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Super Vertex Sum Labeling of Graphs

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Abstract

Let G(p, q) be a graph and f be bijective map $f : V(G) \longrightarrow \{1, 2, ..., |V(G)|\}$. For any two vertices u, $v \in V(G)$ with $d(u, v) = 1$ then $f(u) + f(v) = f(w)$ for some vertex $w \in V(G)$. Let μ_f be the maximum $f(u)$ for vertex $u \in V(G)$. If $\mu_f(G) = |V(G)|$ then f is called Super Vertex Sum Labeling. Super Vertex Sum Labeling will be disconnected for we need isolates to super vertex sum label a graph. The labeling with minimum number of isolates is called *Optimal.* The least number of isolates need to super vertex sum label the graph is called *super vertex sum number* of the graph. It is denoted by $\sigma_{sv}(G)$. In this paper, we will give lower bound of $\sigma_{sv}(G)$ and optimal *Super vertex sum labeling scheme* for super subdivision of path, cycle, star and spider.
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1 Introduction

Let G or $G(V, E)$, $V(G)$ and $E(G)$ be a simple undirected finite graph, it set of vertices and its set of edges respectively. We refer to [1] and [2] for concepts and terminologies that are used in this paper that are not explained in detail. The sum labeling of graphs was introduced by Harary [3] in 1990. MacDougall *et al.*[4] introduced the notion of a vertex-magic total labeling in 1999. Following definitions are useful for the present study.

Definition 1.1. A *sum labeling* is an injective mapping f from the vertices of G into the positive integers such that, for any two vertices u, $v \in V(G)$ with labels $f(u)$ and $f(v)$, respectively, uv is an edge iff $f(u) + f(v)$ is the label of another vertex in $V(G)$. Any graph supporting such a labeling is called a *Sum Graph.*[3]

Definition 1.2. The disconnected component added to sum label a graph is a set of isolated vertices defined as *Isolates.* The labeling scheme that requires the fewest isolates is termed *Optimal.*[3]

Definition 1.3. The number of isolates required for a graph G to support a sum labeling is known as the *Sum Number* of the graph. It is denoted as $\sigma(G)$.[3]

Definition 1.4. For a graph G(V,E) an injective mapping f from V ∪ E to the set $\{1, 2, ..., |V| + |E|\}$ is *vertex-magic total labeling* if there exist constant k called the magic constant, such that for every vertex v, $f(v) + \sum f(uv) = k$, where the sum is over all the vertices u adjacent to v.[4]

Definition 1.5. A *super vertex-magic total labeling* of a graph $G(V, E)$ is defined as *vertex-magic total labeling* f of G with the additional property that $f(V) = \{1, 2, ..., |V|\}$ and $f(E) = \{|V| + 1, |V| + 2, ..., |V| + |E|\}$.[5]

Definition 1.6. Let G be a graph with q edges. A graph H is called a *super subdivision* of G if H is obtained from G by replacing every edge e_i of G by a complete bipartite graph K_{2,m_i} for some m_i, $1 \leq i \leq q$ in such a way that the end vertices of each e_i are identified with the two vertices of 2-vertices-part of K_{2,m_i} after removing the edge e_i from graph G. If m_i is varying arbitrarily for each edge e_i, then super subdivision is called *arbitrary super subdivision* of G.[6]

2 Super Vertex Sum Labeling

Definition 2.1. Let $G(p, q)$ be a graph and f be a bijective map $f : V(G) \longrightarrow \{1, 2, ..., |V(G)|\}$. For any two vertices u, $v \in V(G)$ with $d(u, v) = 1$ then $f(u) + f(v) = f(w)$ for some vertex $w \in V(G)$. Let μ_f be the maximum $f(u)$ for vertex $u \in V(G)$. If $\mu_f(G) = |V(G)|$ then f is called a *super vertex sum labeling.*

Definition 2.2. Super vertex sum labeling will be disconnected for we need isolates to super vertex sum label a graph. The least number of isolates needed to super vertex sum label the graph G is called *super vertex sum number* of the graph, denoted by $\sigma_{sv}(G)$. The labeling with the minimum number of isolates is called *optimal.*

Definition 2.3. A graph that admits super vertex sum labeling is called as *super vertex sum graph.*

Theorem 2.1. *A super vertex sum graph G is disconnected.*

Proof. Let G be a super vertex sum graph with n vertices. i.e., $|V(G)| = n$. By definition, $\mu_f(G) = f(u) = n$ for some vertex $u \in V(G)$. Suppose G is connected. Being connected, u is adjacent to some other vertex x with label less than n.

Then either

$$f(u) = n + 0 \tag{1}$$

or

$$f(u) = n + f(x) > \mu_f(G) \tag{2}$$

Since 0 is not an element of $\{1, 2, ..., |V(G)|\}$, 1 is not possible. Hence, 2 must be true which is a contradiction.

Therefore, G is disconnected. i.e., there exists at least one isolated vertex. □

Theorem 2.2. *Let G is a super vertex sum graph. Then $\sigma_{sv}(G)$ is bounded below by the minimum degree, $\delta(G)$, of the connected component of G.*

Proof. Let $G(p, q)$ be a super vertex sum graph and u be a vertex with the maximum label in the connected component of G. Then u will have at least $\delta(G)$ adjacent vertices. By definition, if d(u,v) = 1 then *f(u)+f(v)=f(w)*, ∀ u,v ∈ V(G). Therefore, we will have at least $\delta(G)$ isolates to super vertex sum label a graph. Hence, $\sigma_{sv}(G)$ is at least as much as $\delta(G)$ of the connected component. □

3 Some Optimal Super Vertex Sum Labeling Schemes

MacDougall *et al.*[4], proved that C_n, P_n $(n > 2)$, $K_{m,m}$ $(m > 1)$, $K_{m,m} - e$ $(m > 2)$ and K_n for n is odd, have vertex-magic total labeling. They also proved that $K_{m,n}$ does not have vertex-magic total labeling if $n > m + 1$. MacDougall *et al.*[5], defined super vertex-magic total labeling and showed that a graph G(p,q) admits the labeling if the magic constant k satisfies the condition $k \geq \frac{(41p+21)}{18}$ and C_n admits the labeling iff n is odd. Sethuraman *et al.*[6] introduced super subdivision of graph and proved that arbitrary super subdivision of any path or cycle is graceful.

In this section, we obtain optimal super vertex sum labeling for super subdivision of path, cycle, star and spider.

Theorem 3.1. *Super subdivision of path is a super vertex sum graph with* $\sigma_{sv} = 2$

Proof. Let G be a path with p vertices and q edges. By the definition of super subdivision of graph, each edge of G is replaced by the complete bipartite graph $K_{2,m}$. The resulting graph H has $p + qm$ vertices and $2mq$ edges.

Let u_1 and u_2 be the isolated vertices. Define

$$f : V(H) \longrightarrow \{1, 2, ..., p + mq + 2\}.$$

Case (i) m = 2

Choose the first vertex as a vertex with degree 2 such that all its adjacent vertices are also of degree 2. Then $f(v_1) = 1$.

For i = 1 to q,

$$\begin{cases} \text{for all vertices } v \text{ adjacent to v}_i \text{ and degree 2,} \\ f(v_{ij}) = i + 1 + j + 3(q - i);\ j = 1, 2; \\ \text{if a vertex } v \text{ is adjacent to all } v_{ij}\ (j = 1, 2) \text{ other than v}_i, \\ f(v_{i+1}) = i+1. \end{cases}$$

Case (ii) m ≥ 3

Choose the first vertex as a vertex with degree m such that all its adjacent vertices are also of degree 2. Then $f(v_1) = 1$.

For i = 1 to q

$$\begin{cases} \text{for all vertices v adjacent to } v_i \text{ and degree 2,} \\ f(v_{ij}) = i+1+j+(m+1)(q-i);\ j = 1 \text{ to } 2; \\ \text{if a vertex } v \text{ is adjacent to all } v_{ij} \text{ (j = 1 to 2) other than } v_i, \\ f(v_{i+1}) = i+1. \end{cases}$$

We label the isolates u_i as $f(u_i) = p + mq + i$, $i = 1$ to 2.

Hence, the super subdivision of path is a super vertex sum graph with $\sigma_{sv} = 2$. □

Theorem 3.2. *Super subdivision of cycle is a super vertex sum graph with* $\sigma_{sv} = 2$.

Proof. Let G be a cycle with p vertices. Let v_i ($1 \leq i \leq p$) be the vertices of G. Let H be the arbitrary super subdivision of G which is obtained by replacing every edge of G with $K_{2,m}$.

The resulting graph H has *p(m+1)* vertices and *2mq* edges.

Let u_1 and u_2 be the isolated vertices. Define

$$f : V(H) \longrightarrow \{1, 2, ..., p(m + 1) + 2\}$$

Choose a vertex of degree *2m* as the first vertex and let $f(v_1) = 1$.

For i = 1 to p,

$$\begin{cases} \text{if } i \neq p \\ \begin{cases} \text{for a vertex with degree 2 and adjacent to } v_i, \\ f(v_{i1}) = i + 1 + (m + 1)(q - i);\ j = 1,\ 2; \\ \text{for an unlabelled vertex of degree 2m and adjacent to } v_{i1}, \\ f(vi + 1) = i + 1; \\ \text{for all vertices with degree 2 and adjacent to } v_1 \text{ and } v_{i+1}, \\ f(v_{ij}) = i + j + (m + 1)(q - i);\ j = 2 \text{ to m.} \end{cases} \\ \\ \text{if } i = p \\ \begin{cases} \text{for all vertices with degree 2 and adjacent to } v_1 \text{ and } v_i, \\ f(v_{ij}) = i + j;\ j = 2 \text{ to m} \end{cases} \end{cases}$$

We label the isolates u_i as $f(u_i)$ = p(m+1)+i, i = 1 to 2.

Hence, super subdivision of a cycle C_n is a super vertex sum graph with $\sigma_{sv} = 2$. □

Theorem 3.3. *Super subdivision of star* $K_{1,n}$ *is a super vertex sum graph with* $\sigma_{sv} = 2$.

Proof. Let G be a star with p vertices and q edges. By the definition of the super subdivision, each edge of G is replaced by the complete bipartite graph $K_{2,m}$. The resulting graph H has *p+mq)* vertices and *2mq*edges.

Let u_1 and u_2 be the isolated vertices. Define

$$f : V(H) \longrightarrow \{1, 2, ..., p + mq + 2\}.$$

Choose the vertex with degree mq as the central vertex c and label it as *f(c)=1*.

Case (i) $m = 1$

For i = 1 to q

$$\begin{cases} \text{if vertex } v \text{ is of degree } m \text{ and } d(c,v)\text{=2,} \\ f(v_i) = i + 1; \\ \text{if a vertex } v \text{ is adjacent to } \mathrm{v}_i, \\ f(v_{ij}) = f(v_i) + j + 3(q-1);\ j = 1,\ 2. \end{cases}$$

Case (ii) $m \geq 3$

For i = 1 to q

$$\begin{cases} \text{if deg(v)=m,} \\ f(v_i) = i + 1; \\ \text{if vertex } v \text{ is adjacent to } v_i, \\ f(v_{ij}) = f(v_i) + j + (m+1)(q-1);\ j = 1 \text{ to m.} \end{cases}$$

We label the isolates u_i as $f(u_i)$ = p+mq+i, i = 1, 2. Hence, super subdivision of star is a super vertex sum graph with $\sigma_{sv} = 2$. □

Theorem 3.4. *Super subdivision of spider is a super vertex sum graph with* $\sigma_{sv} = 2$.

Proof. Let G be a spider with k paths. Let p be the number of vertices and q be the number of edges of graph G. Let l_1,l_2,...,l_n be the length of path P_1,P_2,...,P_k, respectively. Therefore, the number of vertices $p = l_1$+l_2+...+l_k+1 and the number of edges $q = l_1$+l_2+...+l_k. Super subdivide the graph G by replacing each edge by $K_{2,m}$. The resulting graph H has *p+mq* vertices and *2mq* edges.

Define

$$f : V(H) \longrightarrow \{1, 2, ..., p + mq + 2\}\,.$$

Choose the vertex with a maximum degree (i.e., mk) as the central vertex c and label it as $f(c) = 1$. All the remaining vertices are named as follows. Let i = 1.

For x = 1 to k,

$for\ l = 1\ to\ l_x$,

$if\ l = 1$,

for an unlabelled (unvisited) vertex v with degree m or $2m$ and $d(c, v) = 2$,

$f(v_i) = i + 1$;

for an unlabelled (unvisited) vertex v with $d(c, v) = 1$ and degree 2,

$f(v_{ij}) = f(v_i) + j + (m + 1)(q - i)$; j=1 to m;

$i = i + 1$

$l = l + 1$

$if\ l \neq 1$ and l$\leq l_x$,

for an unlabelled (unvisited) vertex v with degree m or $2m$ and $d(v_{i-1}, v) = 2$,

$f(v_i) = i + 1$;

for an unlabelled (unvisited) vertex v with $d(v_{i-1}, v) = 1$ and degree 2,

$f(v_{ij}) = f(v_i) + j + (m + 1)(q - i)$; j=1 to m;

$i = i + 1$

$l = l + 1$

$x = x + 1$

We label the isolates u_i as *f(u_i)* = p+mq+i, i = 1 to 2. Hence, super subdivision of a spider is a super vertex Sum Graph with $\sigma_{sv} = 2$. □

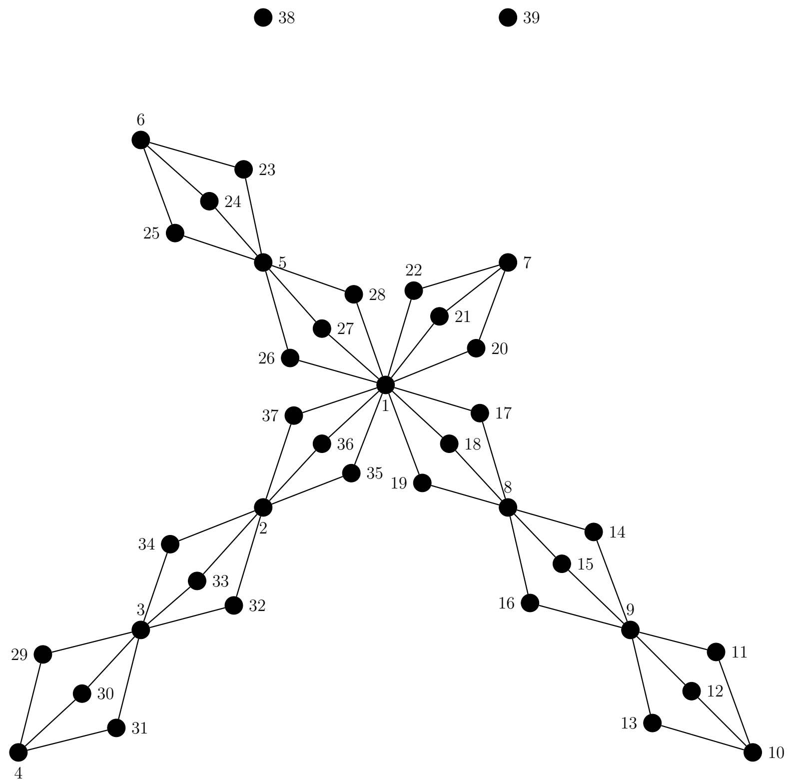

Figure 1: The super subdivided spider

Example

Let G be a spider with four paths of length $l_1 = 3$, $l_2 = 2$, $l_3 = 1$ and $l_4 = 3$. Here, for G, $p = 10$ and $q = 9$.

G is super subdivided by $K_{2,m}$ where $m = 3$.

Therefore, the resulting graph has $p+mq$=10+(9×3) = 37 vertices and $2mq$ = 2×3×9 =54 edges.

The super vertex sum labeling for the super subdivided spider is given in Figure 1.

4 Conclusion

In this paper, we introduced a vertex labeling named as super vertex sum labeling. We proved that any graph that supports the labeling will necessarily be disconnected. Further, we found the lower bound of the super vertex sum number $\sigma_{sv}(G)$ of a graph G. We have also provided optimal super vertex sum labeling for the super subdivision of path, cycle, star and spider. For further studies, optimal super vertex sum labeling for other families of trees, the optimal super vertex sum number for arbitrary super subdivision of the same graphs can be explored.

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