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Zagreb Radio Indices of Graphs

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Abstract

Radio coloring is a distanced based vertex coloring such that the color difference of any two distinct vertices in the graph must be at least the difference of the diameter of the graph and the distance between the vertices plus one. Zagreb index is a topological index defined for the degrees of the vertices of a graph. We introduce the analogical Zagreb radio indices for graphs. The first and second Zagreb radio indices of some popular graphs are computed.

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1 Introduction

Graph labeling[4] is an ever-growing area of research in graph theory. It is an assignment of labels to the vertices or edges or both of a graph. Integers are the most famous labels as operations on them are relatively easier. Graph coloring is a special type of graph labeling. A graph coloring is said to be proper coloring if no two adjacent vertices have the same color. For the easiness of computing, we assign numbers in the place of colors.

There are numerous types of vertex colorings. For basic concepts in graph theory and that of colorings we refer to [1]. Radio coloring is a type of vertex coloring. It can be presented as a function c from V(G) to the set of nonnegative integers such that $|c(u) - c(v)| \geqslant diam(G) + 1 - d(u, v)\ \forall\ u, v \in V(G)$. Here, $d(u, v)$ denotes the distance between the vertices u and v and $diam(G)$ is the maximum of all such distances mentioned as the diameter of the graph G. The maximum value assigned to a vertex by c is the c-span of G. The minimum of all c-spans is defined as the *radio number* ($rn(G)$) of G.[1] The term radio coloring is due to its connection with channel assignment problem.[8]

Graph theory got a lot of applications in the field of chemistry. Structural formulas of covalently bonded atoms can be represented as graphs called molecular graphs or constitutional graphs. For quantitative structure-activity relationships (QSAR), which are important especially for drug design, one looks for correlations between physical, chemical, or biochemical properties (possessing a continuous variation and being expressed numerically) and chemical structures. For this purpose a different approach was proposed by H. Wiener in 1947 with the introduction of topological indices. Topological indices are numerical values associated with a graph which determines the topology of the graphs.[11]

Among a host of topological indices, we deal only with the Zagreb indices of a graph. Zagreb index is a topological index defined for the degrees of the vertices.[5] The first Zagreb index is $M_1(G) = \sum_v d(v)^2$. The second Zagreb index is $M_2(G) = \sum_{uv \in E(G)} d(u)d(v)$.

We, now, introduce the analogical Zagreb radio indices for graphs.

Definition 1. The First Zagreb Radio Index is $M_{rc1}(G) = \sum_{u \in V(G)} c(u)^2$ where c is the radio coloring whose span is the *radio number* of the graph G.

Definition 2. The Second Zagreb Radio Index is $M_{rc2}(\text{G}) = \sum_{uv \in E(G)} c(u)c(v)$ where c is the radio coloring whose span is the *radio number* of the graph and $\sum_{uv \in E(G)} c(u)c(v)$ is the minimum.

The first and second Zagreb radio indices of some popular graphs are computed in the subsequent sections. The $forgotten$ topological index is relatively high in magnitude in comparison with the first and second Zagreb indices.[10] The first and second Zagreb radio indices give a higher quantity closer to the $forgotten$ topological index.

2 Radio Number of Certain Graphs

In this section, we provide the *radio numbers* of certain graphs.

Definition 3. A **gear graph**, G_n, is obtained from a wheel graph, W_n, by adding a vertex to each edge of the cycle C_n.[3]

The *radio numbers* some popular graphs are given in the next table.[1],[3]

Graph	**Vertices**	**Radio Number**
Star graph ($\text{K}_{1,n}$)	$n+1$	$n+1$
Complete graph (K_n)	n	$n-1$
Complete bipartite graph ($\text{K}_{m,n}$)	$m+n$	$m+n$
Wheel graph (W_n)	$n+1$	$n+1$
Gear graph (G_n)	$2n+1$	$4n+2$

3 Main Results

In radio coloring, we use *zero* also as a color. It is to be noted that the conditions of radio coloring obviously imply that every vertex in a graph receives distinct color.

However, they need not be consecutive.

3.1 Zagreb radio indices of stars

Theorem 4. *The first Zagreb radio index of a star, $M_{rc1}(K_{1,n}) = \frac{(n+1)(n+2)(2n+3)}{6} - 1$ for $n \geqslant 2$.*

Proof. We know that for n $\geqslant$ 2, diam($K_{1,n}$) = 2. Moreover, *radio number*, $rn(K_{1,n})$ = n+1. This implies that the colors are $0, 2, 3, \ldots, n+1$. Therefore,

$$\begin{aligned} M_{rc1}(\mathrm{K}_{1,n}) &= \sum_{u\in V(K_{1,n})} c(u)^2 \\ &= 0^2 + 2^2 + ... + (n+1)^2 \\ &= \frac{(n+1)(n+2)(2n+3)}{6} - 1 \end{aligned}$$

Hence, the result. □

Theorem 5. *The second Zagreb radio index of a star graph, $M_{rc2}(K_{1,n}) = 0$ for all n$\geqslant$ 2.*

Proof. We consider star graphs with at least three vertices. We have, diam($K_{1,n}$) = 2. The optimal radio coloring is possible when the central vertex is given color zero and other vertices, the colors 2,3,. . . ,n+1. Hence, $rn(K_{1,n})$ = n+1.

We have, $M_{rc2}(K_{1,n}) = \sum_{uv\in E(K_{1,n})} c(u)c(v)$. Since every edge of $K_{1,n}$ is incident to the central vertex $c(u)c(v)$ is 0 for all uv $\in E(K_{1,n})$. So, $M_{rc2}(K_{1,n}) = 0$ for all n$\geqslant$ 2. □

3.2 Zagreb radio indices of complete graphs

Theorem 6. *The first Zagreb radio index of complete graph, $M_{rc1}(K_{1,n}) = \frac{(n)(n-1)(2n-1)}{6}$ for $n \geqslant 3$.*

Proof. We know that diam(K_n) = 1, since n $\geqslant$ 3. We can rewrite the radio coloring condition as $|c(u) - c(v)| \geqslant 1$. So, the vertices can be colored with the consecutive integers from 0 to $n-1$. Hence, $rn(K_n) = n - 1$. Therefore,

$$\begin{aligned} M_{rc1}(\mathrm{K}_n) &= \sum_{u\in V(K_{1,n})} c(u)^2 \\ &= 0^2 + 1^2 + 2^2 + ... + (n-1)^2 \\ &= \frac{(n)(n-1)(2n-1)}{6} \end{aligned}$$

□

Theorem 7. *The second Zagreb radio index of complete graph, $M_{rc2}(K_n) = \frac{(n-1)(n-2)(3n^2-n)}{24}$ $\forall$ $n \geqslant 3$.*

Proof. Let $v_0, v_1, ..., v_{n-1}$be the vertices of K_n. Let c be a coloring such that c(v_i) = i. As $|c(v_i) - c(v_j)| \geqslant 1$ for i $\neq$ j, the coloring c is a required radio coloring. Since every vertex is adjacent to every the other vertex it will satisfy the minimality condition also.

We have to find $\sum c(v_i)c(v_j)$ for all $v_iv_j \in$ E(K_n). However, $c(v_0)c(v_i) = 0\ \forall$ i. Therefore,

$$\begin{aligned}
M_{rc2}(K_n) &= \sum_{uv\in E(K_n)} c(u)c(v) \\
&= \sum c(v_i)c(v_j)\ where\ i \neq j,\ and\ i\ and\ j \neq 0 \\
&= \sum ij \\
&= \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} ij\ (since\ every\ edge\ is\ counted\ exactly\ once) \\
&= \sum_{i=1}^{n-2} i \sum_{j=i+1}^{n-1} j \\
\text{But, } \textstyle\sum_{j=i+1}^{n-1} j &= \frac{n(n-1)}{2} - \frac{i(i+1)}{2} \\
\text{So, } M_{rc2}(K_n) &= \sum_{i=1}^{n-2} i(\frac{n(n-1)}{2} - \frac{i(i+1)}{2}) \\
&= \sum_{i=1}^{n-2} i(\frac{n(n-1)}{2} - \frac{i(i+1)}{2}) \\
&= \sum_{i=1}^{n-2} i\frac{n(n-1)}{2} - \sum_{i=1}^{n-2} k^2\frac{k^2(k+1)}{2} \\
&= \frac{n(n-1)}{2}\frac{(n-1)(n-2)}{2} - \frac{1}{2}(\sum_{i=1}^{n-2} k^3 + \sum_{i=1}^{n-2} k^2) \\
&= \frac{n(n-1)}{2}\frac{(n-1)(n-2)}{2} - \frac{1}{2}(\frac{(n-1)(n-2)}{2})^2 \\
&- \frac{1}{2}(\frac{(n-1)(n-2)(2n-3)}{6}) \\
&= \frac{(n-1)(n-2)}{24}(3n^2 - n)
\end{aligned}$$

Hence, the result. □

3.3 Zagreb radio indices of complete bipartite graphs

Theorem 8. *The second Zagreb radio index of complete bipartite graphs,* $M_{rc2}(K_{m,n}) = \frac{mn}{4}\ (2n^2 + mn - (m+n) - 1)$ *where* $n \geqslant m \geqslant 2$.

Proof. Consider a complete bipartite graph $K_{m,n}$. Let X and Y be the partite sets of the vertex set. Let $|X|$ = m and $|Y|$ = n. Let the vertices of X be

$v_0, v_1, ..., v_{m-1}$ and the vertices of Y be $u_0, u_1, ..., u_{n-1}$. As in teh previous theorem, we can see that the coloring c given by c(u_i) = i and c(v_i) = i + n + 1, is the required radio coloring. Since each vertex of X is adjacent to all the vertices of Y, this will satisfy the minimality condition also. We compute now $M_{rc2}(K_{m,n})$.

We have,

$$\begin{aligned}
M_{rc2}(K_{m,n}) &= \sum_{uv \in E(K_{m,n})} c(u)c(v) \\
&= \sum_{i=1}^{n-1} c(u_i) \sum_{j=0}^{m-1} c(v_j) \\
&= \sum_{i=1}^{n-1} i \sum_{j=0}^{m-1} (n+1+j) \\
&= \sum_{i=1}^{n-1} i(n+1+n+2+...+n+m) \\
&= \sum_{i=1}^{n-1} i(nm+(1+2+...+m)) \\
&= \frac{n(n-1)}{2}(nm+\frac{m(m+1)}{2}) \\
&= \frac{n^2m(n-1)}{2}+\frac{nm(n-1)(m+1)}{4} \\
&= \frac{2n^2m(n-1+nm(n-1)(m+1)}{4} \\
&= \frac{mn}{4}(2n^2+mn-(m+n)-1)
\end{aligned}$$

Hence the result. □

Theorem 9. *The first Zagreb radio index of complete bipartite graph, $M_{rc1}(K_{m,n})$ $= \frac{2(n+m)^3+3(n+m)^2+(n+m)}{6} - n^2$ where n ⩾ m ⩾ 2.*

Proof. Consider a complete bipartite graph $K_{m,n}$. Given that n ⩾ m ⩾ 2. Let X and Y be the partite sets of the vertex set. Let $|X|$ = m and $|Y|$ = n. Let the vertices of X be $v_0, v_1, \ldots, v_{m-1}$ and the vertices of Y be $u_0, u_1, \ldots, u_{n-1}$. Define a coloring c such that c(u_i) = i and c(v_i) = i + n + 1. It is not difficult to check that this coloring is the required optimal radio coloring. Hence, $rn(K_{m,n})m + n$.

Therefore,

$$\begin{aligned}
M_{rc1}(K_{m,n}) &= \sum_{u\in V(K_{m,n})} c(u)^2 \\
&= 0^2+1^2+...+(n-1)^2+(n+1)^2+...+(m+n)^2 \\
&= 1^2+...+(m+n)^2-n^2 \\
&= \frac{(m+n)(m+n+1)(2(m+n)+1)}{6}-n^2 \\
&= \frac{((m+n)^2+(m+n))(2(m+n)+1)-6n^2}{6} \\
&= \frac{2(n+m)^3+3(n+m)^2+(n+m)}{6}-n^2
\end{aligned}$$

□

3.4 The first Zagreb radio indices of wheel graphs and gear graphs

Theorem 10. *The first Zagreb radio index of wheel graph, $M_{rc1}(W_n) = \frac{(n+1)(n+2)(2n+3)}{6} - 1$ for $n \geqslant 5$.*

Proof. Let z be the central vertex and $v_1, v_2, ..., v_n$ be the vertices on the n-cycle. Consider the coloring c such that c(z) $= 0$, the vertices $v_1, v_2, ..., v_{\lceil \frac{n}{2} \rceil}$ are assigned with consecutive even numbers beginning with 2 and the vertices $v_{\lceil \frac{n}{2} \rceil+1}, .., v_n$ are labeled with consecutive odd integers beginning with 3. It is not difficult to check that this is the required optimal radio coloring. Thus, $rn(W_n) = n+1$. We compute $M_{rc1}(W_n)$ analysing two cases.

Case 1: n = 2k+1
Given $c(z) = 0$.

The vertices $v_1, v_2, ..., v_{k+1}$ are assigned the colors $2, 4, ..., 2(k+1)$ respectively and the vertices $v_{k+2}, ..., v_{k+i}, ..., v_n$ will be given the colors $3, 5, ..., 2i-1, ..., 2k+1$, respectively. Therefore,

$$\begin{aligned}
M_{rc1}(W_n) &= \sum_{u\in V(W_n)} c(u)^2 \\
&= 2^2+3^2+...+(2k+1)^2+(2k+2)^2 \\
&= 2^2+3^2+...+n^2+(n+1)^2 \\
&= \frac{(n+1)(n+2)(2n+3)}{6}-1
\end{aligned}$$

Case 2: n = 2k
Given $c(z) = 0$. The vertices $v_1, v_2, ..., v_k$ are assigned the colors $2, 4, ..., 2k$ respectively and the vertices $v_{k+1}, v_{k+2}, ..., v_{k+i}, ..., v_n$ will be given the colors $3, 5, ..., 2i +$

$1, ..., 2k+1$, respectively. Therefore,

$$\begin{aligned}
M_{rc1}(W_n) &= \sum_{u \in V(W_n)} c(u)^2 \\
&= 2^2 + 3^2 + ... + (2k)^2 + (2k+1)^2 \\
&= 2^2 + 3^2 + ... + n^2 + (n+1)^2 \\
&= \frac{(n+1)(n+2)(2n+3)}{6} - 1
\end{aligned}$$

Hence, $M_{rc1}(W_n) = \frac{(n+1)(n+2)(2n+3)}{6} - 1$. □

Theorem 11. *The first Zagreb radio index of gear graph on 2n+1 vertices is given by,* $M_{rc1}(G_n) = \frac{22n^3+45n^2+35n}{3}$ *for* $n \geqslant 7$.

Proof. Let the central vertex be labeled z and the vertices adjacent to the center are labeled sequentially $v_1, v_2, ..., v_k$. The vertices not adjacent to the center are labeled sequentially $w_1, w_2, ...w_n$, using the same orientation chosen for the v_i?s. If n is odd then we specify that w_1 is adjacent to v_1 and v_2, otherwise w_1 is adjacent to v_1 and v_n.

To make the labeling simple, Fernandez *et al.*[3] relabeled the vertices as $x_0, x_1, x_2, \ldots, x_{2n}$ corresponding to $z, w_1, w_3, \ldots, w_n, w_2, w_4, \ldots, w_{n-1}, v_1, v_2, \ldots v_n$ when n is odd and to $z, w_1, w_3, \ldots, w_{n-1}, w_2, w_4, \ldots, w_n, v_1, v_2, \ldots v_n$ when n is even. The optimal radio coloring c given in [3] is as follows. The central vertex is given the color 0. For $1 \leqslant i \leqslant n$, $c(x_i) = 2 + i$ and for $n+1 \leqslant i \leqslant 2n$, $c(x_i) = n + 1 + 3(i - n)$.

Therefore,

$$M_{rc1}(G_n) = 3^2 + 4^2 + ... + (2k+3)^2 + (n+4)^2 + (n+7)^2 + ... + (4n+1)^2$$

Consider the following summations.

$$3^2 + 4^2 + ... + (2k+3)^2 = \frac{(2k+3)(2k+4)(4k+7)}{6} - 5 \quad \textbf{(1)}$$

$$\begin{aligned}
(n+4)^2 + (n+7)^2 + ... + (4n+1)^2 &= \sum_{i=n+1}^{2n} (n+1+3(i-n))^2 \\
&= \sum_{i=n+1}^{2n} (n+1)^2 + 9(i-n)^2 + 6(n+1)(i-n) \\
&= n(n+1)^2 + \sum_{i=n+1}^{2n} 9(i-n)^2 + 6(n+1)\sum_{i=n+1}^{2n}(i-n) \\
&= n(n+1)^2 - 6n^2(n+1) + \\
&\quad 9\sum_{i=n+1}^{2n}(i^2 + n^2 - 2ni) + 6(n+1)\sum_{i=n+1}^{2n} i \quad \textbf{(2)}
\end{aligned}$$

$$\begin{aligned}
\sum_{i=n+1}^{2n}(i^2+n^2-2ni) &= \frac{2n(2n+1)(4n+1)}{6}-\frac{n(n+1)(2n+1)}{6}+ \\
&\quad n^3-2n(\frac{2n(2n+1)}{2}-\frac{n(n+1)}{2}) \\
&= \frac{2n^3+3n^2+n}{6} \qquad \textbf{(3)}
\end{aligned}$$

Finally,

$$\begin{aligned}
6(n+1)\sum_{i=n+1}^{2n} i &= 6(n+1)(\frac{3n^2+n}{2}) \\
&= 9n^3+12n^2+3n \qquad \textbf{(4)}
\end{aligned}$$

Substituting **(3)** and **(4)** in **(2)** and simplifying we get,

$$\begin{aligned}
(n+4)^2+(n+7)^2+...+(4n+1)^2 &= 7n^3+\frac{25}{2}n^2+\frac{11}{2}n \quad \textbf{(5)} \\
\textbf{(1)}+\textbf{(5)} &\Rightarrow \frac{(2k+3)(2k+4)(4k+7)}{6}-5+7n^3+\frac{25}{2}n^2+\frac{11}{2}n \\
&= \frac{(n+2)(n+3)(2n+5)}{6}-5+7n^3+\frac{25}{2}n^2+\frac{11}{2}n \\
\text{Hence, } M_{rc1}(G_n) &= \frac{22n^3+45n^2+35n}{3}
\end{aligned}$$

□

4 Conclusion

Analogous to the Zagreb indices we have defined the Zagreb radio indices based on the radio coloring of the vertices of a graph. As the indices in many cases are much larger than that of the first and second Zagreb indices and are closer to the $forgotten$ topological index, they might help the QSAR studies of molecular graphs.

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