subseteq W.
\]

So \( B(0, \varepsilon)B(0, \frac{1}{m}) \subseteq W \).] Thus \( U \subseteq W \). \( \square \)

**Note:** (1) In general, separate continuity of multiplication need not imply joint continuity even in algebras which are normed spaces but not complete (see [Rud91], p. 247, 272).

(2) Every ultrabarrelled topological algebra has hypocontinuous multiplication (see Theorem A.6.28 below).

(3) [Fra05, p. 8] If \( (A, \tau) \) is a topological algebra, then its completion \( \hat{A} \) (i.e. the completion of the underlying TVS \( (A, \tau) \)) need not be topological algebra unless the multiplication of \( (A, \tau) \) is jointly continuous.

(4) In some books/papers, a *topological algebra* is assumed to have jointly continuous multiplication; but we shall not use this notion here.

**Definition.** A subset \( U \) of an algebra \( A \) is called *idempotent* (or multiplicative or an *m*-set) if \( U^2 \subseteq U \).

**Definition.** A topological algebra \( (A, \tau) \) is called:

(i) a *locally convex algebra* if it has a base \( \mathcal{W} = \mathcal{W}_A \) of neighborhoods of 0 consisting of convex sets; or equivalently, its topology \( \tau \) is generated
by a family $s_A$ of seminorms such that, for each $p \in s_A$, there exists a $p_1 \in s_A$ such that

$$p(xy) \leq p_1(x)p_1(y)$$

for all $x, y \in A$;

(ii) a locally $m$-convex algebra if it has a base $\mathcal{W}$ of neighborhoods of 0 consisting of convex and idempotent sets (i.e. the sets $U$ such that $U^2 \subseteq U$); or equivalently, if its topology $\tau$ is generated by a family $s_A$ of seminorms which are submultiplicative, i.e. for each $p \in s_A$,

$$p(xy) \leq p(x)p(y)$$

for all $x, y \in A$;

(iii) a locally idempotent algebra if it has a base $\mathcal{W}$ of neighborhoods of 0 consisting of idempotent sets;

(iv) a locally bounded algebra if it has a bounded neighborhood of 0.

**Examples.**

(1) Every normed algebra and, more generally, every locally $m$-convex algebra is locally convex.

(2) Every locally $m$-convex algebra is locally idempotent and every locally idempotent algebra is a topological algebra with jointly continuous multiplication.

(3) The notion of locally idempotent algebras is a strict generalization of the notion of locally $m$-convex algebras. This is shown by the algebra $\ell^q_+ (0 \leq q < 1)$ consisting of all two-sided complex sequences $a = \{a_n\}$ such that $\|a\|_p = \sum_{n=1}^{\infty} |a_n|^p$ is convergent for each $p$ with $q < p \leq 1$ (see [Zel60], p.348).

**Theorem A.6.2.** ([Zel73], p. 32) Let $(A,d)$ be an $F$-algebra with identity $e$. Then the following conditions are equivalent

(a) There exists an equivalent metric $\rho$ on $A$ such that

$$\rho(xy,0) \leq \rho(x,0)\rho(y,0)$$

for all $x, y \in A$.

(b) $A$ is a locally bounded algebra.

(c) The topology of $A$ can be generated by a submultiplicative $k$-norm $\| \cdot \|_k$, $0 < k \leq 1$, with $\|e\|_k = 1$.

**Note.**

(1) If $A$ is a locally $m$-convex algebra with identity $e$, then its topology can be given by a family $\{p_\alpha : \alpha \in I\}$ of submultiplicative seminorms such that $p_\alpha(e) = 1$ for all $\alpha \in I$ ([Fra05]. p. 16).

(2) If $A$ is a locally convex algebra with identity $e$, we do not know whether its topology can be given by a family $\{p_\alpha : \alpha \in I\}$ of seminorms which satisfy $p_\alpha(e) = 1$ for all $\alpha \in I$.

**Definition:** Let $A$ be an algebra. A map $* : x \to x^*$ from $A$ into $A$ is called an involution on $A$ if it satisfies:
(i) \((x + y)^* = x^* + y^*,\)
(ii) \(x^{**} = x,\)
(iii) \((\lambda x)^* = \overline{\lambda} x^*, \) \(\overline{\lambda}\) is the complex conjugate of \(\lambda \in \mathbb{K},\)
(iv) \((xy)^* = y^* x^*,\)
(v) \(e^* = e\) if \(A\) has identity \(e.\)

An algebra \(A\) equipped with an involution \(\ast\) is called an involutive algebra or a \(\ast\)-algebra.

Definition. A normed algebra \((A, \|\cdot\|)\) is called:
(1) a normed \(\ast\)-algebra if \(A\) has an involution such that \(\|x^*\| = \|x\|\) for all \(x \in A;\) in this case, clearly the involution map \(x \mapsto x^\ast\) is continuous.
(2) a \(B^\ast\)-algebra (or equivalently, a \(C^\ast\)-algebra) if \(A\) is a Banach \(\ast\)-algebra whose norm \(\|\cdot\|\) satisfies
\[
\|xx^\ast\| = \|x^\ast x\| = \|x\|^2 \text{ for all } x \in A;
\]
In this case, the norm \(\|\cdot\|\) is called a \(C^\ast\)-norm.

Definition. A complete locally \(m\)-convex \(\ast\)-algebra \((A, \tau)\) is called a locally \(C^\ast\)-algebra if its topology \(\tau\) is determined by a family \(\{p_\alpha : \alpha \in I\}\) of submultiplicative seminorms satisfying
\[
p_\alpha(x^\ast x) = [p_\alpha(x)]^2 \text{ for all } x \in A.
\]
The seminorms \(p_\alpha\) are called \(C^\ast\)-seminorms.

Example. (Arens Algebra)

([Are46]; [Mal86], p. 12)

Let
\[
L^\omega[0, 1] = \bigcap_{p=1}^\infty L^p[0, 1],
\]
where each \(L^p[0, 1], p \geq 1,\) is a normed space with respect to the norm:
\[
\|f\|_p = (\int_0^1 |f(t)|^p \, dt)^{1/p}, \quad f \in L^p[0, 1].
\]
Then:
(a) \(L^\omega[0, 1]\) is an algebra and \(\|f\|_1 \leq \|f\|_2 \leq \ldots \forall f \in L^\omega[0, 1].\) Note that: \(\|fg\|_p \leq \|f\|_q \|g\|_r\) for \(1/p = 1/q + 1/r;\) hence \(L^\omega[0, 1]\) is non-normed algebra.
(b) For any \(p \geq 1\) and \(\varepsilon > 0,\) let \(B_p(0, \varepsilon) = \{f \in L^\omega[0, 1] : \|f\|_p < \varepsilon\}.\) Then the collection \(\{B_p(0, \varepsilon) : p \in \mathbb{N}, \varepsilon > 0\}\) forms a base of neighborhoods of 0 for a Hausdorff locally convex topology \(\tau_\omega\) on \(L^\omega[0, 1].\)
(c) \((L^\omega[0,1], \tau_\omega)\) is a complete metric space w.r.t. the metric:

\[
d_\omega(f, g) = \sum_{p=1}^{\infty} 2^{-p} \frac{||f - g||_p}{1 + ||f - g||_p}, \quad f, g \in L^\omega[0,1].
\]

(d) \((L^\omega[0,1], \tau_\omega)\) is not locally \(m\)-convex.

**Definition:** Let \(A\) be an algebra with identity \(e\) and let \(x \in A\). Then \(x\) is said to be invertible (or regular) if there exists some \(y \in A\), called the inverse of \(x\), such that \(xy = yx = e\); in this case, we write \(y = x^{-1}\).

**Definition:** Let \(A\) be an algebra. For any \(x, y \in A\), the \(q\)-product \(x \circ y\) of \(x, y\) is defined by

\[
x \circ y = x + y - xy.
\]

An element \(x \in A\) is said to be \(q\)-invertible if there exists some \(y \in A\), called the \(q\)-inverse of \(x\) such that \(x \circ y = y \circ x = 0\); in this case, we write \(y = x^{-1}\).

**Definition:** Let \(A\) be an algebra, and let \(x \in A\).

1. If \(A\) has an identity \(e\), then the spectrum of \(x\) is defined by

\[
\sigma_A(x) = \{\lambda \in \mathbb{C} : \lambda e - x \text{ is not invertible in } A\}.
\]

2. If \(A\) is without identity, then the spectrum of \(x\) is defined as

\[
\sigma_A(x) = \{\lambda \in \mathbb{C} : \lambda \neq 0, \frac{x}{\lambda} \text{ is not } q\text{-invertible in } A\} \cup \{0\}.
\]

3. The spectral radius of \(x\) is defined as

\[
r_A(x) = \sup\{|\lambda| : \lambda \in \sigma_A(x)\}.
\]

Clearly, \(0 \leq r_A(x) \leq \infty\), \(r_A(0) = 0\), \(r_A(ax) = |a|r_A(x)\) (\(a \in \mathbb{C}\)), and \(r_A(e) = 1\) if \(A\) has an identity \(e\).

**Example.** Let \(X\) be a locally compact Hausdorff space.

1. If \(X\) is compact, for any \(f \in C_0(X) = C(X)\),

\[
\sigma_{C(X)}(f) = \{f(x) : x \in X\} = f(X).
\]

2. If \(X\) is non-compact, then

\[
\sigma_{C^*_0(X)}(f) = f(X) \cup \{0\} \text{ for all } f \in C_0(X);
\]

\[
\sigma_{C(X)}(f) = f(X) \subseteq f(\beta X) = f(\overline{X}) = \sigma_{C^*_0(X)}(f) \text{ for all } f \in C_0(X),
\]

where \(\beta X\) is the Stone-Cech compactification of \(X\).
Definition. A topological algebra $A$ with identity (resp. without identity) is said to be a $Q$-algebra if the set $G(A)$ of all invertible elements of $A$ (resp. $G^q(A)$ of all quasi-invertible elements of $A$) is open in $A$.

Examples. (1) Every complete locally bounded Hausdorff topological algebra (hence every Banach algebra) is a $Q$-algebra ([Mal86], p.42).

(2). An incomplete normed algebra need not be a $Q$-algebra, since neither $G(A)$ nor $G^q(A)$ need be an open set.

(3). A complete metrizable locally $m$-convex algebra need not be a $Q$-algebra (e.g. $(C(R), k)$, with $k$ the compact-open topology, is a complete metrizable locally $m$-convex algebra but not a $Q$-algebra ([Hus83], p. 8).

Theorem A.6.3. ([Mal86], p. 43-44, 59) Let $A$ be a topological algebra. Then the following are equivalent:

(a) $A$ is a $Q$-algebra, i.e. $G^q(A)$ is open in $A$.
(b) $[G^q(A)]^0 \neq \emptyset$.
(c) $G^q(A)$ is a neighborhood of 0 in $A$.
(d) $S(A) = \{x \in A : r_A(x) \leq 1\}$ is a neighborhood of 0 in $A$.

While considering the spectrum and related notions in the sequel, we shall consider all algebras over the complex field $C$.

Definition: A topological algebra $A$ with identity (resp. without identity) is said to have a continuous inversion (resp. continuous quasi-inversion) if the inversion map $x \rightarrow x^{-1}$ from $G(A) \rightarrow G(A)$ (resp. the quasi-inversion map $x \rightarrow x_{-1}$ from $G^q(A) \rightarrow G^q(A)$) is continuous.

In the sequel, for convenience, we shall consider mostly the results for the more general notion of ”quasi-inversion”.

Theorem A.6.4. (a) ([Mal86], p. 52) Every locally $m$-convex algebra (in particular, every normed algebra) $A$ has a continuous quasi-inversion. Conversely:

(b) ([Pa78], p. 277) Let $A$ be a commutative locally convex $F$-algebra with a continuous quasi-inversion. Then $A$ is a locally $m$-convex algebra.

Example. [Are46] The Arens algebra $L^\infty[0,1] = \cap_{p \geq 1} L_p[0,1]$ is a locally convex $F$-algebra, but not a locally $m$-convex algebra and it does not have a continuous inversion.

Definition: ([Mal86], p. 85; [Hus83], p.10) Let $(A, \tau)$ be a locally $m$-convex algebra whose topology $\tau$ is generated by a family $s_A = \{p : p \in s_A\}$ of submultiplicative seminorms. For each seminorm $p \in s_A$, $N_p = p^{-1}(0)$ is a two-sided ideal of $A$ and so $A/N_p$ is a normed algebra with the norm: $||\pi_p(x)||_p = p(x)$, where $\pi_p : A \rightarrow A/N_p$ is the canonical
map given by
\[ \pi_p(x) = x_p = x + N_p, \ x \in A. \]
Let \( A_p = \overline{A/N_p} \) denote the completion of the normed algebra \( A/N_p \). Then \( A_p \) is a Banach algebra and each \( \pi_p \) can be regarded as the map from \( A \) to \( A_p \). Each \( \pi_p \) is a continuous homomorphism.

We define the partial ordering \( \leq \) on \( s_A \) by
\[ p, q \in s_A, \ p \leq q \iff p(x) \leq q(x) \text{ for all } x \in A. \]
Then, for \( p \leq q \), we have \( N_q \subseteq N_p \) and so there exists an algebra homomorphism \( \varphi_{pq} : A/N_q \to A/N_p \) which has the unique extension to \( \tilde{\varphi}_{pq} : A \to A_p \) for \( p \leq q \), clearly, each \( \varphi_{pq} \) is a continuous homomorphism. Let \( \lim_{\leftarrow} A/N_p \) (resp. \( \lim_{\leftarrow} A_p \)) be the projective limit of \( A/N_p \)’s (resp. \( A_p \)’s).

**Theorem A.6.5.** ([Mal86], p. 88; [Mi52], p. 13; [Zel73], p.33) Let \( (A, \tau) \) be a locally m-convex algebra whose topology \( \tau \) is generated by a family \( s_A = \{ p : p \in s_A \} \) of submultiplicative seminorms. Then
\[ A \subseteq \lim_{\leftarrow} A/N_p \subseteq \lim_{\leftarrow} A_p \subseteq \bar{A}, \]
within topological algebraic isomorphism. If, in addition, \( A \) is complete, then
\[ A = \lim_{\leftarrow} A/N_p = \lim_{\leftarrow} A_p, \]
within topological algebraic isomorphism.

**Proof.** With the notations in the above definition, we can define \( \varphi : A \to \lim_{\leftarrow} A/N_p \) by \( \varphi(x) = (\pi_p(x))_p \), i.e. the \( p \)th projection of \( \varphi(x) \) equals \( \pi_p(x) \), \( x \in A \). It is clear that \( \varphi(x) = 0 \) iff \( \pi_p(x) = 0 \) for all \( p \in s_A \) iff \( p(x) = 0 \) for all \( p \in s_A \) iff \( x = 0 \), because \( A \) is Hausdorff. Hence \( \varphi \) is a one-one map. Further, since the topology \( \tau \) of \( A \) is identical with the Cartesian product topology of its image \( \varphi(A) \) in \( \prod_{p \in \mathcal{A}} A/N_p \), it follows that \( \varphi \) is a homeomorphism of \( (A, \tau) \) onto the subalgebra \( \varphi(A) \) of \( \prod_{p \in \mathcal{A}} A/N_p \) and so \( A \) is isomorphic and homeomorphic to a subalgebra of a Cartesian
product of a family of normed algebras. Hence $A$ can be identified with a subalgebra of $\Pi_{p \in s_A} A/N_p$, the product of normed algebras. □

If $A$ is metrizable, then the indexing set $s_A$ is countable and so each metrizable locally m-convex algebra is identifiable with a subalgebra of a countable product of normed algebras.

**Theorem A.6.6.** ([Hus83], p. 11; [Zel73], p. 33) Let $(A, \tau)$ be a locally m-convex F-algebra. Then $A$ is isomorphic and homeomorphic to a closed subalgebra of Cartesian product of Banach algebras.

**Theorem A.6.7.** ([Mal86], p. 85; [Hus83], p. 10) Let $(A, \tau)$ be a complete locally m-convex algebra whose topology $\tau$ is generated by a family $s_A$ of submultiplicative seminorms. Then:

(a) $\sigma_A(x) = \cup_{p \in s_A} \sigma_{A_p}(x_p)$, $x \in A$.

(b) $r_A(x) = \sup_{p \in s_A} r_{A_p}(x_p) = \sup_{p \in s_A} \lim_{n \to \infty} [p(x^n)]^{1/n}$, $x \in A$.

In the sequel, most of the results stated for locally m-convex algebras actually hold for the more general class of locally convex spaces having continuous inversion/ quasi-inversion (see [Mal86]).

**Theorem A.6.8.** (1) ([Mal86], p. 58; [Fra05], p. 46) If $A$ is a locally m-convex algebra, then, for each $x \in A$, $\sigma_A(x)$ is a non-empty (possibly non-compact) subset of $\mathbb{C}$; hence the spectral radius $r_A(x)$ is finite if exists.

(2) (∗)

([Mal86], p. 60; [Fra05], p. 74) If $A$ is a Q-algebra, then, for each $x \in A$, $\sigma_A(x)$ is a (possibly empty) compact subset of $\mathbb{C}$; hence the spectral radius $r_A(x)$ is finite if exists.

(3) If $A$ is a locally m-convex Q-algebra, then, for each $x \in A$, $\sigma_A(x)$ is a non-empty compact subset of $\mathbb{C}$; hence $r_A(x)$ is finite and exists.

**Definition A.6.9.** Let $A$ be an algebra. A subalgebra $I$ of $A$ is called a left ideal (resp. right ideal) if $AI \subseteq I$ (resp. $IA \subseteq I$); it is called a two-sided ideal (or simply, an ideal) if it is both a left and right ideal. A left (resp. right, two-sided) $I$ in $A$ is called a proper ideal if $I \neq \{0\}$ and $I \neq A$. A proper left, right or two-sided ideal $M$ in $A$ is called a maximal left, right or two-sided ideal if, for any ideal (left, right or two-sided, respectively) $L$, $M \subseteq L \subseteq A$ implies that $L = M$ or $L = A$.

**Definition.** ([Zel73], p. 27-28; [BD73], p. 45) Let $A$ be an algebra. A left (resp. right) ideal $I$ in $A$ is called a modular left (resp. modular
right) ideal if there exists an element $u = u_I'$ (resp. $u = u_I'$) in $A$, called a modular right (resp. modular left) identity for $A$, such that

$$xu - x \in I \quad \text{(resp. } xu - x \in I)$$

for all $x \in A$.

A two-sided ideal $I$ in $A$ is called a modular ideal if there exists an element $u = u_I \in A$, called a modular identity for $A$, such that

$$ux - x \in I, \quad xu - x \in I$$

for all $x \in A$.

Remarks. (1) ([Zel73], p. 27-28) A two-sided ideal $I$ in $A$ is modular iff $A/I$ has an identity (the identity of $A/I$ is the coset containing $u_I$). By Zorn’s lemma, each proper modular ideal is contained in a maximal modular ideal. Clearly, if an algebra $A$ has already an identity $e$, then every two-sided ideal in $A$ is a modular ideal with $e$ a modular identity for $A$; hence in this case, every ideal in $A$ is modular.

(2) If $A$ be a topological algebra, then the closure of every modular left (resp. right, two-sided) modular ideal in $A$ is a modular left (resp. right, two-sided) ideal.

Theorem

A.6.10. ([Mal86], p. 67; [Fra05], p. 74-75) Every maximal modular left, right or two-sided ideal in a $Q$-algebra $A$ is closed. In particular, every maximal ideal in a $Q$-algebra $A$ with identity in $A$ is closed.

Proof. (see [Mal86], p. 67) We only prove for $M$ a maximal modular two-sided ideal in $A$ with modular identity $u$ for $M$. By definition $M$ is proper and $u \notin M$. Suppose $M$ is not closed then $\overline{M} \neq M$, and so, by maximality of $M$, $\overline{M} = A$. Since $A$ is a $Q$-algebra, $G^q(A)$ is open and so $u + G^q(A)$ is an open neighborhood of $u$. Since $u \in A = \overline{M}$,

$$(u + G^q(A)) \cap M \neq \emptyset.$$ 

Let $z \in (u + G^q(A)) \cap M$. Then there exist some $w \in G^q(A), m \in M$ such that

$$z = (u + w) = m, \quad \text{or } u - m = -w \in G^q(A).$$

Hence $u - m$ has a $q$-inverse $y$ (say) in $A$. Then

$$(u-m) \circ y = 0, \quad \text{or } u - m + y - (u - m)y = 0, \quad \text{or } u - m + y - uy + my = 0,$$

or

$$u = m + (uy - y) - my \in M + M + M = M.$$ 

This is a contradiction since $u \notin M$. Hence $M = \overline{M}$, and so $M$ is closed.

If $A$ has an identity $e$, then each maximal ideal $M$ in $A$ is modular with modular identity $e$. Hence, by above argument, $M$ is closed. □
Note. ([Mal86], p. 67) In general, a maximal non-modular ideal in a locally $m$-convex algebra $A$ may be dense in $A$ and so need not be closed.

Definition: (a) If $A$ and $B$ are two algebras, then a linear mapping $\varphi : A \to B$ is called an algebra homomorphism if

$$\varphi(xy) = \varphi(x)\varphi(y)$$

for all $x, y \in A$.

In particular, a non-zero algebra homomorphism $\varphi : A \to \mathbb{K}$ is called a multiplicative linear functional or a character on $A$.

Note. (1) If $\varphi$ is a multiplicative linear functional on an algebra $A$ with identity $e$, then $\varphi$ is non-zero iff $\varphi(e) = 1$. [Suppose $\varphi$ is non-zero. Clearly, $e^2 = e$ implies $[\varphi(e)]^2 = \varphi(e)$ and so either $\varphi(e) = 0$ or $\varphi(e) = 1$. If $\varphi(e) = 0$, then, for any $x \in A$, $\varphi(x) = \varphi(xe) = \varphi(x)\varphi(e) = 0$; hence $\varphi$ is identically zero, a contradiction.]

(2) ([Mal86], p. 68) Any multiplicative linear functional $\varphi$ on an algebra is onto. [Let $\lambda \in \mathbb{C}$. Choose $x \in A$ with $\varphi(x) = \alpha \neq 0$. Then $y = \frac{1}{\alpha}x \in A$ with $\varphi(y) = \lambda$.]

Recall that the co-dimension of a vector subspace $M$ of a vector space $E$ is defined as the dimension of its complementary subspace (or equivalently, the dimension of the quotient space $E/M$). Further, a vector subspace $M$ of a vector space $E$ is maximal iff $M$ is of co-dimension one iff $M = \varphi^{-1}(0)$ for some non-zero linear functional $\varphi$ on $E$.

Theorem A.6.11. Let $A$ be an algebra. Then:

(a) If $\varphi$ is a multiplicative linear functional on $A$, then $M = \varphi^{-1}(0)$ is a maximal modular ideal of co-dimension one in $A$.

(b) If $M \subseteq A$ is a maximal modular ideal of co-dimension one, then there exists a unique multiplicative linear functional $\varphi$ on $A$ such that $M = \varphi^{-1}(0)$. (This need not hold if $M$ is not of co-dimension one.)

(c) There is a one-one correspondence between the set $\Delta(A)$ of multiplicative linear functional on $A$ and the set $\mathcal{M}(A)$ of all maximal modular ideals of co-dimension one in $A$ determined by parts (a) and (b).

Proof. We give the proof only for the case of $A$ an algebra with identity $e$.

(a) If $\varphi$ is a nonzero multiplicative linear functional, then it is easy to see that $M = \varphi^{-1}(0)$ is a maximal ideal. To see that $M$ is of co-dimension (i.e. $A/\varphi^{-1}(0)$ is one-dimensional), let $x \in A$. Then

$$\varphi(x - \frac{\varphi(x)}{\varphi(e)}e) = \varphi(x) - \varphi(x) = 0,$$

or that $x - \frac{\varphi(x)}{\varphi(e)}e \in \varphi^{-1}(0) = M$. 

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hence

\[ x \in M + \frac{\varphi(x)}{\varphi(e)} e \subseteq M \oplus Ce. \]

Therefore, \( A = M \oplus Ce \) (direct sum). Hence \( M = \varphi^{-1}(0) \) is a maximal ideal of codimension one.

(b) Let \( M \) be a maximal ideal of \( A \) of codimension one. Then \( e \notin M \) because otherwise \( M = A \), which contradicts the fact that \( M \) is a proper subset of \( A \) by definition. Hence \( A = M \oplus Ce \). Thus, for each \( x \in A \), there is a unique \( \lambda \in \mathbb{C} \) and \( m \in M \) with \( x = m + \lambda e \). Now define \( \varphi : A \to \mathbb{C} \) by

\[ \varphi(x) = \lambda \text{ for } x = m + \lambda e \in M \oplus Ce = A. \]

It is easy to verify that \( \varphi \) is a multiplicative linear functional with \( M = \varphi^{-1}(0) \). Further, \( \varphi(e) = 1 \), and so \( \varphi \) is nonzero.

(c). This follows from (a) and (b).

\[ \square \]

\textbf{Note.} (1) If \( M \) is a maximal ideal of an algebra \( A \), then \( M \) need not be of co-dimension one; in fact, \( M \) may be of infinite co-dimension ([Zel73], p. 37, Exercise 8.4.a.).

(2) However, if \( A \) is a complex Banach algebra, every maximal modular ideal \( M \) of \( A \) is automatically closed and of co-dimension one ([BD73], p. 79); also then the corresponding multiplicative linear functional \( \varphi \) is automatically continuous. Consequently, in this case, an analogue of the above theorem holds for commutative Banach algebras without mentioning the condition of co-dimension one.

We now state two results on the automatic continuity of multiplicative linear functionals.

\textbf{Theorem A.6.12.} ([BD73], p. 77; [Zel73], p. 37)

\begin{itemize}
  \item[(i)] \( \varphi \) is continuous on \( A \) and \( ||\varphi|| \leq 1 \).
  \item[(ii)] If \( A \) has an identity \( e \) with \( ||e|| = 1 \), then \( ||\varphi|| = 1 \).
\end{itemize}

\textbf{Theorem A.6.13.} ([Mal86], p. 72-73)

If \( A \) is a topological algebra with the property that every maximal modular two-sided ideal in \( A \) is closed, then every multiplicative linear functional \( \varphi \) on \( A \) is continuous. In particular, each multiplicative linear functional \( \varphi \) on a \( Q \)-algebra \( A \) is continuous.

\textbf{Remark.}
A multiplicative linear functional $\varphi$ need not be continuous on a non-complete complex normed algebra $A$.

**Open Problem.** As noted above, every multiplicative linear functional $\varphi$ on a Banach algebra (or, more generally, on a $Q$-algebra) is continuous. In general, a multiplicative linear functional on a locally $m$-convex algebra $A$ need not be continuous. It is still an open problem (known as the Michael-Mazur problem): whether every multiplicative linear functional on a commutative locally $m$-convex $F$-algebra $A$ is continuous (see ([Mic52, Hus83, Dal00, Fra05])).

**Notation.** The set of all multiplicative linear functionals on an algebra $A$ is denoted by $\Delta(A)$. Clearly, $\Delta(A) \subseteq A'$, the algebraic dual of $A$, but $\Delta(A)$ need not be a vector subspace of $A'$. If $A$ is a topological algebra, let $\Delta_c(A)$ denote the set of all (non-zero) continuous multiplicative linear functionals on $A$. In general, $\Delta(A)$ and $\Delta_c(A)$ may be empty. For example, if $A$ is the Arens algebra $L^\omega[0,1]$, then $\Delta(A) = \emptyset$ (see [Zel73], p. 65). However, we have:

**Theorem A.6.14.** ([Zel73], p. 38, 65; [Fra05], p. 55) If $A$ is a commutative complete locally $m$-convex algebra (in particular, a commutative Banach algebra) with identity, then $\Delta_c(A) \neq \emptyset$.

**Theorem A.6.15.** Let $A$ be a commutative locally $m$-convex algebra. Then there is a one-one correspondence between the set $\Delta_c(A)$ and the set $M_c(A)$ of all closed maximal modular ideals of co-dimension one in $A$ given by $\varphi \leftrightarrow M$, where $M = \varphi^{-1}(0)$.

**Definition.** Let $A$ be an algebra. The intersection of all maximal modular left ideals of $A$ is called the radical (or Jacobson radical) of $A$ and is denoted by $\text{rad}(A)$.

**Theorem A.6.16.** ([Fra05], p. 60-61) Let $A$ be an algebra. Then:

(a) $\text{rad}(A)$ is an ideal.

(b) $\text{rad}(A) \subseteq \{x \in A : r_A(x) = 0\}$.

(c) If $A$ is a commutative complete locally $m$-convex algebra, then

$$\text{rad}(A) = \{x \in A : r_A(x) = 0\}.$$  

Also, in this case, if the topology of $A$ is generated by a family $s_A$ of continuous seminorms,

$$\text{rad}(A) = \{x \in A : \lim_{n \to \infty} [p(x^n)]^{1/n} = 0 \text{ for all } p \in s_A\}.$$  

(d) If $A$ is a topological algebra with $\Delta_c(A) \neq \emptyset$, then

$$\text{rad}(A) \subseteq \cap \{\ker(\varphi) : \varphi \in \Delta_c(A)\} = \{x \in A : \varphi(x) = 0 \text{ for all } \varphi \in \Delta_c(A)\};$$
if, in addition, $A$ is a commutative complete locally m-convex $Q$-algebra,

$$\text{rad}(A) = \cap \{\ker(\varphi) : \varphi \in \Delta_c(A)\};$$

hence, in this case, $\text{rad}(A)$ is closed.

Note. For a non-locally m-convex $F$-algebra $A$, $\text{rad}(A)$ need not be closed.

**Definition.** An algebra $A$ is called **semisimple** if $\text{rad}(A) = \{0\}$; $A$ is called **simple** if it has no two-sided ideal other than $\{0\}$ and $A$. The quotient algebra $A/\text{rad}(A)$ is semisimple. Every locally $C^*$-algebra and every topological algebra with an orthogonal basis is semisimple. (Recall that a topological basis $\{x_n\}$ for a topological algebra $A$ is called orthogonal if $x_n x_m = 0$ if $n \neq m$ and $x_n^2 = x_n$ [Hus83, p. 62]. If $A$ is a commutative complete locally m-convex $Q$-algebra with $\Delta_c(A) \neq \emptyset$, then, by above theorem, $A$ is semisimple iff

$$\{x \in A : \varphi(x) = 0 \text{ for all } \varphi \in \Delta_c(A)\} = \{0\}.$$

**Definition:** Let $A$ be a topological algebra with $\Delta_c(A) \neq \emptyset$, and let $w(A^*, A)$ be the weak* topology of $A^*$. Then the relative weak* topologies $s = w(A^*, A)|_{\Delta_c(A)}$ of $A^*$ on $\Delta_c(A)$ is called the Gelfand topology. $\Delta_c(A)$ with this topology is called the maximal ideal spaces of $A$ (or the topological spectrum of $A$) and is denoted by $(\Delta_c(A), s)$. In general, $(\Delta_c(A), s)$ need not be compact or even locally compact.

Recall that, if $A$ is a TVS, the every equicontinuous subset of $A^*$ is relatively $w(A^*, A)$-compact (Alaoglu-Bourbaki theorem; see [Hor66], p. 201).

**Theorem A.6.17.** Let $A$ be a topological algebra. Then:

(a) ([Mal86], p. 140) $(\Delta_c(A), s)$ is a completely regular Hausdorff space.

(b) ([Mal86], p. 75) If $A$ is a $Q$-algebra, then $(\Delta_c(A), s)$ is equicontinuous and hence locally compact.

(c) ([Mal86], p. 143) If $A$ is an $F$-algebra, then $(\Delta_c(A), s)$ is compact iff $A$ is a $Q$-algebra.

(d) ([Hus83], p. 12) If $A$ is a complete locally m-convex algebra, then $\Delta_c(A) \cup \{0\}$ is a $w(A^*, A)$-closed subset of $A^*$.

(e) ([Hus83], p. 12) If $A$ is a complete locally m-convex algebra with identity, then $0$ is an isolated point of $\Delta_c(A) \cup \{0\}$ and $\Delta_c(A)$ is a $w(A^*, A)$-closed subset of $A^*$.

**Definition A.6.18.** ([Mal86], p. 142)
Let $A$ be a topological algebra. Then $\Delta_c(A)$ is called \textit{locally equicontinuous}, a subset of $A^*$ if, for each $f \in \Delta_c(A)$, there exists a neighborhood $U$ of $f$ in $\Delta_c(A)$ such that $U$ is an equicontinuous subset of $(\Delta_c(A), s) \subseteq (A^*, w(A^*, A))$.

\textbf{Definition A.6.19.} ([Mal86], p.73) Let $A$ be a topological algebra.
For each $x \in A$, define $\hat{x} : \Delta_c(A) \to \mathbb{C}$ by

$$\hat{x}(\varphi) = \varphi(x), \ \varphi \in \Delta_c(A).$$

Then $\hat{x}$ is continuous on $\Delta_c(A)$; hence $\hat{x} \in C(\Delta_c(A)) = C(\Delta_c(A), s))$.
The function $\hat{x}$ is called the \textit{Gelfand transform} of $x$ on $A$. Setting $\hat{A} = \{\hat{x} : x \in A\}$, we have $\hat{A} \subseteq C(\Delta_c(A))$. The map $G : x \to \hat{x}$ is called the \textit{Gelfand representation} of $A$ into $C(\Delta_c(A))$.

\textbf{Theorem A.6.20.} ([Mal86], p. 143) Let $A$ be a topological algebra. Then:
\begin{enumerate}[(a)]
\item $(\Delta_c(A), s)$ is locally equicontinuous iff $\Delta_c(A)$ is locally compact and $G : A \to \hat{A} \subseteq C(\Delta_c(A))$ is continuous.
\item If $A$ is a locally $m$-convex $Q$-algebra and $(\Delta_c(A), s)$ is locally compact, then $\Delta_c(A)$ is locally equicontinuous.
\end{enumerate}

\textbf{Theorem A.6.21.} ([Mal86], p. 140, 266) Let $A$ be a topological algebra. Then:
\begin{enumerate}[(a)]
\item The map $G : x \to \hat{x}$ is a continuous algebraic homomorphism of $A$ onto the subalgebra $\hat{A}$ of $C(\Delta_c(A))$.
\item $G$ is one-one iff $A$ is semisimple.
\end{enumerate}

The commutative version of the Gelfand-Naimark theorem for topological algebras is as follows.

\textbf{Theorem A.6.22.} (Gelfand-Naimark) ([Mic52], p. 34-35; [Mal86], p. 488)
\begin{enumerate}[(a)]
\item If $A$ is a commutative complete semisimple locally $m$-convex algebra, then $A$ is both algebraically and topologically isomorphic to a separating subalgebra $\hat{A}$ of $(C(\Delta_c(A), k), k$ being the compact-open topology; if $A$ has an identity $e$, then $\hat{A}$ contains the identity function.
\item If $A$ is a commutative locally $C^*$-algebra and that the Gelfand map $G : x \to \hat{x}$ is continuous, then $A$ is algebraically $*$-isomorphic to $(C_0(\Delta^*_c(A), k)$, where $\Delta^*_c(A) = \Delta_c(A) \cup \{0\}$.
\end{enumerate}

\textbf{Definition.} Let $A$ be a topological algebra.
(1) A net \( \{ e_\lambda : \lambda \in I \} \) in \( A \) is called a left (resp. right) approximate identity of \( A \) if
\[
\lim_\lambda e_\lambda x = x \quad (\text{resp. } \lim_\lambda xe_\lambda = x)
\]
for all \( x \in A \).

(2) \( \{ e_\lambda : \lambda \in I \} \) is called a two-sided approximate identity (or an approximate identity) of \( A \) if it is both a left and right approximate identity.

Every locally \( C^* \)-algebra \( A \) has a uniformly bounded approximate identity \( \{ e_\lambda : \lambda \in I \} \).

Definition. An algebra \( A \) is called
(a) left faithful if, for any \( a \in A \), \( aA = \{ 0 \} \Rightarrow a = 0 \).
(b) right faithful if, for any \( a \in A \), \( Aa = \{ 0 \} \Rightarrow a = 0 \).
(c) faithful if it is both left and right faithful.

Other equivalent terms for “faithful” are “proper”, “without order”, “without annihilators”.

Theorem A.6.23. An algebra \( A \) is faithful in each of the following cases:
(a) \( A \) has an identity \( e \).
(b) \( A \) is a topological algebra having an approximate identity \( \{ e_\lambda : \lambda \in I \} \) (e.g. \( A \) is a \( C^* \)-algebra or locally \( C^* \)-algebra).
(c) \( A \) is a topological algebra with an orthogonal basis.
(d) \( A \) is a commutative semisimple Banach algebra.

Proof. (a) This is clear.
(b) Suppose \( A \) is a topological algebra with an approximate identity \( \{ e_\lambda : \lambda \in I \} \). Let \( a \in A \) with \( ax = 0 \) for all \( x \in A \). Then \( ae_\lambda = 0 \) for all \( \lambda \in I \); hence \( a = \lim_\lambda ae_\lambda = 0 \). Similarly, \( Aa = \{ 0 \} \Rightarrow a = 0 \).
(c) Suppose \( A \) is a topological algebra with an orthogonal basis \( \{ x_n \} \). Let \( a \in A \) with \( ax = 0 \) for all \( x \in A \). We can write \( a = \sum_{n \geq 1} t_n x_n \). Then, since each \( x_m \in A \),
\[
ax_m = 0, \; \text{or} \; \sum_{m \geq 1} t_n(x_n x_m) = 0, \; \text{or} \; t_m x_m^2 = 0,
\]
and so \( t_m = 0 \) for all \( m \geq 1 \). Hence \( a = \sum_{m \geq 1} t_m e_m = 0 \). Similarly, \( Aa = \{ 0 \} \Rightarrow a = 0 \).
(d) Suppose \( A \) is a semisimple. Let \( a \in A \) with \( ax = 0 \) for all \( x \in A \). Then, \( a.a = 0 \) and so, by induction, \( a^n = 0 \) for all \( n \geq 1 \). Hence
\[ \lim_{n \to \infty} \|a^n\|^{\frac{1}{n}} = 0, \text{showing that} \]
\[ a \in \{ x \in A : \lim_{n \to \infty} \|x^n\|^{\frac{1}{n}} = 0 \} = \text{rad}(A). \]

Since \( A \) is semisimple, \( \text{rad}(A) = \{0\} \) and so \( a = 0 \). \( \square \)

**Example.** Let \( A \) be any vector space (resp. TVS) over \( \mathbb{C} \) with \( A \neq \{0\} \). Then \( A \) becomes a commutative algebra (resp. topological algebra) without an identity under the trivial multiplication: \( x, y = 0 \) for all \( x, y \in A \). Then, for any non-zero \( x \in A \), \( xA = \{0\} \). However \( x \neq 0 \), and so \( A \) is not faithful.

Recall that if \( E \) is a TVS, then \( \text{CL}(E) = \text{CL}(E, E) \) is an algebra under composition and has identity \( I : E \to E \) given by \( I(x) = x \) \( (x \in E) \).

**Theorem A.6.24.** Let \( E \) be a TVS (in particular, a topological algebra). Then \( (\text{CL}(E), t_u) \) and \( (\text{CL}(E), t_p) \) are topological algebras.

**Proof.** We need to show that the composition map \((S, T) \to ST\) is separately continuous. Fix \( S_0 \in \text{CL}(E) \), and let \( \{ T_\alpha : \alpha \in I \} \subseteq \text{CL}(E) \) with \( T_\alpha \xrightarrow{t_u} T \in \text{CL}(E) \). To show that \( S_0T_\alpha \xrightarrow{t_u} S_0T \), let \( U(D, W) \) be any \( t_u\)-neighborhood of \( 0 \) in \( \text{CL}(E) \). By continuity of \( S_0 : E \to E \), there exists \( V \in W_E \) such that \( S_0(V) \subseteq W \). Since \( T_\alpha \xrightarrow{t_u} T \), there exists \( \alpha_0 \in I \) such that
\[ (T_\alpha - T)(D) \subseteq V \text{ for all } \alpha \geq \alpha_0. \]
Then, for any \( \alpha \geq \alpha_0 \),
\[ S_0(T_\alpha - T)(D) \subseteq S_0(V) \subseteq W. \]
Hence \( S_0T_\alpha \xrightarrow{t_u} S_0T \).

Next, fix \( T_0 \in \text{CL}(E) \), and let \( \{ S_\alpha : \alpha \in I \} \subseteq \text{CL}(E) \) with \( S_\alpha \xrightarrow{t_u} S \in \text{CL}(E) \). To show \( S_\alpha T_0 \xrightarrow{t_u} ST_0 \), let \( U(D, W) \) be any \( t_u\)-neighborhood of \( 0 \) in \( \text{CL}(E) \). Since \( T_0 : E \to E \) is continuous, \( T_0 \) is bounded and so \( T_\alpha(D) \) is a bounded set in \( E \). Then \( U(T_\alpha(D), W) \) is a \( t_u\)-neighborhood of \( 0 \) in \( \text{CL}(E) \). Since \( S_\alpha \xrightarrow{t_u} S \), there exists \( \alpha_0 \in I \) such that
\[ S_\alpha - S \in U(T_\alpha(D), W) \text{ or } (S_\alpha - S)T_\alpha(D) \subseteq W \text{ for all } \alpha \geq \alpha_0. \]
Hence \( S_\alpha T_0 \xrightarrow{t_u} ST_0 \). Thus \( (\text{CL}(E), t_u) \) is a topological algebra. By the same argument, \( (\text{CL}(E), t_p) \) is also a topological algebra. \( \square \)

Recall that a subcollection \( \mathcal{B} \) of the collection \( b(E) \) of all bounded subsets of a TVS \( E \) is called *fundamental system of bounded sets* if, for each bounded set \( D \subseteq E \), there exists a \( D_i \in \mathcal{B} \) such that \( D \subseteq D_i \) ([Hus65], p. 20).
**Theorem A.6.25.** Let $E$ be an $F$-space (in particular, $F$-algebra). If $E$ has a countable fundamental system of bounded sets, then $(\text{CL}(E), t_u)$ is a complete metrizable topological algebra with jointly continuous multiplication.

**Proof.** Let $D = \{D_1, D_2, \ldots\}$ be a countable fundamental system of bounded sets in $E$. Since $E$ is metrizable, it has a countable base $W_E = \{W_1, W_2, \ldots\}$ (say) of neighborhoods of 0. Then clearly the collection $\{U(D_i, W_j) : i, j = 1, 2, 3, \ldots\}$ is a countable base of neighborhoods of 0 in $(\text{CL}(E), t_u)$. Also $(\text{CL}(E), t_u)$ is Hausdorff. Hence, by the metrization theorem of TVSs, $(\text{CL}(E), t_u)$ is metrizable. Therefore $(\text{CL}(E), t_u)$ is a complete metrizable topological algebra. Thus, by the Aren’s theorem, $(\text{CL}(E), t_u)$ is a topological algebra with jointly continuous multiplication. □

**Definition:** Let $(A, q)$ be an $F$-normed algebra. For any $T \in \text{CL}(A)$, let

$$\|T\|_q = \sup_{x \in A, q(x) \leq 1} q(T(x)).$$

(\*)

In general, $\|T\|_q$ need not exist since the set $\{x \in A : q(x) \leq 1\}$ may not be bounded in the $F$-algebra $(A, q)$. However, for $q$ a $k$-norm ($0 < k \leq 1$), the existence and other useful properties of $\|\cdot\|_q$ are summarized in the following theorem (see [Rol85], p. 101-102; [Bay03], p. 3-5; [ARK11]).

**Theorem A.6.26.** Let $(A, q)$ be a $k$-normed algebra, where $0 < k \leq 1$. Then:

(a) A linear mapping $T : A \rightarrow A$ is continuous $\iff \|T\|_q < \infty$.

(b) $\|\cdot\|_q$ is a $k$-norm on $\text{CL}(A)$.

(c) For any $T \in \text{CL}(A)$,

$$\|T\|_q = \sup_{x \in A, x \neq 0} \frac{q(Tx)}{q(x)}.$$  

(d) For any $T \in \text{CL}(A)$, $q(Tx) \leq \|T\|_q \cdot q(x)$ for all $x \in A$.

(e) For any $S, T \in \text{CL}(A)$, $\|ST\|_q \leq \|S\|_q \|T\|_q$; hence $(\text{CL}(A), \|\cdot\|_q)$ is an $k$-normed algebra.

(f) If $A$ is complete, then $(\text{CL}(A), \|\cdot\|_q)$ is also complete.

(g) If $A$ has a minimal approximate identity $\{e_\lambda : \lambda \in I\}$, then, for any $a \in A$,

$$\|L_a\|_q = \|R_a\|_q = q(a),$$

where $L_a, R_a : A \rightarrow A$ are the maps given by $L_a(x) = ax$ and $R_a(x) = xa$, $x \in A$.  

On the other hand, since $q$ so $Hence$

It is easy to verify that, for any $S, T \in CL(A)$ and $\lambda \in K$,

$$\|\lambda T\|_q = \sup_{q(x) \leq 1} q(\lambda T(x)) \frac{q(\lambda T(x))}{q(x)} = |\lambda|^k \sup_{q(x) \leq 1} q(T(x)) = |\lambda|^k \|T\|_q.$$

Proof. (a) This is clear.

(b) Clearly $\|T\|_q \geq 0$ and $\|T\|_q = 0$ iff $T = 0$. For any $T \in CL(A)$ and $\lambda \in K$,

$$\|\lambda T\|_q = \sup_{q(x) \leq 1} q(\lambda T(x)) \frac{q(\lambda T(x))}{q(x)} = |\lambda|^k \sup_{q(x) \leq 1} q(T(x)) = |\lambda|^k \|T\|_q.$$

(c) For any $x (\neq 0) \in A$, if $y = \frac{x}{q(x)^{\frac{1}{k}}}$, then

$$q(y) = q\left[\frac{x}{q(x)^{\frac{1}{k}}}\right] = \frac{1}{[q(x)^{\frac{1}{k}}]^k} q(x) = 1;$$

hence

$$\sup_{x \in A, x \neq 0} \frac{q(T(x))}{q(x)} = \sup_{x \in A, x \neq 0} \frac{q(T(x))}{q(x)} = \sup_{x \in A, x \neq 0} \frac{q[T(\frac{x}{q(x)^{\frac{1}{k}}})]}{q(x)} = \sup_{y \in A, q(y) = 1} q[T(y)] = \|T\|_q.$$

(d) Let $T \in CL(A)$. Using (c), clearly $q(Tx) \leq \|T\|_q q(x)$ for each $x \neq 0$ in $A$. Clearly, this also holds for $x = 0$.

(e) Let $S, T \in CL(A)$. Using (d),

$$\|ST\|_q = \sup_{q(x) \leq 1} q[S(T(x))] \leq \sup_{q(x) \leq 1} \|S\|_q q(T(x)) \leq \sup_{q(x) \leq 1} \|S\|_q \|T\|_q \cdot q(x) = \|S\|_q \|T\|_q.$$

(f) By Theorem A.6.25, $(CL(A), \|\cdot\|_q)$ is complete and so it is an $F$-algebra. See also ([Bay03], p. 5; [Rol85], p. 41).

(g) Let $a \in A$. Then

$$\|L_a\|_q = \sup_{q(b) \leq 1} q(L_a(b)) = \sup_{q(b) \leq 1} q(ab) = \sup_{q(b) \leq 1} q(a) = q(a).$$

On the other hand, since $q(\epsilon_\lambda) \leq 1$ for all $\lambda \in I$,

$$\|L_a\|_q = \sup_{q(ab) \leq 1} q(ab) \geq q(\epsilon_\lambda) = q(\lim_\lambda \epsilon_\lambda) = q(\lim_\lambda a\epsilon_\lambda) = q(a).$$

Hence $\|L_a\|_q = q(a)$. Similarly, $\|R_a\|_q = q(a)$. □
Remark. In the definition of \( \|T\|_q \) given by (\( \ast \)), we cannot make the blanket assumption that, for \( q \) an \( F \)-norm on \( E \), \( \|T\|_q \) always exists for each \( T \in CL(E) \). This assumption cannot be justified in view of the following counter-examples.

(1) First, \( \|T\|_q \) need not be finite for a general \( F \)-norm. For example, let \( E = \mathbb{R}^2 \); \( q(x_1, x_2) = |x_1| + |x_2|^{1/2} \); \( T(x_1, x_2) = (x_2, x_1) \). Then \( q \) is an \( F \)-norm on \( E \), but \( \|T\|_q = \infty \): for any \( n \in \mathbb{N} \),

\[
\|T\|_q \geq \frac{q[T(n, n^2)]}{q(n, n^2)} = \frac{n^2 + n^{1/2}}{2n} \to \infty.
\]

(2) Even when considering the subspace of those \( T \) for which \( \|T\|_q < \infty \), then \( \|T\|_q \) need not always be an \( F \)-norm, since (\( F_5 \)) need not hold. For example, for a fixed sequence \( (p_n) \) with \( 0 < p_n \leq 1 \), \( p_n \to 0 \), consider the \( F \)-algebra \( E \) of sequences \( (x_n) \subseteq \mathbb{R} \) with \( |x_n|^{p_n} \to 0 \) and \( q((x_n)) = \sup_{n \geq 1} |x_n|^{p_n} \). Then \( \|T\|_q < \infty \) for all multipliers of \( E \), but \( \|T\|_q \) is not an \( F \)-norm, i.e. it makes the space \( CL(E) \) into an additive topological group but not into a topological vector space (as it would lack the continuity of scalar multiplication in the absence of (\( F_5 \)) (cf. [Rol85], Example 1.2.3, p. 8)).

In view of the above remark, we shall need to assume that \( (E, q) \) is a \( k \)-normed space (or a \( k \)-normed algebra) whenever \( \|T\|_q \) is considered for \( T \in CL(E) \).

**Theorem A.6.27.** Let \( (A, q) \) be an \( k \)-normed algebra with \( q \) submultiplicative and \( A \) having an identity \( e \). Then \( A \) is isometrically isomorphic to a closed subalgebra of \( (CL(A), t_u) \).

**Proof.** Define \( q' : A \to \mathbb{R} \) by

\[
q'(x) = \sup_{z \in A, z \neq 0} \frac{q(xz)}{q(z)}, \quad x \in A
\]

Then, for any \( x \in A \),

\[
q'(x) = \sup_{x \in A} \frac{q(xz)}{q(z)} \leq \sup_{z \in A, z \neq 0} \frac{q(x)q(z)}{q(z)} = q(x),
\]

and also

\[
q(x) = q(xe) = q(e) \frac{q(xe)}{q(e)} \leq q(e) \sup_{z \neq 0} \frac{q(xz)}{q(z)} = q(e)q'(x).
\]

Hence, \( q' \) is equivalent to \( q \).
Now for each $x \in A$, consider $L_x : A \rightarrow A$ given by $L_x(y) = xy$, $y \in A$. Then clearly $\{L_x : x \in A\}$ is a subalgebra of $CL(A)$. Further,

$$\|L_x\|_q = \sup_{z \in A, z \neq 0} \frac{q(L_x(z))}{q(z)} = \sup_{z \in A, z \neq 0} \frac{q(xz)}{q(z)} = q/ (x).$$

Hence $(A, q')$ is isometrically isomorphic to the subalgebra $\{L_x : x \in A\}$ of $CL(A)$ under the mapping $x \rightarrow L_x$.

Next $\{L_x : x \in A\}$ is closed in $(CL(E), t_u)$, as follows. Let $T \in \{L_x : x \in A\}\|\|_q$. Then there exists a sequence $\{L_{x_n}\} \subseteq \{L_x : x \in A\}$ such that $L_{x_n} \rightarrow T$. We show that $T = L_{x_0}$ for some $x_0 \in A$. Now, for any $y \in A$,

$$T(y) = \lim_{n \rightarrow \infty} L_{x_n}(y) = \lim_{n \rightarrow \infty} x_n y = \lim_{n \rightarrow \infty} (x_n e)y = L_{x_0}(e)y;$$

i.e. $T = L_{x_0}$, where $x_0 = T(e) \in A$. Hence $T \in \{L_x : x \in A\}$, and so $\{L_x : x \in A\}$ is closed in $(CL(E), t_u)$. □

**Definition.** Let $Y$ be a TVS and $A$ a topological algebra, both over the same field $K$ ($= \mathbb{R}$ or $\mathbb{C}$). Then $Y$ is called a topological left $A$-module if it is a left $A$-module and the module multiplication $(a, x) \rightarrow ax$ from $A \times Y$ into $Y$ is separately continuous.

**Definition:** Let $Y$ be a topological left $A$-module, and let $b(A)$ (resp. $b(Y)$) denote the collection of all bounded sets in $A$ (resp. in $Y$). Then $Y$ is called a topological left $A$-module if it is a left $A$-module and the module multiplication $(a, x) \rightarrow ax$ from $A \times Y$ into $Y$ is separately continuous.

(a) The module multiplication $(a, x) \rightarrow ax$ is called $b(A)$- (resp. $b(Y)$-) hypocontinuous if, given any neighborhood $G$ of 0 in $Y$ and any $D \in b(A)$ (resp. $B \in b(Y)$), there exists a neighborhood $H$ of 0 in $Y$ (resp. $V$ of 0 in $A$) such that $DH \subseteq G$ (resp. $VB \subseteq G$).

(b) The module multiplication $(a, x) \rightarrow ax$ is called jointly continuous if, given any neighborhood $G$ of 0 in $Y$, there exist neighborhoods $U$ of 0 in $A$ and $H$ of 0 in $Y$ such that $UH \subseteq G$.

Clearly, joint continuity $\Rightarrow$ hypocontinuity $\Rightarrow$ separate continuity. The following theorem shows that, under some additional conditions, separate continuity $\Rightarrow$ hypocontinuity.

**Theorem A.6.28.** Let $(Y, \tau)$ be a topological left $A$-module. If $Y$ is ultrabarrelled, then the module multiplication is $b(A)$-hypocontinuous. In particular, every ultrabarrelled topological algebra has a hypocontinuous multiplication.
Proof. Let $G$ be a neighborhood of $0$ in $Y$ and $D \in b(A)$. For any $a \in A$, define $L_a : Y \to Y$ by $L_a(x) = a.x$, $x \in Y$. Clearly, each $L_a$ is linear and also continuous (by separate continuity of the module multiplication). Further, $\{L_a : a \in D\}$ is pointwise bounded in $CL(Y,Y)$. 

[Let $x \in Y$ and $G_1$ a neighborhood of $0$ in $Y$. Since $D$ in bounded in $A$, by separate continuity of the module multiplication, $D.x$ is bounded in $Y$ and so there exists $r > 0$ such that $D.x \subseteq rG_1$. So $\{L_a(x) : a \in D\} = \{a.x : a \in D\} = D.x \subseteq rG_1$, showing that $\{L_a : a \in D\}$ is pointwise bounded in $CL(Y,Y)$.] Since $Y$ is ultrabarralled, by the principle of uniform boundedness, $\{L_a : a \in D\}$ is equicontinuous. Hence, given any $G \in W_Y$, there exists a neighborhood $H$ of $0$ in $Y$ such that $L_a(H) \subseteq G$ for all $a \in D$; i.e. $D.H \subseteq G$. □
7. Measure Theory

In this section, we present some measure theoretic terminology (see [Hal50, Var65, Whe83, Bog07]).

**Definition.** Let $X$ be a given set. A collection $A(X)$ of subsets of $X$ is called an algebra if it satisfies the following conditions:

(a) $X \in A(X)$;
(b) The union of finite number of sets in $A(X)$ is also in $A(X)$;
(c) For any $A, B \in A(X)$, the difference set $A \setminus B$ is also in $A(X)$.

Clearly $\emptyset \in A(X)$. Further, $A(X)$ is closed under finite intersections. An algebra $A(X)$ is called a $\sigma$-algebra if it closed under countable unions (hence also under countable intersections).

**Definition.** If $A(X)$ is an algebra of subsets of $X$, then a function $\mu: A(X) \to \mathbb{R}^+$ is called

(i) **finitely additive** if, for any $A_1, \ldots, A_n \in A(X)$ with $A_i \cap A_j = \emptyset$ for $i \neq j$,

$$\mu(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mu(A_i);$$

(ii) **countably additive** if, for any sequence of sets $A_1, A_2, \ldots \in A(X)$ with $\bigcup_{i=1}^{\infty} A_i \in A(X)$ and $A_i \cap A_j = \emptyset$ for $i \neq j$,

$$\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i).$$

Now, let $X$ be a completely regular Hausdorff space, and let $c(X)$ (resp. $k(X), z(X), cz(X)$) the collection of all closed (resp. compact, zero, cozero) subsets of $X$ (see Section A.1). We refer to [GJ60] for general facts about $z(X)$ and $cz(X)$.

The family of zero sets is preserved by finite unions and countable intersections. A compact $G_\delta$-set is a zero set. Any zero set is closed and a $G_\delta$-set; the converse is true in a normal space.

The family of cozero sets is preserved by finite intersections and countable unions, and forms a base for the topology of $X$. Disjoint zero sets are contained in disjoint cozero sets.

**Definition.** Let $Ba(X)$ denote the smallest $\sigma$-algebra of subsets of $X$ containing $z(X)$, and let $Bo(X)$ denote the smallest $\sigma$-algebra of subsets of $X$ containing $c(X)$. Clearly, the $\sigma$-algebras $Ba(X)$ and $Bo(X)$ are closed under countable unions and intersections. The members of $Ba(X)$ (resp. $Bo(X)$) are called Baire (resp. Borel) sets. Since every zero set
is closed, $\mathcal{B}\sigma(X) \subseteq \mathcal{B}o(X)$, but, in general, $\mathcal{B}\sigma(X) \neq \mathcal{B}o(X)$ even if $X$ is compact. For instance, in the compact Hausdorff space $[0, 1]^{\omega_1}$, singleton sets are closed sets but are not zero sets. It is true that every compact set in the Baire $\sigma$-algebra $\mathcal{B}\sigma(X)$ is a zero set, but not every compact $K \subseteq X$ is in $\mathcal{B}\sigma(X)$. See ([Whe83], p. 109). However, if $X$ is a metric space or, more generally, a perfectly normal space, then $\mathcal{B}\sigma(X) = \mathcal{B}o(X)$ ([Whe83], p. 109; [Bog07], p. 13). (A topological space $X$ is called perfectly normal if every closed set $F \subseteq X$ has the form $Z = f^{-1}(0)$ for some $f \in C(X)$, or equivalently, if it is normal and every closed subset of $X$ is a $G_\delta$-set.)

**Definition.** If $\Sigma$ is a subfamily of $\mathcal{B}\sigma(X)$, then we say that a finite, real-valued, finitely additive set function $\mu$ on $\mathcal{B}\sigma(X)$ is $\Sigma$-regular if, for each $A \in \mathcal{B}\sigma(X)$ and $\varepsilon > 0$, there exists $F \in \Sigma, F \subseteq A$, such that

$$|\mu(H)| < \varepsilon \text{ for all } H \in \Sigma \text{ with } H \subseteq A \setminus F.$$ 

A positive Baire measure $\mu$ on $X$ is a finite, real-valued, non-negative, finitely additive set function on the algebra $\mathcal{B}\sigma(X)$ which is $z(X)$-regular (i.e., equivalently, for any $B \in \mathcal{B}\sigma(X)$,

$$\mu(B) = \sup\{\mu(Z) : Z \subseteq B, Z \in z(X)\}.$$ 

A (signed) Baire measure is the difference of two positive Baire measures.

The spaces of all Baire measures and positive Baire measures are denoted by $M(\mathcal{B}\sigma(X))$ and $M^+(\mathcal{B}\sigma(X))$, respectively.

If $\mu \in M(\mathcal{B}\sigma(X))$, then the set functions $\mu^+$ and $\mu^-$ defined on $\mathcal{B}\sigma(X)$ defined by

$$\mu^+(A) = \sup\{\mu(B) : B \in \mathcal{B}\sigma^*(X), B \subseteq A\},$$

$$\mu^-(A) = -\inf\{\mu(B) : B \in \mathcal{B}\sigma^*(X), B \subseteq A\},$$

are members of $M^+(\mathcal{B}\sigma(X))$ and $\mu = \mu^+ - \mu^-$. The set function $|\mu| = \mu^+ + \mu^-$ is again in $M^+(\mathcal{B}\sigma(X))$ and is called the total variation of $\mu$. If $A \in \mathcal{B}\sigma(X)$,

$$|\mu|(A) = \sup\left\{\sum_{i=1}^{n} |\mu(B_i)| : B_i \in \mathcal{B}\sigma^*(X), A = \bigcup_{i=1}^{n} B_i\right\}.$$ 

The inner and outer inner measures determined by $\mu \in M^+(\mathcal{B}\sigma(X))$ are

$$\mu_*(A) = \sup\{\mu(Z) : Z \in z(X), Z \subseteq A\},$$

$$\mu^*(A) = \inf\{\mu(U) : U \in cz(X), A \subseteq U\},$$

for each subset $A$ of $X$.
\( M(\mathcal{B}a(X)) \) is a Banach space with the norm \(||\mu|| = |\mu|(X)\) and \(M^+(\mathcal{B}a(X))\) is a closed subset of \(M(\mathcal{B}a(X))\).

**Definition.** If \( \mu \in M^+(\mathcal{B}a(X)) \) and \( f \in C_b(X) \). If \( f \geq 0 \), we define the integral
\[
\int_X f d\mu = \sup \{ \int_X h d\mu : 0 \leq \varphi \leq f, \varphi \text{ a simple function w.r.t. } \mathcal{B}o(X) \}.
\]
For arbitrary \( f \in C_b(X) \), we can write \( f = f^+ - f^- \), where \( f^+(x) = \max\{f(x), 0\} \) and \( f^-(x) = \max\{-f(x), 0\} \). Then we define
\[
\int_X f d\mu = \int_X f^+ d\mu - \int_X f^- d\mu.
\]
Now if \( \mu \in M(\mathcal{B}a(X)) \), \( \mu = \mu^+ - \mu^- \) with \( \mu^+, \mu^- \in M^+(\mathcal{B}a(X)) \), and \( f \in C_b(X) \), we define
\[
\int_X f d\mu = \int_X f^+ d\mu - \int_X f^- d\mu.
\]
For a full discussion of integration with respect to a finitely additive measures, see [Hal50, Roy88, DS58].

The following version of the classical Riesz representation theorem is well-known (see, [Var65, Whe83]).

**Theorem A.7.1** (Riesz-Alexandroff) Let \( X \) be a completely regular Hausdorff space. Then \((C_b(X), u)^* = M(\mathcal{B}a(X)) \) via the linear isomorphism \( L \leftrightarrow \mu \), where
\[
L(f) = \int_X f d\mu \text{ for all } g \in C_b(X) \text{ and } ||L|| = |\mu|(X).
\]

**Remark.** ([Whe83], p. 117) The Borel measures \( \mu \) on \( \mathcal{B}o(X) \) arise in the usual Riesz Representation Theorem for compact spaces. For a short and elegant proof of this result, see Garling [Gar73]. In the completely regular space, several difficulties arise if we attempt to represent members of \((C_b(X), u)^* \) by Borel measures. Each \( L \in (C_b(X), u)^* \) can be represented by a finitely additive \( c(X) \)-regular measure \( \nu \) on \( \mathcal{B}o(X) \), but \( \nu \) need not be unique; unfortunately, there may be uncountably many \( \nu \) which work. Uniqueness of the representing Borel measure is restored if \( X \) is normal.

In view of the Riesz-Aleksandrov theorem, any \( L \in (C_b(X), u)^* \) can be identified with a unique \( \mu \in M(\mathcal{B}a(X)) \) and may write \( \mu(f) \) for \( \int_X f d\mu \) for brevity. In his fundamental paper, Varadarajan [Var65] analyzed three subspaces of \( M(\mathcal{B}a(X)) \), as follows:
Definition. (1) A measure $\mu \in M(\mathcal{B}a(X))$ is called $\sigma$-additive if, for any sequence $\{Z_n\}$ of zero sets of $X$ with $Z_n \downarrow \emptyset$, $|\mu(Z_n)| \rightarrow 0$. It is equivalent to that $\mu$ be countably additive on $\mathcal{B}a(X)$, so that $\mu$ can be extended uniquely to a countably additive set function on $\mathcal{B}a(X)$. We shall always assume that this extension has been made. Any countably additive measure on $\mathcal{B}a(X)$ must be $\sigma$-regular (using the fact that every cozero set is a countable union of zero sets), hence is $\sigma$-additive.

(2) $\mu$ is called $\tau$-additive if it is $c(X)$-regular and, for any net $\{Z_\alpha\}$ of zero sets of $X$ with $Z_\alpha \downarrow \emptyset$, $|\mu(Z_\alpha)| \rightarrow 0$. This behavior is not preserved by decreasing net of cozero sets. If $X$ is locally compact, every $\tau$-additive measure is compact zero set regular. Any $\tau$-additive measure $\mu$ on $\mathcal{B}a(X)$ can be extended to a unique $c(X)$-regular Borel measure $\nu$ such that $F_\alpha \downarrow F$, $\nu(F_\alpha) \rightarrow \nu(F)$. If every $\sigma$-additive measure in $M(\mathcal{B}a(X))$ is $\tau$-additive, then $X$ is called measure compact.

(3) $\mu$ is called tight if, given any $\varepsilon > 0$, there exits a compact set $K \subseteq X$ such that $|\mu|(X) < |\mu|^*(K) + \varepsilon$. Any tight Baire measure can be extended to a unique $c(X)$-regular Borel measure $\nu$. We shall always assume that this extension has been made.

Let $M_\sigma(\mathcal{B}a(X))$ (resp. $M_\tau(\mathcal{B}o(X))$, $M_t(\mathcal{B}o(X))$) denote the space of all bounded real-valued $\sigma$-additive (resp. $\tau$-additive, tight) measures on $\mathcal{B}a(X)$ (resp. $\mathcal{B}o(X)$). Restricting the above measures to $\mathcal{B}a(X)$, we clearly have $M_t(\mathcal{B}a(X)) \subseteq M_t(\mathcal{B}a(X)) \subseteq M_\sigma(\mathcal{B}a(X)) \subseteq M(\mathcal{B}a(X))$. Note that $M_t(\mathcal{B}o(X)) \subseteq M_\sigma(\mathcal{B}a(X))$ need not hold in general as an $\mu \in M_\sigma(\mathcal{B}o(X))$ is not necessarily $z(X)$-regular. Every tight $\mu \in M(\mathcal{B}a(X))$ is $\tau$-additive and hence it has a unique extension $\mu' \in M_\tau(\mathcal{B}o(X))$. 
8. Uniform Spaces and Topological Groups

Definition. [Will70, p. 238] Let $Y$ be a non-empty set.

1. The subset $\Delta = \Delta(Y) = \{(x,x) : x \in Y\}$ of $Y \times Y$ is called the diagonal on $Y$.
2. For any $A \subseteq Y \times Y$, we define $A^{-1}$ by
   \[ A^{-1} = \{(y,x) : (x,y) \in A\}. \]
   If $A = A^{-1}$, then $A$ is called symmetric.
3. For any $A,B \subseteq Y \times Y$, we define $A \circ B$ by
   \[ A \circ B = \{(x,y) \in Y \times Y : \exists z \in Y \text{ such that } (x,z) \in A \text{ and } (z,y) \in B\}. \]
   If $A = B$, we shall write $A \circ A = A^2$. Note that,
   \[ (x,z) \in A, (z,y) \in B \Rightarrow (x,y) \in A \circ B. \]

Definition. [Will70, p. 238] A subset $\mathcal{H}$ of subsets of $Y \times Y$ is called a uniformity (or uniform structure) on $Y$ if

\begin{enumerate}
\item $\Delta \subseteq \mathcal{H}$ for all $A \in \mathcal{H}$.
\item $A \in \mathcal{H} \implies A^{-1} \in \mathcal{H}$, i.e. each $A \in \mathcal{H}$ is symmetric.
\item If $A \in \mathcal{H}$, $\exists$ some $B \in \mathcal{H}$ such that $B^2 \subseteq A$.
\item If $A,B \in \mathcal{H}$, $\exists$ some $C \in \mathcal{H}$ such that $C \subseteq A \cap B$.
\item If $A \in \mathcal{H}$ and $A \subseteq B$, then $B \in \mathcal{H}$.
\end{enumerate}

In this case, the pair $(Y, \mathcal{H})$ is called a uniform space and we shall often denote it by $\mathcal{H}(Y)$. The members of $\mathcal{H}$ are called vicinities of $\mathcal{H}$.

Definition. [Will70, p. 238-239] (1) Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two uniformities on a set $Y$. If $\mathcal{H}_1 \subseteq \mathcal{H}_2$, then $\mathcal{H}_1$ is said to be weaker than $\mathcal{H}_2$ and $\mathcal{H}_2$ is said to be stronger than $\mathcal{H}_1$.

2. The weakest uniformity on a set $Y$ is $\mathcal{I} = \{Y \times Y\}$, called the indiscrete uniformity.

3. The strongest uniformity on a set $Y$ is $\mathcal{D} = \{A \subseteq Y \times Y : \Delta \subseteq A\}$, called the discrete uniformity.

Definition: Let $\mathcal{H}$ be a uniformity on a set $Y$. Then a subcollection $\mathcal{B} \subseteq \mathcal{H}$ is called a base (or basis) for $\mathcal{H}$ if, for any $A \in \mathcal{H}$, there exists $B \in \mathcal{B}$ such that $B \subseteq A$.

Theorem A.8.1. Let $\mathcal{B}$ be a base for a uniformity $\mathcal{H}$ on $Y$. Then it has the following properties:

\begin{enumerate}
\item $\Delta \subseteq B$ for all $B \in \mathcal{B}$.
\item If $B \in \mathcal{B}$, $\exists$ some $C \in \mathcal{B}$ such that $C^{-1} \subseteq B$.
\item If $B \in \mathcal{B}$, $\exists$ some $C \in \mathcal{B}$ such that $C^2 \subseteq B$.
\end{enumerate}
(b₄) If \( A, B \in \mathcal{B} \), \( \exists \) some \( C \in \mathcal{B} \) such that \( C \subseteq A \cap B \).

Conversely, if a collection \( \mathcal{B} \) of subsets of \( Y \times Y \) satisfies the above conditions (b₁)-(b₄), then the collection

\[
\mathcal{H} = \{ A \subseteq Y \times Y : A \text{ contains some } B \in \mathcal{B} \}
\]

is a uniformity on \( Y \) whose base is \( \mathcal{B} \).

We next define a topology on a set \( Y \) induced by a uniformity \( \mathcal{H} \) on \( Y \).

**Definition:** [Edw65, p. 26] Let \( \mathcal{H} \) be a uniformity on a set \( Y \). For any \( x \in Y \) and \( A \in \mathcal{H} \), let

\[
N_A(x) = \{ y \in Y : (x, y) \in A \}.
\]

Then, for each \( x \in Y \), the collection \( \{ N_A(x) : A \in \mathcal{H} \} \) satisfies all the conditions of a neighborhood system at \( x \). Consequently, there exists a unique topology \( \tau_\mathcal{H} \) on \( Y \) such that, for each \( x \in Y \), the collection \( \{ N_A(x) : A \in \mathcal{H} \} \) forms a local base at \( x \). In this case, the topology \( \tau_\mathcal{H} \) on \( Y \) is said to be the **topology induced** by (or derived from) the uniformity \( \mathcal{H} \).

**Notation.** For any \( K \subseteq Y \) and \( A \in \mathcal{H} \), let

\[
N_A(K) = \bigcup \{ N_A(x) : x \in K \} = \{ y \in Y : (x, y) \in A \text{ for some } x \in K \}.
\]

As we shall see below, every semi-metric (in particular, metric) space is a uniform space and every uniform space is a topological space.

**Theorem A.8.2.** [Edw65, p. 30] Let \((Y, d)\) be a semi-metric space (in particular, a metric space). For each \( r > 0 \), let

\[
D_r = \{ (x, y) \in Y \times Y : d(x, y) < r \}.
\]

Then the collection \( \mathcal{B} = \{ D_r : r > 0 \} \) is a base for a uniformity on \( Y \).

More generally, we have:

**Theorem A.8.3.** [Edw65, p. 27] Let \( D = \{ d_\alpha : \alpha \in I \} \) be a family of semi-metrics on a set \( Y \), and let

\[
D^+ = \{ \max\{d_{\alpha_1}, \ldots, d_{\alpha_n} \} \text{ for all } \{\alpha_1, \ldots, \alpha_n\} \subseteq I \}.
\]

For each \( r > 0 \), let

\[
D_{d,r} = \{ (x, y) \in Y \times Y : d(x, y) < r \}, \quad d \in D^+.
\]

Then the collection \( \mathcal{B} = \{ D_{d,r} : d \in D^+, r > 0 \} \) is a base for a uniformity on \( Y \). (Note that \( D \subseteq D^+ \).)

**Definition:** [Edw65, p. 28]
A topological space \((Y, \tau)\) is said to be \textit{uniformizable} if its topology can be derived from a uniformity, i.e., there exists a uniformity \(\mathcal{H}\) on \(Y\) such that \(\tau = \tau_\mathcal{H}\).

**Theorem A.8.4.** (a) [Will70, p. 256] A topological space \((Y, \tau)\) is uniformizable \(\iff\) it is completely regular.

(b) [Will70, p. 257] A uniformity \(\mathcal{H}\) on \(Y\) is pseudometrizable \(\iff\) it has a countable base.

(b) [Edw65, p. 27-28] A topological space \((Y, \tau)\) is uniformizable with uniformity \(\mathcal{H}\) \(\iff\) there exists a family \(\{d_\alpha : \alpha \in I\}\) of semi-metrics on \(Y\), which generates the uniformity \(\mathcal{H}\).

**Note.** In particular, every metric space and every completely regular space is a uniform space.

**Definition:** [Edw65, p. 28]

Let \((Y, \mathcal{H})\) be uniform space.

(1) A net \(\{x_\alpha : \alpha \in I\}\) in \(Y\) is said to be \textit{convergent} to \(x \in Y\) if, given any \(A \in \mathcal{H}\), there exists an \(\alpha_0 = \alpha_0(A) \in I\) such that

\[
(x_\alpha, x) \in A \quad \text{for all } \alpha \geq \alpha_0;
\]

in this case, we write \(x_\alpha \rightarrow x\).

(2) A net \(\{x_\alpha : \alpha \in I\}\) in \(Y\) is said to be a \textit{Cauchy net} if, given any \(A \in \mathcal{H}\), there exists an \(\alpha_0 = \alpha_0(A) \in I\) such that

\[
(x_\alpha, x_\beta) \in A \quad \text{for all } \alpha, \beta \geq \alpha_0.
\]

**Definition:** A uniform space \((Y, \mathcal{H})\) is said to be \textit{complete} if every Cauchy net in \(Y\) is convergent to a point in \(Y\).

**Definition:** [Edw65, p. 30] Let \((Y, \mathcal{H})\) be a uniform space. The set \(S \subseteq Y\) is said to be \textit{totally bounded} if, given any \(A \in \mathcal{H}\), there exists a finite set

\[
\{x_1, x_2, \ldots, x_n\} \subseteq Y
\]

such that

\[
S \subseteq \bigcup_{i=1}^n N_A(x_i) = \bigcup_{i=1}^n \{y \in Y : (y, x_i) \in A\}.
\]

Equivalently, [Will70, p. 262], a set \(S \subseteq Y\) is \textit{totally bounded} if, given any \(A \in \mathcal{H}\), there exists a finite cover \(\{U_1, U_2, \ldots, U_n\}\) of \(S\) such that

\[
\{\bigcup_{i=1}^n U_i \times U_i \subseteq A\}.
\]
Theorem A.8.5 [Will70, p. 262] A subset \( S \) of a uniform space \((Y, H)\) is compact \(\iff\) it is complete and totally bounded.

Definition. [Will70, p. 243] (a) Let \((Y, H_Y)\) and \((Z, H_Z)\) be uniform spaces and \(x_o \subseteq Y\). A function \(f : Y \to Z\) is said to be continuous at \(x_o\) if, for any \(B \in H_Z\), there exists an \(A \in H_Y\) such that 
\[
(f(x), f(x_o)) \in B \text{ whenever } x \in Y \text{ with } (x, x_o) \in A.
\]
(b) Let \((Y, H_Y)\) and \((Z, H_Z)\) be uniform spaces and \(S \subseteq Y\). A function \(f : Y \to Z\) is said to be uniformly continuous on \(S\) if, for any \(B \in H_Z\), there exists an \(A \in H_Y\) such that 
\[
(f(x'), f(x'')) \in B \text{ whenever } x', x'' \in Y \text{ with } (x', x'') \in A.
\]

Note. If \(f\) is uniformly continuous on \(S \subseteq Y\), then clearly \(f\) is continuous on \(S\); the converse holds if \(S\) is compact.

We next recall the notion of a "topological group" and show these spaces are uniform spaces.

Notations: Let \(G = (G, \cdot)\) be a group.

For any \(V, W \subseteq G\), let
\[
V \cdot W = \{a \cdot b : a \in V, b \in W\},
\]
\[
V^{-1} = \{a^{-1} : a \in V\}.
\]

Definition: Let \(G = (G, \cdot)\) be a group, written multiplicatively, and \(\tau\) a topology on \(G\). Then the \(G\) is called a topological group if

\((T_{G1})\) The mapping \(P : (x, y) \to x \cdot y\) of \(G \times G \to G\) is jointly continuous (i.e. given any \(x, y \in G\) and any neighborhood \(U\) of \(xy\), there exist neighborhoods \(V\) and \(W\) of \(x\) and \(y\) respectively such that \(P(V, W) \subseteq U\) or \(V \cdot W \subseteq U\)).

\((T_{G2})\) The inverse mapping \(I : x \to x^{-1}\) of \(G \to G\) is continuous (i.e. given any \(x \in G\) and any neighborhood \(U\) of \(x^{-1}\), there exists a neighborhood \(V\) of \(x\) such that \(I(V) \subseteq U\) or \(V^{-1} \subseteq U\)).

In this case, we denote \(G = (G, \cdot, \tau)\).

Note. The two conditions \((T_{G1} & T_{G2})\) are equivalent to the single condition:

\((T_{G})\) The map \((x, y) \to xy^{-1}\) of \(G \times G \to G\) is jointly continuous.

Examples:
(1) \((\mathbb{R}, +)\) with the usual topology is a TG.
(2) \((\mathbb{R} \setminus \{0\}, \cdot)\) with the usual topology is a TG.
(3) \((\mathbb{Q},+), (\mathbb{Q}\setminus\{0\},\cdot)\) with the induced topologies are topological groups.

(4) \((\mathbb{C},+), (\mathbb{C}\setminus\{0\},\cdot)\) are TG.

(5) Every TVS \((X,\tau)\) is a TG under the operation of +; in fact, by (TVS\(_1\)), \((x,y) \rightarrow x + y\) is continuous and, by (TVS\(_2\)), the inversion \(x \rightarrow x^{-1} = -x\) is continuous. In particular, every normed vector space \((X,\|\cdot\|)\) is a TG under the operation of +.

(6) \(T = \{\lambda \in \mathbb{C} : |\lambda| = 1\}\) is a TG under multiplication; here \(\lambda^{-1} = \overline{\lambda}\).

It is called a torus group [Pa78, p. 88].

Remark. By definition, any TVS \((E,\tau)\) is topological group under +.

Lemma A.8.6. Any group \(G = (G,\cdot)\) with the discrete topology \(\tau_I\) is a TG.

Proof. (TG\(_1\)) We verify that the map \((x,y) \rightarrow x \cdot y\) is continuous. Let \(x,y \in G\). Let \(U\) be any neighborhood of \(xy\) in \((G,\tau_I)\). Since \((G,\tau_I)\) is a discrete space \(V = \{x\}\) and \(W = \{y\}\) are \(\tau_I\)-open neighborhoods of \(x\) and \(y\), respectively. Clearly

\[ V \cdot W = \{x \cdot y\} \subseteq U.\]

Hence the map \((x,y) \rightarrow x \cdot y\) is continuous.

(TG\(_2\)) We show that the map \(I : x \rightarrow x^{-1}\) is continuous. Let \(x \in G\) and \(U\) any neighborhood of \(x^{-1}\) in \((G,\tau_I)\). Now \(V = \{x\} \in \tau\), and

\[ I(V) = V^{-1} = \{x^{-1}\} \subseteq U.\]

Thus \(I = x \rightarrow x^{-1}\) is contains on \(G\). Hence \((G,\cdot,\tau_I)\) is a TG.

Lemma A.8.7. Any group \(G = (G,\cdot)\) with the indiscrete topology \(\tau_D\) is a TG.

Proof. (TG\(_1\)) We show that the map \((x,y) \rightarrow x \cdot y\) is continuous. Let \(x,y \in G\). Since \((G,\tau_D)\) is an indiscrete space, \(G\) is the only non-empty open set in \(G\) and so \(U = G\) is the only \(\tau_D\)-neighborhood of \(G\). Take \(V = G\) and \(W = G\) as \(\tau_D\)-neighborhoods of \(x\) and \(y\) respectively. Then

\[ V \cdot W = G \cdot G = G = U.\]

Hence \((x,y) \rightarrow x \cdot y\) is continuous.

(TG\(_2\)) We show that \(I = x \rightarrow x^{-1}\) is continuous. Let \(x \in G\). Then \(U = G\) is the only \(\tau_D\)-open neighborhood of \(x^{-1}\). Take \(V = G\) as \(\tau_D\)-neighborhood of \(x\). Then

\[ I(V) = V^{-1} = G^{-1} = G = U.\]

So \(I : x \rightarrow x^{-1}\) is continuous.
Thus \((G, \cdot, \tau_D)\) is a TG.

\[\square\]

**Theorem A.8.8.** Let \(G = (G, \cdot, \tau)\) be a TG. Then, for any fixed \(a \in G\), the maps \(L_a : x \rightarrow a \cdot x\) and \(R_a : x \rightarrow x \cdot a\) are homeomorphisms of \(G\) to \(G\).

**Proof.** We prove the theorem for \(L_a\).

\(L_a\) is 1-1: Let \(x, y \in G\) with \(L_a(x) = L_a(y)\). Then
\[
a \cdot x = a \cdot y \implies a^{-1} \cdot (a \cdot x) = a^{-1} \cdot (a \cdot y) 
\implies e \cdot x = e \cdot y \implies x = y.
\]

\(L_a\) is onto: Let \(y \in G\) (domain) and
\[
L_a(a^{-1} \cdot y) = a \cdot (a^{-1} \cdot y) = e \cdot y = y.
\]

This shows that \(L_a\) is onto.

\(L_a\) is continuous: Let \(x_o \in G\). By \((TG_1)\), the map \((a, x) \rightarrow a \cdot x\) is continuous at \(x_o\) or, equivalently, \(L_a : x \rightarrow ax\) is continuous at \(x_o\).

\((L_a)^{-1}\) is continuous: Now \(L_a(x) = a \cdot x = z\) (say), and so
\[
L_a^{-1}(z) = x = a^{-1} \cdot z = L_a^{-1}(z).
\]

Thus \(L_a^{-1} = L_{a^{-1}}\). By above part, \(L_{a^{-1}}\) is continuous. Hence \((L_a)^{-1} : G \rightarrow G\) is continuous.

Thus \(L_a : x \rightarrow a \cdot x\) is a homeomorphism. Similarly \(R_a : x \rightarrow x \cdot a\) is a homeomorphism.

\[\square\]

**Corollary A.8.9.** Let \(G = (G, \cdot, \tau)\) be a TG and \(U\) an open set in \(G\). Then, for any \(a \in G\), \(a \cdot U\) and \(U \cdot a\) are also open. Moreover, if \(A \subseteq G\), \(A \cdot U\) and \(U \cdot A\) are open.

**Proof.** Since the mapping \(L_a : x \rightarrow a x\) is homeomorphism, it is, in particular, an open map. Hence \(L_a(U)\) or \(a \cdot U\) is open. Similarly \(R_a : x \rightarrow x \cdot a\) is open and so \(U \cdot a\) is an open set.

Next, if \(A \subseteq G\), \(A \cdot U = \cup_{a \in A} a \cdot U\) which is open; also \(U \cdot A = \cup_{x \in A} U \cdot a\) is open.

\[\square\]

**Lemma A.8.10.** Let \(G = (G, \cdot, \tau)\) be a TG. Then:

(i) The map \(x \rightarrow x^{-1}\) is a homeomorphism.

(ii) If \(a, b \in G\), the map \(x \rightarrow a \cdot x \cdot b\) is a homeomorphism.

(iii) If \(a \in G\), the map \(x \rightarrow x \cdot a \cdot x^{-1}\) is continuous.

(iv) If \(Z\) is a base of neighborhoods at \(e\), then, for any \(x \in G\), \(\{x \cdot U : U \in Z\}\) is a base of neighborhoods at \(x\), and so is \(\{U \cdot x : U \in \mathcal{U}\}\).

**Theorem A.8.11.** Let \(G = (G, \cdot, \tau)\) be a TG with identity \(e\) and let \(\mathcal{U}\) be a base of neighborhoods at \(e\). Then:

(i) \(e \in U\) for each \(U \in \mathcal{U}\).
(ii) For each \( U \in Z \), there exists some \( V \in Z \) such that \( V^2 = V \cdot V \subseteq U \).

(iii) For each \( U \in Z \), there exists some \( W \in Z \) such that \( W^{-1} \subseteq U \).

(iv) For each \( U \in Z \) and \( a \in U \), there exists some \( V \in Z \) such that \( a \cdot V \subseteq U \).

(v) For each \( U \in Z \) and \( x \in G \), there exists some \( V \in Z \) such that \( x \cdot V^{-1} \subseteq U \).

(vi) For each \( U, V \in Z \), there exists some \( W \in Z \) such that \( W \subseteq U \cap V \).

(vii) \( \bigcap_{U \in \mathcal{U}} U = \{ e \} \).

Proof. (i) Since each \( U \in \mathcal{U} \) is a neighborhood of \( e \) and so \( e \in U \) for each \( U \in \mathcal{U} \).

(ii) Let \( U \in \mathcal{U} \). Since the mapping \( (x, y) \to x \cdot y \) is continuous at \((e, e)\) and \((e, e) \to e \cdot e = e\), there exist neighborhoods \( V_1 \) and \( V_2 \) of \( e \) such that \( V_1 \cdot V_2 \subseteq U \). Put \( V = V_1 \cap V_2 \). Then \( V \) is a neighborhood of \( e \), \( V \subseteq V_1 \) and \( V \subseteq V_2 \). Hence
\[
V^2 = V \cdot V \subseteq V_1 \subseteq U.
\]

(iii) Let \( U \in \mathcal{U} \). Choose an open neighborhood \( W_1 \) of \( e \) with \( W \subseteq U \).

Since \( x \to x^{-1} \) is a homeomorphism, \( W_1^{-1} \) is an open neighborhood of \( e^{-1} = e \). Let \( W = W_1 \cap W_1^{-1} \). Then \( W \in \mathcal{U} \) and
\[
W^{-1} = W \subseteq U.
\]

(iv) Let \( U \in \mathcal{U} \) and \( a \in U \). Since the map \( (x, y) \to x \cdot y \) is continuous at \((a, e)\) and \((a, e) \to a \cdot e = a\), \( \exists \) a neighborhood \( V \) of \( e \) in \( X \) such that \( a \cdot V \subseteq U \).

The remaining proofs are omitted. \( \square \)

Theorem A.8.12. [Ber74, p.17] Let \( G = (G, \cdot, \tau) \) be a topological group with a base \( \mathcal{U} \) of neighborhoods of \( e \) in \( G \). Then the following are equivalent.

(a) \( G \) is Hausdorff.

(b) \( \{ e \} \) is a closed subset of \( G \).

(c) \( \bigcap_{U \in \mathcal{U}} U = \{ e \} \).

Theorem A.8.13. (a) Let \( G \) be a TG. Then, for any neighborhood \( U \) of \( e \) in \( G \), there exists a neighborhood \( V \) of \( 0 \) such that \( \overline{V} \subseteq U \). Consequently, every topological group is regular.

(2) Every Hausdorff TG is completely regular.

Definition. [Ber74, p. 26] A metric \( d \) on a group \( (G, \cdot) \) is called:
(i) *left invariant*

if \( d(a \cdot x, a \cdot y) = d(x, y) \) for all \( x, y, a \in G \).

(ii) *right invariant* if \( d(x \cdot a, y \cdot a) = d(x, y) \) for all \( x, y, a \in G \).

**Metrization Theorem A.8.14.** (Birkhoff-Kakutani)

([Ber74, p. 28], [Hus66, p. 49]) (a) Let \( G = (G, \cdot, \tau) \) be a TG with identity element \( e \). Then \( G \) is metrizable iff it is Hausdorff and has a countable base of \( \tau \)-neighborhoods of \( e \).

(b) A metrizable TG admits a left invariant (or right invariant) compatible metric.

**Theorem A.8.15.** (a) Every topological group \( (G, \tau) \) is a uniform space.

(b) Every topological vector space \( (E, \tau) \) is a uniform space.

Proof. (a) Let \( B \) be any base of \( \tau \)-neighborhoods of the identity \( e \) in \( G \). For each \( V \in B \), let

\[
A_V = \{(x, y) \in G \times G : xy^{-1} \in V \}.
\]

Then \( \mathcal{H} = \{A_V : V \in B\} \) is a uniformity on \( G \), as follows.

\[
\rho_1 \triangle (G) = \{(x, x) : x \in G \} \subseteq A_V \quad \text{for all } A_V \in \mathcal{H}. \quad \text{[Let } A_V \in \mathcal{H} \text{ with } V \in B. \text{ For any } x \in G,]
\]

\[
xx^{-1} = e \in V \quad \text{and so } (x, x) \in A_V.
\]

Hence \( \Delta \subseteq A_V \) for all \( A_V \in \mathcal{H} \).

\[
\rho_2 \quad A_V \in \mathcal{H} \implies (A_V)^{-1} \in \mathcal{H}. \quad \text{[Let } A_V \in \mathcal{H} \text{ with } V \in B. \text{ We note that } (A_V)^{-1} = A_{V^{-1}} :]
\]

\[
(A_V)^{-1} = \{(x, y) : (y, x) \in A_V \} = \{(x, y) : yx^{-1} \in V \} = \{(x, y) : xy^{-1} = (yx^{-1})^{-1} \in V^{-1} \} = A_{V^{-1}}.
\]

Hence, if \( A_V \in \mathcal{H} \), then \( (A_V)^{-1} = A_{V^{-1}} \in \mathcal{H} \) since \( V^{-1} \in B \).

\[
\rho_3 \quad \text{If } A_V \in \mathcal{H}, \exists \text{ some } A_W \in \mathcal{H} \text{ such that } (A_W)^2 \subseteq A_V : \quad \text{[Let } A_V \in \mathcal{H} \text{ with } V \in B. \text{ Choose } W \in B \text{ such that } WW \subseteq V. \text{ We note that } (A_W)^2 \subseteq A_{WW}, \text{ as follows}]
\]

\[
(A_W)^2 = A_W \circ A_W
\]

\[
= \{(x, y) : \exists \text{ some } z \in G \text{ such that } (x, z) \in A_W \text{ and } (z, y) \in A_W\}
\]

\[
= \{(x, y) : \exists \text{ some } z \in G \text{ such that } xz^{-1} \in W \text{ and } zy^{-1} \in W\}
\]

\[
\subseteq \{(x, y) : \exists \text{ some } z \in G \text{ such that } (xz^{-1})(zy^{-1}) \in WW\}
\]
\[
\begin{align*}
\ &= \ \{ (x, y) : \{ (x, y) : xy^{-1} \in WW \} = A_{WW}.
\end{align*}
\]

Hence \( A_W^2 \subseteq A_{WW} \subseteq A_V. \)

\((u_4)\) If \( A_V, A_W \in \mathcal{H}, \exists \ \text{some } A_U \in \mathcal{H} \text{ such that } A_U \subseteq A_V \cap A_W : [\]

Let \( A_V, A_W \in \mathcal{H} \) with \( V, W \in \mathcal{B} \). Choose \( U \in \mathcal{B} \) such that \( U \subseteq V \cap W. \)

We note that \( A_V \cap A_W = A_{V \cap W}, \) as follows:

\[
\begin{align*}
A_V \cap A_W &= \{ (x, y) : xy^{-1} \in V \} \cap \{ (x, y) : xy^{-1} \in W \} \\
&= \{ (x, y) : xy^{-1} \in V \cap W \} = A_{V \cap W}.
\end{align*}
\]

Hence \( A_U \subseteq A_V \cap W \subseteq A_V \cap A_W \).

\((u_5)\) If \( A_V \in \mathcal{H} \) and \( A_V \subseteq A_W \), then \( A_W \in \mathcal{H} : [\)

Let \( A_V \in \mathcal{H} \) and \( A_V \subseteq A_W \) with \( V, W \in \mathcal{B} \). We note that \( A_V \subseteq A_W \) implies \( V \subseteq W \), as follows. Let \( x \in V \). Then \( xe^{-1} = xe = x \in V \), and so \( (x, e) \in A_V \). Since \( A_V \subseteq A_W \), \( (x, e) \in A_W \). Hence \( xe^{-1} \in W \) or \( x \in W \). So \( V \subseteq W \). Since \( W \in \mathcal{B} \), we have \( A_W \in \mathcal{H} \).

(b) This follows as in (a). In fact, let \( (E, \tau) \) be a TVS, and let \( \mathcal{W} \) be a base of neighborhoods of 0 in \( X \). In this case, for each \( V \in \mathcal{W} \), let

\[
A_V = \{ (x, y) \in E \times E : x - y \in V \}.
\]

Then \( \mathcal{H} = \{ A_V : V \in \mathcal{W} \} \) is a uniformity on \( E \), as follows:

\((u_1)\) \( \Delta = \{ (x, x) : x \in E \} \subseteq A_V \) for all \( A_V \in \mathcal{H} \) (since, for any \( x \in E \),

\[
\begin{align*}
x - x &= 0 \in V \text{ and so } (x, x) \in A_V.
\end{align*}
\]

\((u_2)\) For any balanced \( V \in \mathcal{W} \),

\[
\begin{align*}
A_V^{-1} &= \{ (x, y) : (y, x) \in A_V \} = \{ (x, y) : y - x \in V \} \\
&= \{ (x, y) : x - y \in -V \} = A_{-V}.
\end{align*}
\]

Hence, if \( A_V \in \mathcal{H} \), then \( A_V^{-1} = A_{-V} \in \mathcal{H} \text{ since } -V \in \mathcal{W} \).

\((u_3)\) Let \( A_V \in \mathcal{H} \). Choose a balanced \( W \in \mathcal{W} \) such that \( W + W \subseteq V \).

Then

\[
\begin{align*}
A_W^2 &= A_W \circ A_W \\
&= \{ (x, y) : \exists \ some \ z \in E \ such \ that \ (x, z) \in A_W \\
\text{and } (z, y) \in A_W \}, \\
&= \{ (x, y) : \exists \ some \ z \in E \ such \ that \ x - z \in W \text{ and } z - y \in W \} \\
&\subseteq \{ (x, y) : x - y = (x - z) + (z - y) \in W + W \} = A_{W + W}.
\end{align*}
\]

Hence \( A_W^2 \subseteq A_{W + W} \subseteq A_V. \)

\((u_4)\) Let \( A_V, A_W \in \mathcal{H} \). Choose \( G \in \mathcal{W} \) such that \( G \subseteq V \cap W \). Then

\[
\begin{align*}
A_V \cap A_W &= \{ (x, y) : x - y \in V \} \cap \{ (x, y) : x - y \in W \} \\
&= \{ (x, y) : x - y \in V \cap W \} = A_{V \cap W}.
\end{align*}
\]
Hence \( A_G \subseteq A_{V \cap W} \subseteq A_V \cap A_W \).

(\( u_5 \)). Let \( A_V \in \mathcal{H} \) and \( A_V \subseteq A_W \). This implies \( V \subseteq W \), as follows.

[Let \( x \in V \). Then \( x - 0 = x \in V \), and so \( (x, 0) \in A_V \). Since \( A_V \subseteq A_W \), \( (x, 0) \in A_W \). Hence \( x - 0 \in W \) or \( x \in W \). So \( V \subseteq W \).] Since \( W \in \mathcal{W} \), we have \( A_W \in \mathcal{H} \). \( \Box \)
9. Notes and Comments

Section A.1. This contains some basic definitions and results from General Topology. A full discussion of the results can be found in standard text books. See, for example, Kelley [Kel55], Gillman and Jerison [GJ60], Dugundji [Dug66], Willard [Will70] and Husain [Hus77].

Section A.2. This is a collection of basic definitions and results from topological vector spaces. See, for example, Dunford-Schwartz [DS58], Edwards [Edw65], Horvath [Hor66], Husain [Hus65], Kelley and Namioka [KN63], Köthe [Ko69], A. Robertson & W. Robertson [RR64], Rudin [Rud91], and Schaefer [Scha71].

Section A.3. Here some linear topologies on $CL(E, F)$, known as the $G$-topologies are defined. The most important cases of the $G$-topology are: the topology $t_u$ of uniform convergence on bounded sets, the topology $t_{pc}$ of precompact convergence, the topology $t_c$ of compact convergence and the topology $t_p$ of pointwise convergence.

This section also includes some fundamental theorems of Functional Analysis such as the principle of uniform boundedness, Banach-Steinhaus, open mapping theorem, and closed graph theorem. All these are given in the non-locally convex setting.

Section A.4. The notion of a shrinkable neighborhood of 0 in a TVS $E$ is presented. It had first appeared in the Ph.D. thesis of R. T. Ives in 1957 (see [Kl60a]), and further studied by Klee ([Kl60a, Kl60b]).

Important facts about shrinkable neighborhoods included in this section are:

(A) Every Hausdorff TVS $E$ has a base of shrinkable neighborhoods of 0.

(B) If $W$ is a shrinkable neighborhood of 0 in a TVS $E$, then its Minkowski functional $\rho_W$ is continuous and positively homogeneous.

The proofs of these important results, due to Klee [Kl60a], may not be found in the text books and therefore we have included them here.

Section A.5. This includes the notions of non-Archimedean valued fields, non-Archimedean TVSs and locally $\mathbb{F}$-convex TVSs. Most of these results are collected from the monograph of Prolla [Pro82] (see also [Pro77, NBB71, vR78]).

We mention that an NA TVS over $\mathbb{F}$ or equivalently, a locally $\mathbb{F}$-convex TVS (for non-trivially valued field $\mathbb{F}$) may be considered as an analogue of the usual locally convex TVS over $\mathbb{K}$. 
Section A.6. This section contains basic definitions and results from topological algebras and modules. The proofs of these results except for a few selected theorems are omitted; a full discussion of the results can be found in standard text books. See, for example, [Mal86, Mic52, Hus83, Dal00, Fra05, Zel71, Zel73].

Section A.7. Some measure theoretic terminology is considered. For a full discussion, see Halmos [Hal50], Dunford-Schwartz [DS58], Royden [Roy88]. We also include some material on the classes of tight, $\tau$-additive and $\sigma$-additive measures on a topological spaces $X$. The detail may be found in the survey papers of LeCam [Lec57], Varadarajan [Var65] and Wheeler [Whe83].

Section A.8. Uniform spaces are the carriers for the notions of uniform convergence, uniform continuity, equicontinuity and like this. These notions are easily defined in metric spaces; the important quality of metric spaces for this purpose being that the distance is a notion which can be applied uniformly to pair of points without regard to their location. This quality is not possessed by topological spaces, where the neighborhoods of a point depend on the location of the point. Therefore uniform spaces will require somewhat more structure than a topology provides. The general notion of a general uniform space was introduced by A. Weil in 193; another approach is due to W.J. Tuckey (1940). See also J.R. Isbell [Isb64]. For detail of the definitions and results of this section, see [Edw65, Will70].

In this section, we have also introduced here the notion of a "topological group" which is more general than the notion of a "topological vector space. It is shown that every topological group is a uniform spaces. For detail see, [Hus66, Ber74].
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APPENDIX B

List of Symbols

\( \overline{A} \) or \( cl(A) \), the closure of a subset \( A \) of a topological space, A.1, 1.1
\( A^0 \) or \( int A \), the interior of a subset \( A \) of a topological space, A.1, 1.1
\( A(x) = \{ f(x) : f \in A \} \), A.3, 4.1
\( B(X) \), the algebra of all bounded functions \( \varphi : X \to \mathbb{K} (= \mathbb{R} \text{ or } \mathbb{C}) \), 1.1
\( B_f(X) = \{ \varphi \in B(X) : \varphi \text{ has finite support} \} \), 1.1
\( B_o(X) = \{ \varphi \in B(X) : \varphi \text{ has } \sigma \text{-compact support} \} \), 1.1
\( B_o(X) = \{ \varphi \in B(X) : \varphi \text{ has countable support} \} \), 1.1
\( B_o(X) = \{ \varphi \in B(X) : \varphi \text{ vanishes at infinity} \} \), 1.1
\( B_o(X) = \{ \varphi \in B(X) : \varphi \text{ has compact support} \} \), 1.1
\( B_o(X) = \{ \varphi \in B(X) : \varphi \text{ has } \sigma \text{-compact support} \} \), 1.1
\( B_o(X) = \{ \varphi \in B(X) : \varphi \text{ vanishes at infinity for each } v \in V \} \), 1.2
\( CV_b(X) = \{ f \in C(X) : vf \text{ is bounded on } X \} \), 1.2
\( CV_o(X) = \{ f \in C(X) : vf \text{ is bounded for each } v \in V \} \), 1.2
\( CV_o(X) = \{ f \in C(X) : vf \text{ vanishes at infinity for each } v \in V \} \), 1.2
\( C(X) \), the algebra of all continuous functions \( \varphi : X \to \mathbb{K} \), 1.1
\( C_b(X) = \{ \varphi \in C(X) : \varphi \text{ is bounded on } X \} \), 1.1
\( C_o(X) = \{ \varphi \in C(X) : \varphi \text{ vanishes at infinity} \} \), 1.1
\( C_o(X) = \{ \varphi \in C(X) : \varphi \text{ vanishes at infinity} \} \), 1.1
\( CV_b(X) = \{ f \in C(X) : vf \text{ is bounded on } X \} \), 1.2
\( CV_o(X) = \{ f \in C(X) : vf \text{ vanishes at infinity for each } v \in V \} \), 1.2
\( C(X, E) \), the vector space of all continuous functions \( f : X \to E \), 1.1
\( C_b(X, E) = \{ f \in C(X, E) : f \text{ is bounded on } X \} \), 1.1
\( C_o(X, E) = \{ f \in C(X, E) : f \text{ vanishes at infinity} \} \)
\( C_o(X, E) = \{ f \in C(X, E) : f \text{ vanishes at infinity} \} \)
\( C_o(X, E) = \{ f \in C(X, E) : f \text{ has compact support} \} \)
\( CV_b(X, E) = \{ f \in C(X, E) : f(X) \text{ is relatively compact in } E \} \)
\( CV_o(X, E) = \{ f \in C(X, E) : f(X) \text{ is precompact in } E \} \), 4.2
\( CV_b(X, E) = \{ f \in C(X, E) : vf \text{ is bounded on } X \text{ for each } v \in V \} \), 1.2
$CV_{pc}(X, E) = \{ f \in C(X, E) : vf(X) \text{ is precompact in } E \text{ for all } v \in V \}$,

$CV_{vo}(X, E) = \{ f \in C(X, E) : vf \text{ vanishes at infinity for each } v \in V \}$,

$C(X) \otimes E = \{ \varphi \otimes a : \varphi \in C(X), a \in E \}$,

$C_\phi$, the composition operator on $CV_\phi(X, E)$, $C_\phi(f) = f \circ \phi$ with $\phi : X \to X$,

$c(X)$, the collection of all closed subsets of $X$,

$cZ(X)$, the collection of all cozero subsets of $X$,

$E^*$, the topological (or continuous) dual of a topological vector space $E$,

$F$, a field, $A.4$

$F(X, E)$, the vector space of all functions $X \to E$, $1.1, 1.2, 9.1, 9.2$

$FV_\circ(X, E^*)$, the vector space of all functions $f : X \to E$ with $vf$ vanishes at infinity for each $v \in V$, $1.2$

$F_\sigma$, countable union of closed sets, $A.1$

$f \prec U$ means that there exists a $Z \in z(X)$ such that $Z \subseteq U$ and $\text{supp}(f) \subseteq Z$, $7.1, 7.2$

$\| f \|_W = \| \rho_W \circ f \| = \sup_{x \in X} |\rho_W(f(x))|$, $8.1, 8.2$

$G_\delta$, countable intersection of open sets, $A.1$

$G(\mu, W, \varepsilon) = \{ f \in C_b(X, E) : \mu(\{ x : f(x) \notin W \}) \leq \varepsilon \}$, $7.1, 7.2$

$\gamma$, the weak topology on $(C_b(X, E), \beta)$, $7.1$

$\gamma_V$, the weak topology on $(CV_\circ(X, E), \omega_V)$, $7.2$

$\text{Hom}_A(A, Y)$, the vector space of all continuous left $A$-module homomorphisms of $A$ into $Y$, $10.3$

$\int_x \varphi d\mu$, the integral of $\varphi \in C_b(X)$ with respect to a scalar measure $\mu : X \to \mathbb{K}$, $1.7, 8., 8.2$

$\int_x dm f$, the integral of $f \in C_b(X, E)$ with respect to a vector measure $m : X \to E^*$, $8.1, 8.2$

$k$, the compact-open topology on $C(X, E)$ or $C_b(X, E)$, $1.1$

$\mathbb{K}$, the field of scalars, always either the field $\mathbb{R}$ or the complex field $\mathbb{C}$, $1.1, A.1, A.2$

$k(X)$, the collection of all compact subsets of $X$,

$K^+(X)$, the set of all non-negative constant functions on $X$, $1.2$

$\chi_A$, the characteristic function of a subset $A$, $1.1, 1.2, A.1$

$\chi_c(X) = \{ \lambda \chi_K : \lambda > 0 \text{ and } K \subseteq X, \text{ compact} \}$, $1.2$

$\chi_f(X) = \{ \lambda \chi_F : \lambda > 0 \text{ and } F \subseteq X, \text{ finite} \}$, $1.2$

$\ell_p = \ell_p(\mathbb{N}) = \{ x = (x_n) \subseteq \mathbb{K} : \sum_{n=1}^{\infty} |x_n|^p < \infty \}$, $0 < p < \infty$, $A.2$

$L_p = L_p[a, b] = \{ f : [a, b] \to \mathbb{K} : \int_a^b |f(t)|^p dt < \infty \}$, $0 < p < \infty$, $A.2$
\( \mathcal{M}(A) \), the set of all maximal ideals of codimension 1 in \( A \), A.6, 5.2

\( \mathcal{M}_c(A) \), the set of all maximal closed ideals of codimension 1 in \( A \), A.6, 5.2

\( M_\theta \), the multiplication operator from \( CV_0(X, E) \) into \( F(X, E) \) defined by \( M_\theta(f) = \theta f \) with \( \theta : X \to \mathbb{C} \), 9.1, 9.2

\( M_\varphi \), the multiplication operator from \( CV_0(X, E) \) into \( F(X, E) \) defined by \( M_\varphi(f) = \varphi f \) with \( \varphi : X \to E \), 9.1, 9.2

\( M_\rho(A) \), the algebra of all double multipliers \((S, T)\) on an algebra \( A \), 10.1

\( M_\rho(Ba(X)) \), the vector space of all bounded real-valued \( \sigma \)-additive measures on \( Ba(X) \), A.7, 8.1

\( M_\tau(Bo(X)) \), the vector space of all bounded real-valued \( \tau \)-additive measures on \( Bo(X) \), A.7, 8.1

\( M_\rho(Bo(X)) \), the vector space of all bounded real-valued \( \tau \)-additive measures on \( Bo(X) \), A.7, 8.1

\( M_\rho(Ba(X), E^*) \), the vector space of all bounded \( E^* \)-valued \( \sigma \)-additive measures on \( Ba(X) \), 8.1, 8.2

\( M_\tau(Bo(X), E^*) \), the vector space of all bounded \( E^* \)-valued \( \tau \)-additive measures on \( Bo(X) \), 8.1, 8.2

\( M_\rho(Bo(X), E^*) \), the vector space of all bounded \( E^* \)-valued \( \tau \)-additive measures on \( Bo(X) \), 8.1, 8.2

\( M(\mu, v, W, \varepsilon) = \{ f \in CV_0(X, E) : \mu^s(\{ x \in X : \nu f(x) \notin W \}) \leq \varepsilon \} \), 7.2

\( M(D, G) = \{ x \in Y : D x \subseteq G \} \), 10.2

\( N(\varphi, W) = \{ f \in C_b(X, E) : (\varphi f)(X) \subseteq W \} \), 1.1

\( N(A, W) := N(\chi_A, W) = \{ f \in C_b(X, E) : f(A) \subseteq W \} \), 1.1

\( p \), the pointwise topology on \( C(X, E) \), 1.1

\( p_f \), the seminorm on \( E \) defined by \( p_f(x) = |f(x)|, x \in E \), A.2

\( q_x \), the seminorm on \( E^* \) defined by \( q_x(f) = |f(x)|, f \in E^* \), A.2

\( \mathbb{R} \), the field of real numbers, A.2

\( \rho_A \), the Minkowski functional of \( A \subseteq E \) defined by \( \rho_A(x) = \inf\{ \lambda > 0 : x \in \lambda A \} \), A.2

\( \varphi \otimes a \), the function, 1.1

\( \prod X_\alpha \), the Cartesian product of a family \( \{ X_\alpha : \alpha \in I \} \) of topological spaces with the product topology, A.1

\( \sigma \), the \( \sigma \)-compact-open topology on \( C_b(X, E) \), 1.1

\( \sigma_\tau \), the countable-open topology on \( C_b(X, E) \), 1.1

\( S^+_o(X) \), the set of all non-negative upper semi-continuous functions on \( X \) which vanish at infinity, 1.1

\( S(X, E) \), a vector subspace of \( F(X, E) \), 9.1, 9.1

\( SM(X) \), the collection of all submeasures \( \mu : cz(X) \to [0, \infty] \), 7.1, 7.2
\textit{supp}(f) = \text{closure of the set } \{x \in X : f(x) \neq 0\}, \text{ where } f : X \to E \text{ or } K, \text{ A.1, 1.1}

\text{TA}, \text{ topological algebra, A.6}

\text{TVS}, \text{ topological vector space, 1.1, A.2}

\text{u}, \text{ the uniform topology on } C_b(X,E), \text{ 1.1}

\text{U}(D,W) = \{ T \in \text{CL}(E,F) : T(D) \subseteq W \}, \text{ A.3, A.6, 4.2, 9.1, 10.1}

\text{U}(D,G) = \{ T \in \text{Hom}_A(A,Y) : T(D) \subseteq G \}, \text{ 10.3}

\text{U}_d(D,W) = \{ (S,T) \in M_d(A) : S(D) \subseteq W \text{ and } T(D) \subseteq W \}, \text{ 10.1, } V > 0, \text{ a Nachbin family such that, for each } x \in X, \text{ there is a } v \in V \text{ with } v(x) \neq 0, \text{ 1.2}

\text{VM}_t(\text{Bo}(X),E^*) = \{ vm : v \in V, m \in M_t(\text{Bo}(X),E^*) \}, \text{ 8.2}

\text{w}(E,E^*) \text{ or } \sigma(E,E^*), \text{ the weak topology on a TVS } E, \text{ A.2}

\text{w}(E^*,E) \text{ or } \sigma(E^*,E), \text{ the weak* -topology on the dual space } E^*, \text{ A.2}

\omega, \text{ the first uncountable ordinal, 1.1, 6.1, 6.2}

\omega_V, \text{ the weighted topology on } CV_0(X,E) \text{ or } CV_o(X,E), \text{ 1.2}

\text{W} = \mathcal{W}_E, \text{ a base of neighbourhoods of 0 in a topological vector space } E, \text{ A.2, A.3}

\text{W}_{\pi,\varphi}, \text{ the weighted composition operator on } CV_0(X,E), \text{ } W_{\pi,\varphi}(f) = \pi \cdot f \circ \varphi \text{ with } \varphi : X \to X \text{ and } \pi : X \to L(E), \text{ 9.2}

z(\varphi) = \varphi^{-1}(0) = \{ x \in X : \varphi(x) = 0 \}, \text{ a zero set of a function } \varphi : X \to \mathbb{R}, \text{ A.1, A.7}

z(X), \text{ the collection of all zero sets in } X, \text{ A.7}
APPENDIX C

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