

Interval Valued Gaussian Quadrature Two Point formula for solving Integral Equations

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ABSTRACT: In this paper, new approaches to find the approximate solution of definite integral of the polynomial functions of degree $2n-1$ and compute numerically specified the values of $[a, b]$ is determined by interval analysis method and also used Gaussian quadrature 2 point formula for interval method. In interval method it gives more accurate and approximate solution of real life situation and numerical illustrations are given.

Key Words: Interval Analysis, Gaussian Quadrature, interval Method etc

1. Introduction

In various applications in physics or in engineering, we frequently come across integral which are improper in the sense that, the limit 'a' or 'b' or both may be infinite or integrand may be singular or unbounded at either or both of the limits of integration or else the integrand may be singular at one or more internal points of the interval $[a, b]$.

Gauss developed a procedure in which both positions of the sampling points x_i and weights w_i have been optimized. Gauss realized that by treating both the sample points x_i and the n weights w_i as variables. We make the formula exact and accurate for polynomials with $2n$ coefficients (i.e) polynomial of degree less than or equal to $2n-1$. Formula derived by using this idea is called Gauss- Legendre formula or Gauss- Quadrature formula. This is an open quadrature formula and functional values at the end points need not be known.

In general $\int_{x_0}^{x_1} f(x)dx = \sum_{i=1}^n w_i f(x_i)$, where x_i 's are sampling points and w_i are the

weight at these points. The truncation error is not readily available, because this formula involves evaluation of derivatives of the higher order. To get an estimate of the error, the integral may be evaluated for two different values of n , from which an idea about the error may be formed. If necessary, the integral may be evaluated with larger number of abscissa points. Most common values are $n+1=4$ or $n+1=6$.

2. Preliminaries: interval arithmetic

Let $\tilde{a} = [a_1, a_2]$, $\tilde{b} = [b_1, b_2]$

(I). Addition

$$\tilde{a} + \tilde{b} = [a_1 + b_1, a_2 + b_2]$$

(II). Subtraction

$$\tilde{a} - \tilde{b} = [a_1 - b_2, a_2 - b_1]$$

(III). Multiplication

$$\tilde{a} \tilde{b} = [\min(a_1 b_1, a_1 b_2, a_2 b_1, a_2 b_2), \max(a_1 b_1, a_1 b_2, a_2 b_1, a_2 b_2)]$$

(IV). Division

$$\frac{[i, j]}{[k, l]} = [i, j] \cdot [\frac{1}{l}, \frac{1}{k}] \text{ If } 0 \notin [k, l]$$

(V). $\ll \tilde{a} = [\ll a_1, \ll a_2]$ for $\ll \geq 0$

$$[\ll a_2, \ll a_1] \text{ for } \ll < 0$$

(VI). Inverse

$$[a_1, a_2]^{-1} = [\frac{1}{a_2}, \frac{1}{a_1}], \text{ for } 0 \notin [a_1, a_2]$$

(VII). $[a_1, a_2]^n = [a_1^n, a_2^n]$, if $a_1 \geq 0$

$$=[a_2^n, a_1^n], \text{ if } a_2 < 0$$

$$=[0, \max\{a_1^n, a_2^n\}], \text{ otherwise}$$

3. Proposed Method

Our proposed method, we have choose the interval valued function

$$\int_{x_0}^{x_1} \tilde{f}(x) dx = \sum_{i=1}^n \tilde{w}_i \tilde{f}(x_i), \text{ where } \tilde{x}_i \text{'s are sampling interval points and } \tilde{w}_i \text{ are the weight}$$

of the interval points

$$\text{Gauss quadrature methods are of the form } \int_{-1}^1 \tilde{f}(x) dx = \sum_{i=1}^n \tilde{c}_i \tilde{f}(x_i) \text{ such that } n$$

values of c_i 's and n nodes of interval x_i (totally $2n$ is unknown intervals) must be determined. In such a way that Gaussian quadrature interval method are exact for polynomials of degree $0, 1, 2, \dots, 2n-1$

Derivation for $n=2$

We derive below the gaussian quadrature two point interval method of the form $\int_{-1}^1 \tilde{f}(x) dx = \tilde{c}_1 \tilde{f}(x_1) + \tilde{c}_2 \tilde{f}(x_2) \rightarrow (A)$. This method must be exact for polynomials of degree for $0, 1, 2$ and 3 . There are two constants c_1 and c_2 and two nodes $\tilde{x}_1$, & $\tilde{x}_2$ to be determined, we determine the four unknowns by requiring that the method (1) is exact for polynomial of degree less than or equal to three.

For $\tilde{f}(x) = 1, \tilde{x}, \tilde{x}^2$ and $\tilde{x}^3$

$$\tilde{f}(x) = 1, \quad \int_{-1}^1 dx = \left[\tilde{x} \right]_{-1}^1 = 2 = \tilde{c}_1 + \tilde{c}_2$$

$$\tilde{f}(x) = \tilde{x}, \quad \int_{-1}^1 \tilde{x} dx = \left[\frac{\tilde{x}^2}{2} \right]_{-1}^1 = 0 = \tilde{c}_1 \tilde{x}_1 + \tilde{c}_2 \tilde{x}_2$$

$$\tilde{f}(x) = \tilde{x}^2, \quad \int_{-1}^1 \tilde{x}^2 dx = \left[\frac{\tilde{x}^3}{3} \right]_{-1}^1 = \frac{2}{3} = \tilde{c}_1 \tilde{x}_1^2 + \tilde{c}_2 \tilde{x}_2^2$$

$$f(x) = x^3, \quad \int_{-1}^1 \tilde{x}^3 dx = \left[\frac{\tilde{x}^4}{4} \right]_{-1}^1 = 0 = \tilde{c}_1 \tilde{x}_1^3 + \tilde{c}_2 \tilde{x}_2^3$$

We use the properties of Legendre polynomials to solve this system

$$\text{We have } \tilde{c}_1 + \tilde{c}_2 = \int_{-1}^1 dx \dots\dots(1)$$

$$\tilde{c}_1 \tilde{x}_1 + \tilde{c}_2 \tilde{x}_2 = \int_{-1}^1 \tilde{x} dx \dots\dots(2)$$

$$\tilde{c}_1 \tilde{x}_1^2 + \tilde{c}_2 \tilde{x}_2^2 = \int_{-1}^1 \tilde{x}^2 dx \dots\dots(3)$$

$$\tilde{c}_1 \tilde{x}_1^3 + \tilde{c}_2 \tilde{x}_2^3 = \int_{-1}^1 \tilde{x}^3 dx \dots\dots(4)$$

Multiply (3) by $\frac{3}{2}$ and (1) by $\frac{1}{2}$ and subtract the resulting (1) from (2)

$$\text{This gives } \int_{-1}^1 \left(\frac{3}{2} \tilde{x}^2 - \frac{1}{2} \tilde{x} \right) dx = \tilde{c}_1 \left[\frac{3}{2} \tilde{x}_1^2 - \frac{1}{2} \right] + \tilde{c}_2 \left[\frac{3}{2} \tilde{x}_2^2 - \frac{1}{2} \right]$$

$$\text{But } \tilde{P}_2(x) = \frac{1}{2}(3\tilde{x}^2 - 1), \quad \tilde{P}_0(x) = 1$$

$$\int_{-1}^1 P_2(x) P_0(x) dx = c_1 P_2(x) + c_2 P_2(x) \text{ or } c_1 P_1(x) + c_2 P_2(x) = 0 \text{ by orthogonal Property}$$

Multiply (4) by $\frac{3}{2}$ and (2) by $\frac{1}{2}$ and subtract the resulting (2) from (4)

$$\text{This gives } \int_{-1}^1 \left(\frac{3}{2} \tilde{x}^3 - \frac{1}{2} \tilde{x} \right) dx = \tilde{c}_1 \left[\frac{3}{2} \tilde{x}_1^3 - \frac{1}{2} \tilde{x}_1 \right] + \tilde{c}_2 \left[\frac{3}{2} \tilde{x}_2^3 - \frac{1}{2} \tilde{x}_2 \right]$$

$$\int_{-1}^1 \tilde{x} \left(\frac{3}{2} \tilde{x}^2 - \frac{1}{2} \right) dx = \tilde{c}_1 \tilde{x}_1 \left[\frac{3}{2} \tilde{x}_1^2 - \frac{1}{2} \right] + \tilde{c}_2 \tilde{x}_2 \left[\frac{3}{2} \tilde{x}_2^2 - \frac{1}{2} \right]$$

$$\int_{-1}^1 \tilde{P}_2(x) \tilde{P}_1(x) dx = \tilde{c}_1 \tilde{x}_1 \tilde{P}_2(x_1) + \tilde{c}_2 \tilde{x}_2 \tilde{P}_2(x_2)$$

$$\text{But } \tilde{P}_2(x) = \frac{1}{2}(3\tilde{x}^2 - 1), \quad \tilde{P}_0(x) = 1$$

$$\tilde{c}_1 \tilde{x}_1 \tilde{P}_2(x_1) + \tilde{c}_2 \tilde{x}_2 \tilde{P}_2(x_2) = 0$$

Will be satisfied for any choice of $\tilde{c}_1, \tilde{c}_2$, if we choose $\tilde{x}_1, \tilde{x}_2$ to be zeros of $P_2(x) = 0$

$$\text{Since } \tilde{P}_2(x) = \frac{1}{2}(3\tilde{x}^2 - 1), \quad \tilde{P}_2(x) = 0 \Rightarrow 3\tilde{x}^2 - 1 = 0$$

$$\text{Take } \tilde{x}_1 = \frac{1}{\sqrt{3}} \text{ and } \tilde{x}_2 = -\frac{1}{\sqrt{3}}$$

Substituting in (1) and (2)

$$\tilde{c}_1 + \tilde{c}_2 = 2$$

$$\tilde{c}_1\left(-\frac{1}{\sqrt{3}}\right) + \tilde{c}_2\left(\frac{1}{\sqrt{3}}\right) = 0 \Rightarrow \tilde{c}_1 = \tilde{c}_2$$

$$2\tilde{c}_1 = 2 \Rightarrow \tilde{c}_1 = 1 = \tilde{c}_2$$

Gaussian two point quadrature rules in

$$\int_{-1}^1 [f(x), \bar{f}(x)] dx = \left[f\left(-\frac{1}{\sqrt{3}}\right), \bar{f}\left(-\frac{1}{\sqrt{3}}\right) \right] + \left[f\left(\frac{1}{\sqrt{3}}\right), \bar{f}\left(\frac{1}{\sqrt{3}}\right) \right]. \text{ This is a two ordinate}$$

scheme which is exact for polynomials of degree ≤ 3

If we are given an integral of the form $\int_a^b f(x) dx$ and we want to evaluate it using Gauss

quadrature methods, then we transform $\int_a^b f(x) dx$ to $\int_{-1}^1 f(x) dx$

Using the following put $x = \frac{b-a}{2}t + \frac{b+a}{2}$

$$dx = \frac{b-a}{2} dt$$

$$\int_a^b f(x) dx = \int_{-1}^1 \frac{b-a}{2} f\left(\frac{b-a}{2}t + \frac{b+a}{2}\right) dt$$

$$= \frac{b-a}{2} \int_{-1}^1 f\left(\frac{b-a}{2}t + \frac{b+a}{2}\right) dt$$

4. Numerical Examples

Find the value of the function $\int_{-1}^1 \frac{dx}{[1,1] + [0,2]x^2}$

$$f(x) = \frac{1}{[1,1] + [0,2]x^2}$$

By interval valued Gaussian two point quadrature formula

$$\int_{-1}^1 [f(x), \bar{f}(x)] dx = \left[f\left(-\frac{1}{\sqrt{3}}\right), \bar{f}\left(-\frac{1}{\sqrt{3}}\right) \right] + \left[f\left(\frac{1}{\sqrt{3}}\right), \bar{f}\left(\frac{1}{\sqrt{3}}\right) \right]$$

$$= \left[\frac{1}{1}, \frac{1}{1+2x^2} \right]$$

$$= \left[1, \frac{1}{1+2\left(\frac{1}{3}\right)} + \frac{1}{1+2\left(\frac{1}{3}\right)} \right]$$

$$= \left[1, \frac{3}{5} + \frac{3}{5} \right] = [1, 1.2]$$

Another example for Gaussian quadrature two point formula

Find the value of the equations $\int_{-1}^1 \frac{dx}{[1,3] + [1,1]x^2}$

$$\tilde{f}(x) = \frac{1}{[1,3] + [1,1]x^2}$$

Using Gaussian Formula, we get

$$\begin{aligned} &= \left[\frac{1}{1+x^2}, \frac{1}{3+x^2} \right] \\ &= \left[\frac{3}{5} + \frac{3}{5}, \frac{3}{10} + \frac{3}{10} \right] \\ &= [0.6, 1.2] \end{aligned}$$

5. Conclusion

The proposed interval valued gaussian quadrature two point method is easy to learn and understand work efficiently to obtain the solution of real life situation and also interval valued gaussian quadrature delivers high accuracy at low computational cost. If this method is used as a computer program it is easy to correlate to real life situations.

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