**International Journal of Pure and Applied Mathematics**
**Volume 118 No. 10 2018, 455-463**
ISSN: 1311-8080 (printed version); ISSN: 1314-3395 (on-line version)
**url:** http://www.ijpam.eu
**doi:** 10.12732/ijpam.v118i10.85
Special Issue
ijpam.eu

# Some Properties of Graphs having Induced $nK_2$ Decomposition

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**Abstract**

A decomposition of a graph $G(p, q)$ is a collection of subgraphs $H_1, H_2$, ..., $H_t$ of $G$ such that every edge of G belongs to exactly one $H_i$, for some positive integer $t$. If each $H_i$ is isomorphic to induced $nH$, we say that $G$ has induced $nH$ subgraph decomposition. In this paper we initiate a study on induced $nH$ subgraph decomposition, where $H$ is apparently $K_2$. The bounds for $n$ and the edge-partition number ${\pi_\nu}'(G)$ is attained which helps to estimate an upper bound for the clique number of a graph $G$ having $nK_2$ induced subgraph decomposition.

**AMS Subject Classification:** 05C15, 05C07, 05C42, 05C69

**Key Words and Phrases:** $nK_2$, Induced Subgraph Decomposition, radius and diameter of the graph, edge partition number, the clique number.

## 1 Introduction

A. Bialostocki and Y. Roditty introduced $3K_2$ decomposition of a graph in 1982[1]. Although knotty, it is more appealing to conceptualise induced $nK_2$ subgraph decomposition. Investigation of some inferred ideas from this concept yields powerful results which are devoted to analyse social network, drug prescription and scheduling problems.

By a graph, we refer to a simple graph which is a finite, undirected, non-trivial, connected graph without any multiple edges and loops. The order and size of the graph are denoted by $p$ and $q$ respectively. As in [2], a subgraph $H$ of a graph $G$ is called an *induced subgraph* of $G$ if whenever $u$ and $v$ are vertices of $H$ and $uv$ is an edge of $G$, then $uv$ is an edge of $H$ as well. If $S$ is a nonempty set of vertices of a graph $G$, then the *subgraph of $G$ induced by $S$* is the induced subgraph with vertex set $S$. This induced subgraph is denoted by $\langle S \rangle$. A *decomposition* of a graph $G$ is a collection of edge-disjoint subgraphs $H_1, H_2, ..., H_n$ of $G$ such that every edge of $G$ belongs to exactly one $H_i$. If each $H_i \cong H$, then we say that $G$ has a *$H$-decomposition* and is denoted by $H|G$. If each $H_i$ is an induced subgraph forming $'n'$ copies of $K_2$ in $G$, then we say that, there exist an *$nK_2$ induced subgraph decomposition of $G$*.

We see in [2] that if $u$ and $v$ are vertices of a graph $G$, then the *distance between $u$ and $v$* is the length of any shortest $u - v$ path in $G$ and is denoted by $d(u, v)$. We can see in [4] that *eccentricity of a vertex $u$* is the supremum of distances between $u$ and vertices of $G$. The maximum [minimum] eccentricity of a vertex of $G$ will be called the *diameter* [*radius*] of the graph $G$ and denoted by *diam(G)* [*rad(G)*]. A graph G is *complete*, if every two distinct vertices of G are adjacent.[4] A complete graph of order $r$ is denoted by $K_r$. A *clique* in a graph $G$ is a complete subgraph of $G$. The order of the largest clique in a graph $G$ is its *clique number*, which is denoted by $\omega(G)$.

As seen in [5] if $G$ is a connected graph and $e_1 = (u_1, v_1), e_2 = (u_2, v_2)$ are two edges of $G$, then the *distance between edges or edge distance* of $e_1$ and $e_2$ is defined as $ed(e_1, e_2) = min\{d(u_1, u_2), d(u_1, v_2), d(v_1, u_2), d(v_1, v_2)\}$. If $ed(e_1, e_2) = 0$, then these edges are called *neighbor edges*. In a similar fashion we define the distance between two subgraphs $H_1$ and $H_2$ of $G$ as $d(H_1, H_2) = min\{d(u, v)|u \in V(H_1), v \in V(H_2))\}$. Thereby we can have induced $nH$ subgraph decomposition if the distance between the subgraphs under each subset of decomposition is at least two. Due to inducedness we observe that for $nK_2$ induced subgraph decomposition of $G$, distinct $K_2$'s in each $H_i$ are of edge distance at least two in $G$.

## 2 Graphs with property $\mathcal{P}_n(H)$

For some positive integer $n \geq 2$, we say that a graph $G(p, q)$ has property $\mathcal{P}_n(H)$, similar to the one in [6], if $G$ has induced $nH$ subgraph decomposition, for some subgraph $H$ of $G$. We see that, for every connected graph $G$, Since $G \subseteq G$, $G$ has the property $\mathcal{P}_1(G)$ which is $G$ itself. In this paper we initiate a study on the property $\mathcal{P}_n(H)$, where $H$ is apparently $K_2$. Obviously, every graph of positive size has the property $\mathcal{P}_1(K_2)$. That is, if $G$ is a graph of size $q > 0$, then $G$ has $K_2$ decomposition with $q$ number of subgraphs. Hence, it is worth further to consider only the case $n \geq 2$. Reiterating, for any positive integer $n \geq 2$, we say that a graph $G(p, q)$ has property $\mathcal{P}_n(K_2)$ if $G$ has induced $nK_2$ Subgraph Decomposition.

The following observations readily follows from the afore mentioned definition.

**Observation 1.** *If $G$ is disconnected with $'n'$ components each of size $'t'$, then $G$ has the property $\mathcal{P}_n(K_2)$, with $t$ copies of $nK_2$. For example, if $G$ is disconnected with two components, each being $P_3$, then $G$ has the property $\mathcal{P}_2(K_2)$ with two subgraphs as $2K_2$. Similarly, if $G$ is disconnected with three components each being $C_4$, $P_5$ and $K_{1,4}$, then, $G$ has the property $\mathcal{P}_3(K_2)$ with four subgraphs under decomposition. Also, if $G$ has one more component $T_{3,1}$, then $G$ has the property $\mathcal{P}_4(K_2)$ with four subgraphs under decomposition.*

**Theorem 1.** *Suppose $G(p,q)$ is disconnected with two components $G_1(p_1,q_1)$ and $G_2(p_2,q_2)$ having the property $\mathcal{P}_{n_1}(K_2)$ and $\mathcal{P}_{n_2}(K_2)$ respectively. Then, $G$ has the property $\mathcal{P}_{n_1+n_2}(K_2)$ only if both the components have equal number of subgraphs under decomposition.*

*Proof.* Suppose that the components $G_1(p_1,q_1)$ has $\mathcal{P}_{n_1}(K_2)$ with $t_1$ number of subgraphs and $G_2(p_2,q_2)$ has $\mathcal{P}_{n_2}(K_2)$ with $t_2$ number of subgraphs. Then,

$$q_1 = n_1.t_1 \qquad \text{and} \qquad q_2 = n_2.t_2 \tag{1}$$

Suppose $G$ has $\mathcal{P}_{n_1+n_2}(K_2)$. Then by (1), $q = (n_1+n_2)(t)$ and hence $q_1 + q_2 = (n_1+n_2)(t)$. This implies $n_1t_1 + n_2t_2 = n_1t + n_2t$ and we have,$t_1 = t = t_2$. That is, both the components have equal number of subgraphs under decomposition. □

It is easy to see that $C_6$ is the smallest connected graph with respect to order that has the property $\mathcal{P}_2(K_2)$. This in turn provides an important insight about the radius of the graph holding the property $\mathcal{P}_2(K_2)$. which has to be greater or equal to 3. That is, if $G$ is any connected graph having the property $\mathcal{P}_2(K_2)$, then, $rad(G) \geq 3$. That is, a graph of radius 1 or 2 will not have $\mathcal{P}_2(K_2)$. When radius of the graph is equal to 1, at least one vertex is adjacent to every other vertices. That is, a graph $G$ of radius one has $\Delta(G) = p-1$. Hence, there exists not even a single $2K_2$. When radius of the graph is equal to 2, the longest induced path is $P_5$ and obviously, $P_5$ having size 4 has only one induced $2K_2$. Hence, the graphs with radius 1 or 2 will not have an induced $2K_2$ decomposition. Thus we conclude that if $G$ has $\mathcal{P}_2(K_2)$, then $rad(G) \geq 3$. As far as induced $nK_2$ decomposition is concerned, the following graphs are forbidden.

**Observation 2.** *(Some known graphs which are forbidden)*
*(i)$K_p$, $W_p$, $K_{1,p}$, ... as their radius= $1 < 3$.*
*(ii)$K_{r,s}$, Petersen Graph, Bull graph, $P_4$, $C_4$,...as radius =$2 < 3$.*
*(iii)Any connected graph $G$ having at least one vertex of degree $(p-1)$ or maximum degree $(p-1)$ as radius =1.*
*(iv)Any connected graph $G$ having maximum degree $(p-2)$ as its radius is 2.*

**Observation 3.** *If $G$ is any connected graph with a vertex of maximum degree $\Delta$ having the property $\mathcal{P}_n(K_2)$, we see that $\Delta$ has a crucial role in this decomposition as these $(2\Delta-1)$ edges must lie in different subgraphs. Therefore, if $G$ is any connected graph having the property $\mathcal{P}_n(K_2)$, then the minimum number of subgraphs is at least 3 as $(2\Delta-1) \geq 3$. Also, $q \geq n(2\Delta-1) \Rightarrow q \geq 3n \Rightarrow q \geq 6$ as $n \geq 2$.*

**Observation 4.** *Suppose $G$ is any connected graph with a vertex of maximum degree $\Delta$ having the property $\mathcal{P}_n(K_2)$, then, $q = n.t$, where 't' is the number of subgraphs under decomposition. We have observed that $q \geq n(2\Delta-1)$. So by Hand Shaking Lemma we have, $2q \geq 2.n(2\Delta-1) \Rightarrow \sum_1^p d(v) \geq 2.n.(2\Delta-1) \Rightarrow p.\Delta \geq 2.n.(2\Delta-1) \Rightarrow p \geq \frac{2.n.(2\Delta-1)}{\Delta} \Rightarrow p \geq 3n \Rightarrow p \geq 6$.*

## 3 The Edge-partition number of a Graph

Given a connected graph $G$ and a positive integer $'n'$, we can determine the largest $n$ and correspondingly the fewest number of subgraphs into which $G$ can be decomposed so that each subgraph is according to the property $\mathcal{P}_n(K_2)$ of $G$ . This decomposition is analogous to edge-partitioning of $G$, provided each subgraph is obtained by the property $\mathcal{P}_n(K_2)$ of $G$. Hence, we determine the *edge-partition number* ${\pi_\nu}'(G)$ as the minimum number of subsets into which the edge set of $G$ can be partitioned so that the subgraph induced by each subset is $nK_2$ as defined in II. With respect to each $n$ there exists ${\pi_n}'(G)$, which is the number of subsets into which the edge set of $G$ is partitioned with regard to $\mathcal{P}_n(K_2)$. We call ${\pi_\nu}'(G)$ as the *edge-partition number of* $G$ only if $n$ attains its maximum value $\nu$. As seen in observation(3), ${\pi_\nu}'(G) \geq 3$.

We have observed that if $G$ has $\mathcal{P}_n(K_2)$, then $rad(G) \geq 3$ and so $G$ has at least two subgraphs of diameter 3. For the betterment of extracting the edge-partition number, let us consider all the possible subgraphs of diameter 3 in $G$. Let ${\pi_\nu}'(G)$ be the maximum among the size of all such subgraphs. Each of these ${\pi_\nu}'(G)$ number of edges has to be placed in different copies as they are at edge distance less than 2. Hence, $E(G)$ can be partitioned into at least ${\pi_\nu}'(G)$ number of subsets. Let $\Delta'$ be the maximum degree of the diameter 3 subgraph of size ${\pi_\nu}'(G)$.

We see that this $\Delta'$ too has a crucial role in this decomposition and ${\pi_\nu}'(G)$ completely depends on $\Delta'$ where $2 \leq \Delta' \leq \Delta$. We now establish a result for a tree which yields the upper bound for $n$ in $\mathcal{P}_n(K_2)$ with respect to ${\pi_\nu}'(G)$ for a graph $G$.

## 4 Bounds for $n$ and ${\pi_\nu}'(G)$ of a Graph satisfying $\mathcal{P}_n(K_2)$

As discussed earlier, we see that $n \geq 2$ for a graph $G$ having the property $\mathcal{P}_n(K_2)$. Hence, we now obtain the upper bound for $n$ and the edge partition number through the following proposition and theorems.

**Proposition 1.** *Suppose the graph $G$ having $\mathcal{P}_n(K_2)$ under consideration is a tree. Then, the edge partition number ${\pi_\nu}'(G) = 2\Delta-(i+j+1)$ where $i = 0, 1, 2, ...$ and $j = i, i+2, i+3, ...$ . Also, $2 \leq n \leq \frac{q}{{\pi_\nu}'(G)}$.*

*Proof.* As $G$ has $\mathcal{P}_n(K_2)$, we see that $G$ contains atleast two subgraphs of diameter 3, one of them having maximum size ${\pi_\nu}'(G)$. Since $G$ is a tree, ${\pi_\nu}'(G)$ completely depends on the number of edges adjacent to a vertex of maximum degree and one of its neighbor in the diameter 3 subgraph. We now calculate ${\pi_\nu}'(G)$ in the following manner.

For $k \in N$ let $H_k$ be a maximal subgraph of diameter 3 and $u$ be a vertex of maximum degree in $H$. As $G$ has $\mathcal{P}_n(K_2)$, $u$ has at least one adjacent vertex say, $v$, which is not a pendant vertex. For $i = 0, 1, 2, ..., \Delta - 2$, if

$$\begin{aligned} d(u) &= \Delta - i;\ for\, some\, i \\ d(v) &= \Delta - j;\ for\, some\, j\, where\, j = i, i+1, ... \\ \textit{then, set}\quad x_{ij} &= 2\Delta - (i+j+1) \end{aligned}$$

Where $x_{ij}$ is the size of a diamter 3 subgraph in $G$. Consider the edge partition number of $G$ as ${\pi_\nu}'(G) = max\{x_{ij}\}$. Clearly, the subgraph $H_k$, for some $k$, formed by $u, v$ and its neighbors is of diameter 3 with maximum size ${\pi_\nu}'(G)$ and maximum degree $d(u) = \Delta' = \Delta - i$.

Since $G$ has the property $\mathcal{P}_n(K_2)$, we see that $q = \nu.({\pi_\nu}'(G))$ where, $\nu$ is the largest value of $n$. Hence, we obtain the bound for $n$ as $2 \leq n \leq \frac{q}{{\pi_\nu}'(G)}$, where ${\pi_\nu}'(G) = max\{x_{ij}\}$. □

**Corollary 1.** *The path graph $P_k$ satisfies the property $\mathcal{P}_n(K_2)$ iff $k = n.{\pi_n}'(P_k) + 1$ where $2 \leq n \leq \lfloor \frac{k-1}{3} \rfloor$ and ${\pi_n}'(P_k) \geq 3$.*

*Proof.* Suppose that $P_k$ has the property $\mathcal{P}_n(K_2)$. Then, by the previous proposition we see that $d(u) = \Delta' = \Delta = 2$ and ${\pi_\nu}'(P_k) = 2\Delta - 1 = 3$ or ${\pi_n}'(P_k) \geq 3$. Hence, $2 \leq n \leq \lfloor \frac{q}{2\Delta-1} \rfloor = \lfloor \frac{k-1}{3} \rfloor$. Since $P_k$ has the property $\mathcal{P}_n(K_2)$, we have $k - 1 = \nu.({\pi_\nu}'(P_k))$ or $k = \nu.({\pi_\nu}'(P_k)) + 1$. In general, $k = n.({\pi_n}'(P_k)) + 1$ where, $2 \leq n \leq \lfloor \frac{k-1}{3} \rfloor$ Conversely, consider the path graph $P_k$ such that $k = n.{\pi_n}'(P_k) + 1$ where $2 \leq n \leq \lfloor \frac{k-1}{3} \rfloor$. From the bounds for $n$, we have $k \geq 3n + 1 \Rightarrow k - 1 \geq 6 \Rightarrow rad(P_k) \geq 3$. $P_k$ has at least 6 edges and radius at least 3 implies that $P_k$ has the property $\mathcal{P}_n(K_2)$ for $2 \leq n \leq \lfloor \frac{k-1}{3} \rfloor$. □

**Corollary 2.** *A cycle $C_k$ has the property $\mathcal{P}_n(K_2)$ iff $k = n.\Pi'_n(k)$ where $2 \leq n \leq \lfloor \frac{k}{3} \rfloor$.*

*Proof.* The proof is similar to that of corollary 1. □

**Remark 1.** *The largest value of n in $P_k$ and $C_k$ are $\lfloor \frac{k-1}{3} \rfloor$ and $\lfloor \frac{k}{3} \rfloor$ respectively. However, the Edge-partition number of a path graph and that of a cycle is, $\pi'_\nu(P_k) = \pi'_\nu(C_k) = 3$.*

**Theorem 2.** *If the acyclic graph $T(p, q)$ has the property $\mathcal{P}_n(K_2)$, then*
*(i)* $n \mid q$ *such that* $2 \leq n \leq \frac{q}{{\pi_\nu}'(T)}$.
*(ii)* $diam(G) \geq 3 + (n - 1)[{\pi_\nu}'(T) - \Delta' + 2]$.

*Proof.* As $T(p, q)$ is acyclic having the property $\mathcal{P}_n(K_2)$, by proposition(1), condition (i) is proved.

Now to prove condition(ii). Let $u, v \in V(T)$ are two adjacent vertices in $T$ such that $deg(u) + deg(v)$ is maximum and ${\pi_\nu}'(T)$ be the edge partition number of $T(V, E)$. Consider $V'(T) = N_{[u,v]}$. Let $H(V', E')$ be the subgraph of diameter 3 induced by $V'$ with maximum degree $d(u) = \Delta'$ and $|E'| = {\pi_\nu}'(T)$. Clearly, $V'(T) \subset V(T)$ and $H$ is not a spanning subgraph of $T(V, E)$. Hence, $d(T) \geq d(H)$. To have $nK_2$ decomposition in T, $d(H)$ is increased by $(n - 1)$ for $(\Delta' - 1)$ edges adjacent to $u$ of $H$ and for the remaining $[{\pi_\nu}' - (\Delta' - 1)]$ edges, the diameter is increased $(n-1)$times. Hence we have,

$$\begin{aligned} d(T) &> d(H) \\ d(T) &\geq d(H) + (n-1) + (n-1)[\pi_\nu{}' - (\Delta' - 1)]] \\ &\geq d(H) + (n-1)[\pi_\nu{}' - \Delta' + 2] \\ &\geq 3 + (n-1)[\pi_\nu{}' - \Delta' + 2] \end{aligned}$$

□

**Observation 5.** *From various examples, we have seen that if a graph $G(p,q)$ has the property $\mathcal{P}_n(K_2)$, then the Density of the graph $D = \frac{2q}{p(p-1)} < 0.3$. The Density measures how many edges are in the edge set $E$ compared to maximum possible number of edges between vertices in set $V$. [7] Obviously, as the density approaches 1, q approaches $\frac{p(p-1)}{2}$ which is a forbidden case of $\mathcal{P}_n(K_2)$. Hence it can be inferred that when the density $D \geq 0.3$, the possibility of $G$ having $\mathcal{P}_n(K_2)$ reduces. The average degree of a graph $G$, $\frac{2q}{p}$ is another measure of how many edges are in set $E$ compared to number of vertices in the set $V$. [7] Hence we can conclude that if $G$ has $\mathcal{P}_n(K_2)$, then the average degree is less than $0.3(p-1)$.*

**Theorem 3.** *If the graph $G(p,q)$ is not acyclic, having the property $\mathcal{P}_n(K_2)$, where both $p$ and $q$ are at least six and the average degree is less than $0.3(p-1)$, then*
*(i) $n \mid q$ such that $2 \leq n \leq \frac{q}{\pi_\nu{}'(G)}$*
*(ii)$rad(G) \geq \lfloor \frac{3n}{2} \rfloor$.*

*Proof.* Consider a graph $G(p,q)$ having property $\mathcal{P}_n(K_2)$. Then $G$ has $nK_2$ induced subgraph decomposition. First we prove condition (i). Each subgraph $H_i \approx nK_2 \Rightarrow n \mid q$. Obviously, $n \geq 2$. Since $G$ is not acyclic, $G$ will have at least one of the graphs under case(1), (2) and (3) as a proper subgraph. If $G$ itself is a cycle, then by corollary 2, condition(i) is proved. Else we extract the edge partition number, $\pi_\nu{}'(G)$ from certain maximal subgraphs of diameter 3 in the following manner.

**Case 1:** Suppose $G$ contains a cycle $C_m$ as a proper subgraph. Let $u_i \in V(G)$ lie on a cycle $C_m$ such that $d(u_i) = d(u) = \Delta'$, for some $i$ and $\Delta' = \Delta, \Delta-1, \Delta-2, ..., 3$. Let $w \in V(G)$ such that $d(w)$ is maximum $\forall w \in N(u)$ .
By assuming $s = \sum_{i=1}^{m} d(u_i)$, consider

$$x_{im} = \begin{cases} s - m & ; \quad m = 3 \\ d(u) + d(w) & ; \quad 3 < m < 6 \\ d(u) + d(w) - 1 & ; \quad m \geq 6 \end{cases}$$

**Case 2:** Let $u_i \in V(G)$ lie on a complete graph $K_r$ such that $d(u_i) = d(u) = \Delta$ for some $i$ and $\Delta' = \Delta, \Delta-1, \Delta-2, ..., 3$. Then,

$$x_{ir} = \frac{2s - r^2 + r}{2}$$

where $s = \sum_{i=1}^{m} d(u_i)$.
**Case 3:** Let $u \in V(G)$ lie on a Wheel graph $W_{1,k}$ ; $k \geq 3$ such that $d(u) = \Delta$ for some $i$ and $\Delta^{'} = \Delta, \Delta - 1, \Delta - 2, ..., 3$. Then

$$x_{ik} = \begin{cases} s - k & ; \quad k \leq 5 \\ \Delta^{'} + 3 & ; \quad k \geq 6 \end{cases}$$

where $s = \sum_{i=1}^{m} d(u_i)$. Then $\pi_{\nu}{}^{'}(G) = max\{x_{im}, x_{ir}, x_{ik}\}$ is the required edge partition number. As $G$ has $\mathcal{P}_n(K_2)$, we see that $q \geq n\pi_{\nu}{}^{'}(G) \Rightarrow n \leq \frac{q}{\pi_{\nu}{}^{'}(G)}$. Hence, condition(i) is proved.

We now prove condition (ii). Let $H$ be the diameter 3 induced subgraph of $G$ with $u$ having the maximum degree $\Delta^{'}$ such that $|E(H)| = \pi_{\nu}{}^{'}(G)$. As $G$ has $\mathcal{P}_n(K_2)$, we see that $G$ must have the property $\mathcal{P}_{\nu}(K_2)$ or $\nu K_2$ induced decomposition by condition (i). By the inducedness, only one edge or $K_2$ can be selected from $H$ to be placed under a subset of decomposition. So to have $\nu K_2$ decomposition, $\nu$ copies of $K_2$ cannot be extracted unless we have $\nu$ number of edge disjoint diameter 3 subgraphs. That is,

$$diam(G) \geq 3\nu$$
$$\Rightarrow rad(G) \geq \lfloor \frac{3\nu}{2} \rfloor$$

In general,

$$rad(G) \geq \lfloor \frac{3n}{2} \rfloor$$

□

## 5 The Clique Problem

If $G$ is a graph with property $\mathcal{P}_n(K_2)$, then we can estimate a bound for the clique number, $\omega(G)$. So, we can determine the maximum clique possible in $G$. In social networks cliques represent group of people who know each other. Hence it is fascinating indeed to retrieve the strength of an organisation or community in a particular area.

**Theorem 4.** *Suppose $G$ is a graph having the property $\mathcal{P}_n(K_2)$. Then the clique number $\omega(G) < \frac{1+\sqrt{1+8\pi_{\nu}{}^{'}(G)}}{2}$.*

*Proof.* Suppose $G$ has property $\mathcal{P}_n(K_2)$. Then, by the theorem 2, there exist induced $nK_2$ decomposition of $G$ where, $2 \leq n \leq \frac{q}{\pi_{\nu}^{'}(G)}$. We see that, $\nu = \frac{q}{\pi_{\nu}^{'}(G)}$ is the largest value of $n$ and $\pi_{\nu}^{'}(G)$ is the edge-partition number of $G$. Let $K_r$ be a clique in $G$. Clearly, $G \neq K_r$ ;for $rad(G) = 1$ contradicts the existence of property $\mathcal{P}_n(K_2)$ in $G$. Hence, $K_r \subset G$. Then the maximal subgraph $H^{''}$ induced by $V[K_r]$ and its neighbors has diameter 3. However, the size of $H^{''}$ must be less than or equal to $\pi_{\nu}{}^{'}(G)$ as $\pi_{\nu}{}^{'}(G)$ is the maximum among the size of subgraphs having diameter

3. Clearly, $K_r \subset H^{''}$ and hence

$$|E(K_r)| < |E(H^{"})| \leq {\pi_\nu}'(G)$$
$$\binom{r}{2} < {\pi_\nu}'(G)$$
$$r < \frac{1 + \sqrt{1 + 8{\pi_\nu}'(G)}}{2}$$

That is, $\omega(G) < \frac{1+\sqrt{1+8{\pi_\nu}'(G)}}{2}$. □

## Conclusion

The key condition that is enforced here is 'inducedness' and implemented with $nK_2$ decomposition of a graph $G$ and thus evolving the concept of the Edge-partition number of a graph, $\pi_\nu^{'}(G)$. Also, for the graphs having property $\mathcal{P}_n(K_2)$ the bounds for $'n'$ and the clique number $\omega(G)$ is attained.

### 6 Acknowledgment

We thank our reviewers and referees for their constructive comments and suggestions.

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