A Production-Inventory Model with Delayed Deteriorating Item with Quadratic and Price Dependent Demand

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ABSTRACT

This paper develops an inventory model with delayed deteriorating items in which demand is a deterministic function of selling price. Depletion of these items will depend on demand before the deterioration starts and at the time of deterioration begins, it will depend on both demand and deterioration. Before the deterioration demand is assumed to be quadratic time dependent and after deterioration demand depending on selling price. In this article, the model is considered with finite replenishment. This model assist to minimizing the total inventory cost and maximize the total profit by detecting the optimal cycle length, the optimal time length of replenishment, and the optimal production quantity. With the help of numerical examples the optimal solution of the model is demonstrated and sensitivity analysis is also follow up to show the issue of changes in system parameters.

Keywords: Delayed deterioration, Quadratic demand, selling price.

1. Introduction

In study of inventory system generally assume that the product is stored at stockroom for a long time to fulfill the future customer demand. In general the level of inventory will decrease when demand rate increase. In inventory model this hypothesis is commonly used by researcher to get the trouble-free formulation of the problem. But in real life situation researcher should note that some type of item either deteriorate or become obsolete during storage time. However in real life situation, some products are deteriorated during their stored time. So the decreasing inventory level will depend on both demand rate
and deterioration rate. This is the reason that the many researchers have long been enormously studied the inventory model for the deteriorating item. The traditional EOQ models such as Harris (1913) and Wilson (1934) assume that products can be stored indefinitely to meet future demand. However, many perishable products such as fruits, vegetables, medicines, and volatile liquids degrade or deteriorate continuously due to evaporation, spoilage, and obsolescence, among other reasons. The first research works in this area was done by Whitin (1957) and Ghare and Schrader (1963). Ghare and Schrader (1963) consider deterioration rate is constant. Aggarwal and Jaggi (1995), Jamal et al. (1997), Chang and Dye (1999), Ouyang et al. (2005), and Liao et al. (2000) were developed the model under constant deterioration rate. Later Covert and Philip (1973) extended the Ghare et.al. with deteriorating rate is not constant. That is variable deterioration rate. Then Shah (1977) developed the Ghare and Shrader work with allowing the backorder. Nahmias(1982) provided a review on the perishable inventory theory. Janssen et al.(2016) published the successive advances in deteriorating inventory literature. Later, Raafat (1991) presented a survey of literature on a continuously deteriorating inventory model. Subsequently, Goyal and Giri (2001) provided a survey on the recent trends in modeling of deteriorating inventory. Recently, Bakker et al. (2012) studied a review of inventory systems with deterioration since 2001.

Later some authors consider that the deterioration rate is stochastic. For example, Ghosh and Chaudhuri (2004) developed an inventory model with quadratic demand in which deterioration rate is Weibull distribution. Rajeswari and Vanjikkodi (2012) extend the EOQ model to two parameter Weibull distribution deterioration rate with backorder while Chung et al. (2014) consider an deterioration rate is exponential distribution. But mostly the inventory model with deteriorating items assumes that the deterioration rate per unit time is constant and the deteriorated amount is proportional to the inventory level. The present paper considers this last deterministic perspective. However, there are several deteriorating items that do not start deteriorating immediately they are held in stock. One common problem with inventory models having deterioration is the basic assumption that deterioration starts as soon as the item is in stock. However, this is not the usual situation in real life. Inventories such as, grapefruit, strawberries, gala apples, farm produce, bakery products, and so on would maintain the quality up to some duration of time after that only these product starts to deteriorate. If the deterioration starts the demand of the item will differ from the before deterioration sets in. This is a sensible assumption in the inventory model. Goyal’s (Goyal, 1985) works explain that after the product is produced some items may deteriorate instantaneously. In our case we assumed that the product may not deteriorate immediately, every product has their own time to remain its quality. So we have consider our product will deteriorate with time delay. Ouyang et.al (2006) developed an inventory model for non-instantaneous deteriorating items with permissible delay in payments and where the demand before deterioration starts is the same as that after deterioration begins. Amutha et.al developed an EOQ inventory model for delayed deteriorating
items under permissible delay in payments but where the demand before deterioration starts is different from that after deterioration starts. Later S. Dari & B. Sani (2017) developed the same concept (Amutha et.al) for the production model and where also demand before and after will change. The demand, is assumed quadratic before the deterioration starts and constant after deterioration starts. In our paper we extend S. Dari & B. Sani contribution under the same criteria but after deterioration the demand rate is function of selling price not constant demand.

The demand rate of any product is generally powerful in nature. This circumstance is due to price or discount or season or even launching unfamiliar products in the markets. Especially Goyal and Giri literature review express that the time varying demand not necessary to follow either linear or exponential trend. It is natural that the demand of newly introduced item in the market is very low but as time goes on, the demand will increase (due to awareness, taste or passion) up to its maximum point and later decline. In general, the demand for items such as this are time dependent quadratic. Ghost,S.K, and Chaudhuri,K.S (2004) developed inventory models for time-quadratic demand. AzizulBatenMd and AbdulbasahKamil (2009)expressed inventory model by assuming demand rate as constant., Nita H et.al (2010), Begum R et al (2010,2012) developed inventory models for time-quadratic demand. In some commodities, due to seasonal variations may follow quadratic function of time [i.e., \( D(t) = at^2 + bt + c \), \( a \geq 0 \)]. Kumar et al. developed an inventory model for quadratic demand rate, inflation with permissible delay in payments. In our case depletion of inventory before deterioration is depending upon quadratic demand.

The customer’s demand always decreases and approaches to zero as the product is near to the expiration date. It is commonly seen that a higher selling price has the reverse effect. According to traditional marketing and economic theory, price is a vital factor on the demand of a product. It is evident that the higher the price, the lower the demand. Ladany and Sternleib (1974) discussed the effect of price variation on demand and consequently on EOQ. Following Robinson and Lakhani (1975), Thompson and Teng (1984), Teng and Chang (2005), and Chang et al. (2006), we assume that the demand rate is proportional to an exponential function of the price.

In this paper, we develop an EPQ model for delayed deteriorating items. For our case we assumed two types of demand rate. One is time dependent quadratic demand and another one is price dependent demand. At beginning demand of the inventory will be low as time goes on it will gradually increase depending upon the time. Therefore period before deterioration sets the demand is assumed to be quadratic time depend demand. The period after deterioration sets demand depend on selling price. Shortages are not allowed. Under this condition the mathematical model has been developed to minimize the total cost. Finally the suitable numerical examples have been provided to emphasize the model.
2. Assumptions and Notations

To expand the mathematical model upcoming assumption are being made:

2.1. Assumption

i) We consider a single item over an infinite planning horizon.

ii) All items are inspected and defective ones are discarded

iii) Demand of product exceeds its supply

iv) Demand before deterioration begins \((D1)\) is assumed to be Quadratic and defined by
\[ D_1 = a + bt + ct^2, c \geq 0 \]

v) We assume that the demand rate \(D_2\) is proportional to an exponential function of the price \(s\)
Thus \(D_2\) is proportional to \(\alpha e^{-\lambda s}\). Where \(\alpha\) is the maximum number of potential consumers \(D_2 = \alpha e^{-\lambda s}\)

2.2 Notation

The notations used in this paper are as follows

**Parameters**

\(D_1\) The demand rate in \(t_1 \leq t \leq t_2\)

\(D_2\) The demand rate in \(t_2 \leq t \leq T\)

\(i\) The inventory carrying charge (excluding interest charges)

\(p\) The unit cost of the item

\(S\) Selling price per unit after deterioration sets in

\(S_1\) Selling price per unit before deterioration sets in

\(C\) Deterioration cost

\(A\) The set-up cost per production run

\(\theta\) Deterioration rate of the stock

\(q_1(t)\) The inventory level at time \(t\), \(t_1 \leq t \leq t_2\)

\(q_2(t)\) The inventory level at time \(t\), \(t_2 \leq t \leq T\)

**Decision variable**

\(T\) The production cycle length

\(t_1\) The productions build up period

\(t_2\) The time deterioration sets in

\(K\) Production rate of the items
3. Formulation and solution of the model

3.1. Inventory Level for Before Deterioration Begin

A mathematical inventory model is initiated as follows: At time $t=0$ the production cycle starts with zero inventory and increases at rate $K$ then the production is stopped at $t_1$. During this period the demand rate is zero. Inventory level is increasing only due to production during the interval $[0, t_1]$. During the period $[t_1, t_2]$ the inventory level is decreasing only due to demand rate. During the interval $[t_2, T]$ the inventory level is decreasing due to deterioration and demand rate in which finally reaches to zero inventory at time $T$. Figure (1) shows above model. Based on the above interpretation, during the time interval $[0, t_1]$ the behavior of inventory level is governed by the following system of differential equations. In this period the demand is zero & the inventory level depends only on the production rate $P$.

\[
\frac{dq_1(t)}{dt} = K \quad \text{for } 0 \leq t \leq t_1
\]  

With the boundary conditions $q_1(0) = 0$.

Integrating (1) applying boundary condition, we get

\[
q_1(t) = Kt \quad \text{for } 0 \leq t \leq t_1
\]
3.2. Inventory Level for After Deterioration Begin

In the second interval \([t_1, t_2]\) the inventory level decrease due to demand rate \(D_1\). The differential equation representing the inventory status is given by \((t_1 \leq t \leq t_2)\)

\[
\frac{dq_2(t)}{dt} = -D_1, \quad t_1 \leq t \leq t_2 \tag{3}
\]

With the boundary condition \(q_1(t_1) = q_2(t_1)\). Solving equation (3) we get

\[
q_2(t) = Kt_1 + a(t_1 - t) + \frac{b(t_1^2 - t^2)}{2} + \frac{c(t_1^3 - t^3)}{3}, \quad t_1 \leq t \leq t_2 \tag{4}
\]

Deterioration starts at the time \(t_2\). Due to demand and deterioration the level of inventory bring down during the interval \([t_2, T]\). The inventory status in this interval is represented by the Differential equation

\[
\frac{dq_3(t)}{dt} = -\theta q_3(t) - D_2, \quad t_2 \leq t \leq T \tag{5}
\]

With boundary condition \(q_3(T) = 0\). Solve equation (5) we get,

\[
q_3(t) = \frac{D_2}{\theta} \left[ e^{\theta(T-t)} - 1 \right], \quad t_2 \leq t \leq T \tag{6}
\]

At \(t_2\), \(q_2(t) = q_3(t)\)

\[
K = \frac{1}{t_1} \left[ \frac{D_2}{\theta} \left( e^{\theta(T-t)} - 1 \right) + a(t_2 - t_1) + \frac{b(t_2^2 - t_1^2)}{2} + \frac{c(t_2^3 - t_1^3)}{3} \right] \tag{7}
\]

3.3. Procedure to Find the Inventory Costs

The different inventory cost with the effect of inflation

(i) The cost spent for ordering = \(A\). \(\tag{8}\)

(ii) The cost spent for purchasing is

\[
PC = \int_0^t pKdt = pKt_1 \tag{9}
\]
iii) Inventory holding cost \((HC)\) is

\[
HC = C_t \left[ \int_0^t q_1(t) dt + \int_{t_1}^t q_2(t) dt + \int_{t_2}^T q_3(t) dt \right] \\
= ip \left\{ K_t t_2 - \frac{K_t^2}{2} - \frac{a}{2} (t_2-t_1)^2 + \frac{b}{6} \left(3t_1^3 t_2 - t_2^3 - 2t_1 t_2^2\right) + \frac{c}{3} \left(4t_1^4 t_2 - t_2^4 - 3t_1^4\right) \right\} \\
+ \frac{D_2}{\theta} \left[ t_2 - T + \frac{1}{\theta} (e^{\theta(T-t_2)} - 1) \right] \\
\tag{10}
\]

The number of deteriorated items \(d(t_2)\) can be calculated as

\[
d(t_2) = \int_0^t Kdtdt - \int_{t_1}^t D_1dt - \int_{t_2}^T D_2dt \\
= K_t - a(t_2 - t_1) - \frac{b(t_2^2 - t_1^2)}{2} - \frac{c(t_2^3 - t_1^3)}{3} - D_2(T - t_2) \\
\tag{11}
\]

(iv) Deterioration cost

\[
DC = c_1 \left[ K_t - a(t_2 - t_1) - \frac{b(t_2^2 - t_1^2)}{2} - \frac{c(t_2^3 - t_1^3)}{3} - D_2(T - t_2) \right] \\
\tag{12}
\]

3.4. Total Variable Cost

The total variable cost is the sum of set-up cost, production cost, and inventory holding cost, deterioration cost

Total variable cost per production run

\[
TC = A + PC + HC + DC \\
TC = A + pKt_1 + ip \left\{ K_t t_2 - \frac{K_t^2}{2} - \frac{a}{2} (t_2-t_1)^2 + \frac{b}{6} \left(3t_1^3 t_2 - t_2^3 - 2t_1 t_2^2\right) + \frac{c}{3} \left(4t_1^4 t_2 - t_2^4 - 3t_1^4\right) \right\} \\
+ \frac{D_2}{\theta} \left[ t_2 - T + \frac{1}{\theta} (e^{\theta(T-t_2)} - 1) \right] + \epsilon_1 \left[ K_t - a(t_2 - t_1) - \frac{b(t_2^2 - t_1^2)}{2} - \frac{c(t_2^3 - t_1^3)}{3} - D_2(T - t_2) \right] \\
\tag{13}
\]
Therefore, the Total cost per cycle per unit time is

\[ TC(T) = \frac{TC}{T} \]

\[ TC = \frac{1}{T} [A + PC + HC + DC] \]

\[ TC(T) = \frac{1}{T} \left\{ A + pK_t + ip \left( K_t t - \frac{K_t^2}{2} - \frac{a}{2} (t_z - t_1)^2 + \frac{b}{6} \left( 3t_1^2 - t_z^2 - 2t_1^1 \right) + \frac{c}{3} \left( 4t_1^1 t_z - t_z^3 - 3t_1^1 \right) \right) \right. \]

\[ + \left. \frac{D_2}{\theta} \left( t_z - T + \frac{1}{\theta} (e^{\theta (T-t_z)} - 1) \right) \right\} + \epsilon_c \left( K_t - a(t_z - t_1) - \frac{b(t_z^2 - t_z^1)}{2} - \frac{c(t_z^3 - t_z^1)}{3} - D_2 (T - t_z) \right) \left( 17 \right) \]

Revenue = \( S_i D_i (t_z - t_i) + SD_i (t_z - T) \)

Profit = \( TC - Revenue \) (16)

3.4. Solution procedure

Our aim is to find the optimal cycle length by minimizing the total inventory cost. The necessary conditions to minimize \( TC(T) \) is 

\[ \frac{dTC(T)}{dT} = 0 \]

and the sufficient conditions to minimize \( TC(T) \) is 

\[ \frac{d^2TC(T)}{dT^2} > 0 . \]

\[ \frac{dTC(T)}{dT} = -\frac{1}{T^2} \left\{ A + pK_t + ip \left( K_t t - \frac{K_t^2}{2} - \frac{a}{2} (t_z - t_1)^2 + \frac{b}{6} \left( 3t_1^2 - t_z^2 - 2t_1^1 \right) + \frac{c}{3} \left( 4t_1^1 t_z - t_z^3 - 3t_1^1 \right) \right) \right. \]

\[ + \left. \frac{D_2}{\theta} \left( t_z - T + \frac{1}{\theta} (e^{\theta (T-t_z)} - 1) \right) \right\} + \epsilon_c \left( K_t - a(t_z - t_1) - \frac{b(t_z^2 - t_z^1)}{2} - \frac{c(t_z^3 - t_z^1)}{3} - D_2 (T - t_z) \right) \left( 17 \right) \]

\[ + \frac{1}{T} \left( \frac{ipD_2}{\theta} \left( e^{\theta (T-t_z)} - 1 \right) - D_2 \epsilon_c \right) \]
By solving the equations \( \frac{dTC(T)}{dT} = 0 \), we can get the optimal values of \( T \). Moreover \( T \) satisfies the equations \( \frac{d^2TC(T)}{dT^2} > 0 \).

4. Numerical Examples

**Example 1**

Let \( A = \$100; \ a = 6400; \ b = 17; \ C = 0.9; \ s = \$16; \ s_i = \$24; \ p = \$12; \ C_i = 2; \ i = 0.3; \ \Theta = 0.11; \ \alpha = 7200; \ \lambda = 0.057. \)

Then we obtain the optimal solutions as follows:

\[
T_1^* = 0.7266 \text{ year}, \quad T_2^* = 1.4533 \text{ year}, \quad T^* = 1.7774 \text{ year}, \quad K^* = 7741, \quad TC(T)^* = 47320.0718
\]

maximum profit = \$42916.8152.

5. Sensitive Analysis

We now interpret the issues of changes in the values of the system parameters \( a, b, \alpha, \lambda, \Theta, S \) on the optimal replenishment policy of Example-1. We vary the only one parameter at a time keeping all other parameters as a constant. The results are summarized (compiled) in Table 1.

1. When the parameter \( \alpha \) is increasing, the cycle length, the order quantity are decreasing and the total cost, profit are increasing.
2. When the parameter \( \lambda \) is increasing, the cycle length, the order quantity are increasing and the total cost, profit are decreasing.
3. When the parameter \( \Theta \) is increasing, the cycle length increasing and the order quantity, total cost and profit are decreasing.
4. When the sale price \( S \) of the unit item after deterioration is increasing, the cycle length, profit are increasing and the total cost decreasing, order changes variably.
5. When the setup cost \( A \) is increasing, the order quantity is decreasing, the cycle length, profit and the total cost are changes variably.
Table 1. Sensitive Analysis with respect to each parameter

<table>
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<tr>
<th>Parameter</th>
<th>Value</th>
<th>$t_1^*$</th>
<th>$t_2^*$</th>
<th>$T^*$</th>
<th>$K^*$</th>
<th>$TC(T)^*$</th>
<th>Profit</th>
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Conclusion

In this paper, we develop an economic production quantity model for delayed deteriorating items with quadratic time dependent demand and selling price dependent demand. In this paper, we determine the optimal cycle length $T$, the total variable cost per unit time $TC(T)$ and the economic production quantity (EPQ) and maximum profit.

This paper can be expanded in various approach, for example, we could enlarge this model in view of time-dependent holding cost and rate of deterioration is dependent on time. Finally, we could extend this model by allowing shortages.
REFERENCES


