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Solutions of Partial Fractional Integro-Differential Equations by using Multi-Dimensional Modification of He's Variational Iteration Method

Wasan Ajeel Ahmood [*1] **and Maad Kamal Al-anni** [†2]

[1,2]**Al-Iraqia University, Faculty of Education for Women, Department of Al-Quran Science, Baghdad, Iraq**

Abstract

In this paper, we consider the multi-dimensional modification of He's variational iteration method which is developed to solve multi-dimensional fractional integro-differential equations with the non-local boundary conditions. Then we find the exact solution using multi-dimensional iteration by initial and boundary approximation. We also show that modified method leads to highly accurate solutions.

Keywords: Fractional integro-differential equations, classical ordinary derivatives, modification of He's variational iteration method, non-local boundary condition, fractional and fractal-order models in mechanical engineering

1 Introduction

The vibration in dynamical systems can be divided into two main categories such as discrete and distributed. The variables in the discrete systems depend on time, whereas in distributed systems variables depend on time and space such as in beams, plates etc. Thus, the equations of motion of discrete systems are described by ordinary differential equations, while equations of motion in distributed systems are described by partial differential equations, see [Meirovitch L. 1980]. In the literature, there are many works and studies have been made in the area of vibration problems and several techniques such as; finite element method, finite difference method, perturbation techniques, series solutions(DTM), etc. have

[*]wahsmu5@gmail.com
[†]alannimaad@gmail.com

been used to handle vibration problems. The variational iteration method(VIM) was first proposed by [He JH., 1997–2014] and the method has been applied to study many nonlinear partial differential equations, autonomous and singular ordinary differential equations such that solitary wave solutions, rational solutions, compacton solutions and other types of solution some could be found in [Abdou and Soliman, 2005].
In [Wang SQ. and He JH., 2008], they used the variational iteration method for rigorous derivation of temperature distribution of a reaction-diffusion process with non-local boundary conditions by constructing an initial trial-function without unknown parameters, so that a few iterations lead to exact solution. The solution indicates that this method is effective and convenient, later [Ghorbani A. and Saberi-Nadjafi J., 2009] introduced the modified version of VIM.

In [He JH et al. 2010], they gave the physical interpretations of the fractional calculus and the fractal derivative for discrete lattice equations by discussing some applications, also examined the acceleration of some iteration formulae with particular emphasis being placed on the exponential Pade approximant for solitary solutions.It is also possible that in some cases, it may not be any physical meaning for many non-linear equations, there are also some difficulty to find approximate solutions that satisfy both the equations and the appropriate initial/boundary conditions. Thus the variational iteration method together with approximate initial/boundary conditions and point boundary initial/conditions for solving this kind of problems simultaneously were proposed, and thus [He JH., 2011] introduced the variational iteration method and its fractional modification of Riemann–Liouville derivative and the results are compared.
Now recall the the variational iteration method, see the details in [Asma A et al. 2012] and in order to illustrate the basic concepts of VIM,

$$L(u) + N(u) = g(x)$$

where L is a linear operator; N is nonlinear operator, and $g(x)$ is an non-homogeneous term. By VIM, the one constructs a correction functional is:

$$y_{n+1} = y_n + \int_0^x \lambda[Ly_n(s) - N\widetilde{y_n}(s)]ds$$

where λ is a general Lagrange multiplier, and $\widetilde{y_n}$ denotes restricted variation that is $\delta\widetilde{y_n} = 0$, [Irandoust-Pakchin S. and Abdi-Mazraeh S., 2013] applied (1.D.V.I.M) for solving these problems exactly, [Ahmood W. and Kilicman A., 2016] studied the multidimensional space-time fractional derivative, determined the exact solution via the Laplace transform by some applications and developed mathematical foundations of the respective operators.
The numerical results illustrate that the methods can be implemented easily and accurately when applied to the nonlinear space-time fractional derivatives

Klein-Gordon equation. This approach presents a promising tool for solving many space-time fractional partial differential equations. This method is effective and powerful in solving wide classes of nonlinear evolution fractional order equations. For different type of applications, see [He JH et al. 2014], [Khalida A. and Mouffak B., 2014], [Mohamed S.and Sherif E., 2014], [Wu GC et al. 2015], [Doha EH et al. 2015] and [Jafari H et al. 2015].

[Bildik N. and Deniz S., 2014] they presented a comparison between the adomian decomposition method and Taylor matrix method by solving some well-known partial differential equations. In order to illustrate the analysis, they examined the Telegraph equation, which is considered one of the most significant partial differential equations, described wave propagation of electric signals in a cable transmission line and Klein-Gordon equation which is encountered in several applied physics fields such as, quantum field theory, fluid dynamics , optoelectronic devices design and numerical analysis. [Bildik N. and Deniz S., 2015] considered the solution of systems of delay differential equations together with initial conditions, using numerical methods, including the Taylor collocation method, the Lambert W function and the variational iteration method. We have endeavored to show the most appropriate method by comparing the solutions of this system of equations with different types of methods. [Bhrawy A.H., 2016] adapted an operational matrix formulation of the collocation method for one and two-dimensional non-linear fractional subdiffusion equations. They also used both double and triple shifted Jacobi polynomials as basis functions to elucidate approximate solutions of one and two-dimensional cases. The space-time fractional derivatives given in the underline problems are expressed by Jacobi operational matrices. It helps investigate spectral collocation schemes for both temporal and spatial discretizations.

1.1 Basic Definitions:-

We start this section by the providing some definitions of the fractional derivative and its properties:-

Definition (1): The fractional derivative can define in this form:

$$D_t^\alpha f = D_t^\alpha \frac{d^n}{dt^n}(I^n f) = \frac{d^n}{dt^n}(I^{n-\alpha} f) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_{t_0}^{t} (s-t)^{n-\alpha-1}[f_0(s) - f(s)]ds,$$

And $f_0(t)$ is continuous and differentiable in the above equation can be written as:-

$$D_t^\alpha f = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_{t_0}^{t} (s-t)^{n-\alpha-1} \left[\sum_{i=0}^{n-1} \frac{1}{i!}(s-t_0)^i f^{(i)}(t_0) - f(s) \right] ds,$$

if $f_0(t)$ is continuous but not differentiable in the above equation can be written as:-

$$D_t^\alpha f = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_{t_0}^{t}(s-t)^{n-\alpha-1}\left[\sum_{i=0}^{n-1}\frac{(s-t_0)^i}{\Gamma(1+i\alpha)}f^{(i\alpha)}(t_0)-f(s)\right]ds,$$

In fact the fractional derivative which was defined in the format

$$D_t^\alpha f = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_{t_0}^{t}(s-t)^{n-\alpha-1}[f(t_0)-f(s)]ds,$$

also known as the variational iteration algorithm-I, where f can be continuous and need not necessarily differentiable. Then the variational iteration algorithm-II is given by

$$\begin{aligned} u_{m+1}(t) &= u_0(t)-(-1)^n\int_{t_0}^{t}\frac{1}{(n-1)!}(s-t)^{n-1}f_m(s)ds \\ &= u_0(t)-\frac{(-1)^n}{\Gamma(n)}\int_{t_0}^{t}(s-t)^{n-1}f_m(s)ds, \end{aligned}$$

where u_0 is the initial and boundary conditions

$$u(t)=u_0(t)-\frac{(-1)^n}{\Gamma(n)}\int_{t_0}^{t}(s-1)^{n-1}f(s)ds$$

In the literature there are several different type of fractional derivative definition and the following is known as [Caputo Fractional Derivative:-]

$$D_x^\alpha(f(x)) = \frac{1}{\Gamma(n-\alpha)}\int_0^x(x-t)^{n-\alpha-1}\frac{d^nf(t)}{dt^n}dt,$$

similarly, the next is called as [Riemann-Liouville Fractional Derivative:-]

$$D_x^\alpha(f(x)) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_0^x(x-t)^{n-\alpha-1}f(t)dt,$$

Later it was modified as [Modification of Riemann-Liouville Fractional Derivative:-]

$$D_x^\alpha(f(x)) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_0^x(x-t)^{n-\alpha-1}f[(t)-f(0)]dt,$$

where f is a continuous (but not necessarily differentiable) function. Jumarie's fractional derivative meets the following simple rules:

$$\frac{\partial^\alpha c}{\partial x^\alpha}=0.\frac{\partial^\alpha[cf]}{\partial x^\alpha}=c\frac{\partial^\alpha f}{\partial x^\alpha}.\frac{\partial^\alpha x^\beta}{\partial x^\alpha}=\frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)}x^{\beta-\alpha},\beta\geq\alpha>0$$

Definition (2), [The Local Fractional Derivative:-]

$$f^\alpha(X_0))=\frac{d^\alpha f(x)}{dx^\alpha}=x\overset{\lim}{\to}x_0\ \frac{\Delta^\alpha(f(x)-f(x_0))}{(x-x_0)^\alpha}$$

Where $\Delta^\alpha(f(x)-f(x_0))\cong\Gamma(1+\alpha)\Delta(f(x)-f(x_0))$.

Now, some properties of Local Fractional Derivative:-

$$\begin{aligned} \frac{\partial^{k\alpha}f(x)}{\partial x^{k\alpha}} &= \frac{\partial^\alpha}{\partial x^\alpha}...\frac{\partial^\alpha f(x)}{\partial x^\alpha} \\ \frac{\partial^\alpha f(g(x))}{\partial x^\alpha} &= f^{(1)}(g(x))g^{(\alpha)}(x) \end{aligned}$$

Now, we start this section by the following definitions and some properties of the fractional calculus theory, used in this paper:-

Definition [3]: A real function $f(x), x > 0$, is said to be in the space $C_\mu, m \in R$, if there exist a real number $P > \mu$ such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C[0,1)$. Clearly $C_\mu \subset C_\beta$if $\beta \leq \mu$.

Definition [4]: A real function $f(x), x > 0$, is said to be in the space $C_\mu^m, m \in N \cup 0$, if $f^m \in C_\mu$.

Definition [5], [He H., 2006]: Let $f \in C_{-1}^m, m \in N \cup 0$, then the Caputo fractional derivative of f(x) is defined.

$$\begin{aligned} D^\alpha f(x) &= J^{m-\alpha} f^{(m)}(x), m-1 < \alpha < m, m \in N, \\ &= \frac{D^m f(x)}{Dx^m}, \alpha = m. \end{aligned}$$

Now, some properties of fractional integro-differential equations :

1. $J^\alpha J^v f = J^{\alpha+v} f, \alpha, v > 0, f \in C_\mu, \mu > 0.$

2. $J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}, \alpha > 0, \gamma > -1, x > 0.$

3. $J^\alpha D^\alpha f(x) = f(x) - \sum_{k=0}^{m-1} f^{(k)(0^+)} \frac{x^k}{k!}, x > 0, m-1 < \alpha \leq m.$

1.2 Some Basic Methods for Solving Fractional Integro-Differential Equations:-

In this section, we propose the M.V.I.M for solving the fractional integro-differential equations with the nonlocal boundary conditions by an initial trial-function without unknown parameters. Fractional integro-differential equations appear in various research areas and scientific applications. An effective and very helpful to use method for solving such equations.

Definition (2.1.1), [Irandoust-pakchin S. and Abdi-Mazraeh S., 2013]:-

Consider fractional order integro-differential equation of the form:

$$\begin{aligned} \sum_{i=1}^{n} a_i D^v y(x) = I^{-v} y(x) &= \frac{1}{\Gamma(v)} \int_0^x (x-t)^{v-1} y(t) dt \\ D^\alpha u(x) &= g(x) + \int_a^x h_1(x,t) u(t) dt + \int_a^b h_2(x,t) u(t) dt \end{aligned}$$

Where $n-1 < \alpha \leq n, a < x < b$, and $n \in N$, with the non-local boundary conditions

$$\sum_{j=1}^{n} \left(\gamma_{ij} u^{(j-1)}(a) + \mu_{ij} u^{(j-1)}(a) \right) + \lambda_i \int_a^x H_i(t) u(t) dt = d_i$$

where D^α denotes a differential operator with fractional order $\alpha, g(x)$ and $h_i(x,t)$ for each i=1,2 are holomorphic functions. $H_i(t)$ are continuous functions, $\gamma_{ij}, \mu_{ij}, \lambda_i$ and $d_i, (i = 1, 2, ..., m)$ are constants and u is the unknown function that must be determined. In the next, we give some of these methods:

1.2.1 Solution of the Ordinary Fractional Integro-Differential Equations Via the (MVI):-

Consider the fractional order functional equation of the form:

$$Lu + Ru + Nu = g(x)$$

where L is the fractional order derivative, R is a linear differential operator, N represents the non-linear terms and g is the source term. By using the properties and the inverse operator L_x^{-1} to both sides of the fractional functional equation and using the given conditions, we obtain:-

$$u = f - L_x^{-1}[Ru] + L_x^{-1}[Nu]$$

where f is a term arising from integrating the source term g. By the construction of the correction of functional for fractional functional equation, one can get:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(s)[Lu_n(s) + R\widetilde{u}_n(s) + N\widetilde{u}_n(s) - g(s)]ds,$$

where λ is a Lagrange multiplier, u_n is the n-th approximate solution and $\widetilde{u}_n(s)$ is a variation, i.e., $\delta\widetilde{u}_n(s) = 0$. Now, by the VIM, we determine λ via integration by parts. Then, $u_n(x)$ of the solution $u(x)$ and using u_0 that satisfies at least the initial and boundary conditions. Now, we have $u_0(x)$ as an arbitrary initial guess and then we have;

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(s)[Lu_n(s) + Ru_n(s) + Nu_n(s) - g(s)]ds.$$

By [Xu L., 2006] the above equation is equivalently: That is $u_0(x)$ is an arbitrary initial guess and $u_{n+1}(x) = f(x) - L_x^{-1}[Ru_n(x)] - L_x^{-1}[Nu_n(x)]$, where λ has been identified. Now, the function $f(x)$ can be divided into two parts, namely $f_0(x)$ and $f_1(x)$ then, we can obtain:

$$f(x) = f_0(x) + f_1(x).$$

Further by using the properties, we obtain:-

$$\begin{aligned} u_0(x) &= f_0(x) \\ u_1(x) &= f(x) - L_x^{-1}[Rf_0(x)] - L_x^{-1}[Nf_0(x)] \\ u_{n+1}(x) &= f(x) - L_x^{-1}[Ru_n(x)] - L_x^{-1}[Nu_n(x)]. \end{aligned}$$

From the above equation we obtained the exact solution. In order to illustrate the method, consider the following examples.

Example [1], [Nazari D. and Shahmorad S., 2010]:-
Consider the linear fractional integro-differential equation with given non-local conditions.

$$D^{\frac{1}{2}}y(x) = -x^2\frac{e^x}{3}y(x) + \frac{1}{\Gamma\left(\frac{3}{2}\right)}x^{\frac{1}{2}} - \frac{1}{2}x^2 + e^x\int_0^x ty(t)dt + \int_0^1 x^2y(t)dt,$$

$$y(0) + y(1) - 3\int_0^1 ty(t)dt = 0.$$

Then, by taking $L = D^{\frac{1}{2}}$ and the inverse operator L_x^{-1} to both sides of the above equation, we have:-

$$y(x) = J^{\frac{1}{2}}\left[\frac{1}{\Gamma\left(\frac{3}{2}\right)}x^{\frac{1}{2}} - \frac{1}{2}x^2\right] + J^{\frac{1}{2}}\left[-x^2\frac{e^x}{3}y(x)\right] + J^{\frac{1}{2}}\left[e^x\int_0^x ty(t)dt + \int_0^1 x^2y(t)dt\right].$$

Now, by the VIM and the recursive scheme, we obtain:

$$f(x) \quad = f_0(x) + f_1(x) = J^{\frac{1}{2}}\left[\frac{1}{\Gamma(\frac{3}{2})}x^{\frac{1}{2}} - \frac{1}{2}x^2\right] = \quad x - \frac{x^{\frac{5}{2}}}{\Gamma\left(\frac{7}{2}\right)},$$

by assuming $f_0(x) = x$, then $f_1(x) = -\frac{x^{\frac{5}{2}}}{\Gamma(\frac{7}{2})}$, and by starting of the initial approximation $y_0(x) = f_0(x) = x$, we have:

$$\begin{aligned}
y_0(x) &= x, \\
y_1(x) &= x - \frac{x^{\frac{5}{2}}}{\Gamma\left(\frac{7}{2}\right)} + L_x^{-1}[f_0(x)], \\
y_1(x) &= x - \frac{x^{\frac{5}{2}}}{\Gamma\left(\frac{7}{2}\right)} + J^{\frac{1}{2}}[\frac{-x^3}{3}e^x] + J^{\frac{1}{2}}[\frac{x^3}{3}e^x] + J^{\frac{1}{2}}[\frac{x^2}{2}] = x, \\
y_{n+1}(x) &= x - \frac{x^{\frac{5}{2}}}{\Gamma\left(\frac{7}{2}\right)} + L_x^{-1}[y_n(x)] \Rightarrow y_n(x) = x, n \geq 1.
\end{aligned}$$

In the above equation, it is obtained:

$$y(x) = x$$

is the exact solution of fractional integro-differential equation.

Example [2], [Nazari D. and Shahmorad S., 2010]:-
Consider the linear fractional integro-differential equation with the given non-local conditions:-

$$D^{\frac{5}{4}}y(x) \quad = \quad (\cos(x) - \sin(x))y(x) + g(x) + \int_0^x \sin(t)y(t)dt,$$

with the nonlocal conditions

$$\begin{aligned}
y(0) + y(1) + \frac{e+1}{e+2}y^{'}(0) + \frac{1}{2}y^{'}(1) - 8\int_0^1 ty(t)dt &= 0, \\
2y(0) + 2y(1) + \frac{e}{e+1}y^{'}(0) - y^{'}(1) &= 0.
\end{aligned}$$

By, $f(x) = \frac{8}{3}\frac{x^{\frac{3}{4}}}{\Gamma(\frac{3}{4})} - 2\cos(x) - 2x\sin(x) + x^2\sin(x) + 2$, the exact solution of the fractional integro-differential equation with the non-local conditions would be given by $y(x) = x^2$. Then by using the second properties, we have:

$$\begin{aligned}
f(x) \quad &= \quad L_x^{-1}[g(x)] = J^{\frac{5}{4}}g(x) = \frac{8}{3}\frac{1}{\Gamma(\frac{3}{4})}\frac{\Gamma(\frac{7}{4})}{\Gamma(3)}x^2 - 2J^{\frac{5}{4}}\cos(x) - 2J^{\frac{5}{4}}x\sin(x) - J^{\frac{5}{4}}x^2\sin(x) + -J^{\frac{5}{4}}2 \\
&= \quad x^2 - 2J^{\frac{5}{4}}\cos(x) - 2J^{\frac{5}{4}}x\sin(x) - J^{\frac{5}{4}}x^2\sin(x) + -J^{\frac{5}{4}}2.
\end{aligned}$$

By $f_0(x) = x^2$, where is satisfies in nonlocal boundary conditions also, we have

$f_0(x) = x^2$,
$f_1(x) = 2J^{\frac{5}{4}}\cos(x) - 2J^{\frac{5}{4}}x\sin(x) + J^{\frac{5}{4}}x^2\sin(x) + J^{\frac{5}{4}}2$,

following the recursive relationship, we obtain:-

$y_0(x) = f_0(x)$,
$y_1(x) = f(x) - L_x^{-1}[(\cos(x) - \sin)x^2 + \int_0^x \sin(t)t^2dt]$,
$y_{n+1}(x) = f(x) - L_x^{-1}[(\cos(x) - \sin)y_n(x) + \int_0^x \sin(t)y_n(x)dt], n \geq 1$.

Hence

$y_0(x) = x^2$,
$y_1(x) = x^2$,
$y_{n+1}(x) = x^2, n \geq 1$.

So $y(x) = \lim_{n\to\infty} y_n(x) = x$, is the exact solution of problem.

Example [3]:-
Consider the linear fractional integro-differential equation of $2 < v = \frac{8}{3} < 3$ with the given non-local conditions.

$$D^{\frac{8}{3}}y(x) = \frac{1}{3}e^{x^3} + x^{\frac{1}{3}} - 1 - x^3e^{(x-1)}y(x) + \int_0^x e^t y(t)dt$$

$$y(0) + y''(1) - 4\int_0^1 e^t y(t)dt = 0.$$

Then, by taking $L = D^{\frac{8}{3}}$ and taking the inverse operator L_x^{-1} to both sides of the above equation, we have:-

$$y(x) = J^{\frac{8}{3}}\left[\frac{1}{3}e^{x^3} + x^{\frac{1}{3}} - 1\right] - J^{\frac{8}{3}}\left[x^3e^{(x-1)}y(x)\right] + J^{\frac{8}{3}}\left[\int_0^x e^t y(t)dt\right].$$

Now, by the VIM and the recursive scheme, we obtain:

$$f(x) = f_0(x) + f_1(x) = J^{\frac{8}{3}}\left[\frac{1}{3}e^{x^3} + x^{\frac{1}{3}} - 1\right]$$

By the second properties, can get:

$$J^{\frac{8}{3}}x^{\frac{1}{3}} = \frac{\Gamma(\frac{1}{3}+1)}{\Gamma(\frac{8}{3}+\frac{1}{3}+1)}x^{\frac{8}{3}+\frac{1}{3}} = \frac{\Gamma(\frac{4}{3})!}{\Gamma(4)}x^3$$

by assuming

$$f_0(x) = \frac{\Gamma(\frac{4}{3})!}{\Gamma(4)}x^3, f_1(x) = J^{\frac{8}{3}}\left[\frac{1}{3}e^{x^3} - 1\right]$$

and by starting of the initial approximation $y_0(x) = f_0(x) = x$, we have:

$$\begin{aligned} y_0(x) &= x^3, \\ y_1(x) &= x^3 + J^{\frac{8}{3}}\left[\frac{1}{3}e^{x^3} - 1\right] - J^{\frac{8}{3}}\left[x^3e^{(x-1)}y(x)\right] + L_x^{-1}\left[\int_0^x e^t \frac{1}{(\frac{35}{12})!}x^{(\frac{35}{12})}dt\right] \\ y_{n+1}(x) &= x^3 \end{aligned}$$

In the above equation, it is obtained:

$$y(x) = x^3$$

is the exact solution of fractional integro-differential equation.

Example [4]:-
Consider the linear fractional integro-differential of $0 < v = \frac{1}{3} < 1$ equation with the given non-local conditions.

$$D^{\frac{1}{3}}y(x) = \frac{3}{2}\frac{x^{\frac{2}{3}}}{\Gamma(\frac{2}{3})} - 1 + e^{x^2} - x^2e^{(x^2)} + \int_0^x x^2e^{xt}y(t)dt,$$

$$y(0)+2y(1)+3\int_0^1 ty(t)dt \quad = \quad 0.$$

Then, by applying $L=D^{\frac{1}{3}}$ and L_x^{-1} to both sides of the above equation, we have:-

$$y(x) \quad = \quad J^{\frac{1}{3}}\left[\frac{3}{2}\frac{x^{\frac{2}{3}}}{\Gamma(\frac{2}{3})}-1+e^{x^2}-x^2e^{(x^2)}\right]+J^{\frac{1}{3}}\left[\int_0^x x^2e^{xt}y(t)dt\right].$$

Now, by the VIM and the recursive scheme, we obtain:

$$f(x)=f_0(x)+f_1(x)=J^{\frac{8}{3}}\left[\frac{1}{3}e^{x^3}+x^{\frac{1}{3}}-1\right]$$

By the second properties, can get:

$$J^{\frac{1}{3}}x^{\frac{2}{3}}=\frac{\Gamma(\frac{2}{3}+1)}{\Gamma(\frac{1}{3}+\frac{2}{3}+1)}x^{\frac{1}{3}+\frac{2}{3}}=x$$

by assuming

$$\begin{aligned} f_0(x) &= x, \\ f_1(x) &= J^{\frac{8}{3}}\left[-1+e^{x^2}-x^2e^{(x^2)}\right] \end{aligned}$$

and by starting of the initial approximation $y_0(x)=f_0(x)=x$, we have:

$$\begin{aligned} y_0(x) &= x, \\ y_1(x) &= x+J^{\frac{1}{3}}\left[-1+e^{x^2}-x^2e^{(x^2)}\right]+L_x^{-1}\left[\int_0^x x^2e^{xt}y(t)dt\right]=x \\ y_{n+1}(x) &= x \end{aligned}$$

In the above equation, it is obtained:

$$y(x)=x$$

is the exact solution of fractional integro-differential equation.

We note that in particular next example, fractional order models in Mechanical Engineering:

Example [5]:-

Consider the following two-dimensional wave-like equation of fractional integro-differential equation with variable coefficients and non-local conditions:

$$D_t^{\frac{3}{4}}u(x,y,t) \quad = \quad \frac{1}{12}(x^2u_{xx}+y^2u_{yy})+\frac{6x^{\frac{9}{4}}}{\Gamma\left(\frac{13}{4}\right)}+\int_0^x te^xu(t)dt,$$

$$u(0)+u(1)-\frac{e+1}{e-2}u'(0)+\frac{1}{6}u''(1)-5\int_0^1 tu(t) \quad = \quad 0.$$

Subject to the Neumann conditions

$$\begin{aligned} u_x(0,y,t) &= u_y(x,0,t)=0 \\ u_x(1,y,t) &= 4\cos(ht), \quad u_y(x,1,t)=4\sin(ht) \end{aligned}$$

By $f_0(x) = x^3$ and $f_1(x) = 0$, we select $u_0(x) = f_0(x) = x^3$ to find

$$\begin{aligned} u_0(x) &= x^3, \\ u_1(x) &= x^3 + J^{\frac{3}{4}}\frac{1}{12}(x^2 u_{xx} + J^{\frac{3}{4}}\frac{6x^{\frac{9}{4}}}{\Gamma\left(\frac{13}{4}\right)} + J^{\frac{3}{4}}\int_0^x te^x t^3 dt = x^3, \\ u_{n+1}(x) &= x^3, \quad n \geq 1. \end{aligned}$$

In view of the above equation, we can get $u(x) = x^3$, is the exact solution of problem the fractional integro-differential equation with the non-local conditions.

2 Solution of the Partial Fractional Integro-Differential Equations:

Consider the boundary value problem which consists of the m-th Fractional order Multi-Dimensional Integro-Differential equation with constant coefficient:

$$\sum_{\substack{\alpha_1,\alpha_2,\ldots,\alpha_n=0 \\ 0\leq\sum_{i=1}^n \alpha_i\leq m}}^{m} a_{\alpha_1,\alpha_2,\ldots,\alpha_n}\frac{\partial^{\sum_{i=1}^n \alpha_i}u(x_1,x_2,...,x_n)}{\prod_{i=1}^n \partial x_i^{\alpha_i}} = g(x_1,x_2,...,x_n) +$$
$$\int_{a_n}^{x_n}\int_{a_{n-1}}^{x_{n-1}}...\int_{a_1}^{x_1} h_1(x_1,t_1,x_2,t_2,...,x_n,t_n)u(t_1,t_2,...,t_n)dt_1dt_2...dt_n +$$
$$\int_{a_n}^{b_n}\int_{a_{n-1}}^{b_{n-1}}...\int_{a_1}^{b_1} h_2(x_1,t_1,x_2,t_2,...,x_nt_n,)u(t_1,t_2,...,t_n)dt_1dt_2...dt_n,$$

where $n-1 < \alpha_i \leq n, a_i < x_i < b_i$ for each $i = 1,2,...,n$. Together with appropriate non-local boundary condition, where

$$\frac{\partial^{\sum_{i=1}^n \alpha_i}u(x_1,x_2,...,x_n)}{\prod_{i=1}^n \partial x_i^{\alpha_i}} + \frac{\partial^{\sum_{i=1}^n \alpha_i}u(x_1,x_2,...,x_n)}{\prod_{i=1}^n \partial x_i^{\alpha_i}} +$$
$$\prod_{i=1}^n \lambda_{i_j}\int_{a_n}^{x_n}\int_{a_{n-1}}^{x_{n-1}}...\int_{a_1}^{x_1}\prod_{i=1}^n \lambda_{i_j}H_{i_j}(t_1,t_2,...,t_n)u(t_1,t_2,...,t_n)dt_1dt_2...dt_n = A_i.$$

Denotes multi-dimensional differential operator with fractional order α_i, for each $i = 1,2,...,n; \alpha_1,\alpha_2,...,\alpha_n$ is a known constant for each $\alpha_1,\alpha_2,...,\alpha_n = 0,1,...,m.$ such as $0 \leq \sum_{i=1}^n \alpha_i \leq m$ and $a_{\alpha_1,\alpha_2,...,\alpha_n} \neq 0$, g is a known function of $x_1,x_2,...,x_n; h_1,h_2$ are a known function of x_i,t_i for each $i = 1,2,...,n$ and u is the unknown function that must be determined. Then, by taking or using the (MVIM) of the fractional multi-dimensional integro-differential equation and by using the property:

$$\prod_{i=1}^n J_i^{\alpha_i}x_i^{\gamma_i} = \prod_{i=1}^n\left[\frac{\Gamma(\gamma_i+1)}{\Gamma(\alpha_i+\gamma_i+1)}x_i^{\alpha_i+\gamma_i}\right],$$

$\alpha_i > 0, \gamma_i > -1, x_i > 0$. For each $i = 1,2,...,n$. By assuming $L_n = D^{\sum_{i=1}^n \alpha_i}$ is the partial or multi-dimensional order derivative and by applying the inverse multi-dimensional operator L_n^{-1} to both sides of the multi-dimensional fractional integro-differential equation, we have:

$$\sum_{\substack{\alpha_1,\alpha_2,\ldots,\alpha_n=0 \\ 0\leq\sum_{i=1}^n \alpha_i\leq m}}^{m} a_{\alpha_1,\alpha_2,...,\alpha_n}L_n^{-1}\left\{\frac{\partial^{\sum_{i=1}^n \alpha_i}u(x_1,x_2,...,x_n)}{\prod_{i=1}^n \partial x_i^{\alpha_i}}\right\} = L_n^{-1}\left\{g(x_1,x_2,...,x_n)\right\} +$$
$$L_n^{-1}\left\{\int_{a_n}^{x_n}\int_{a_{n-1}}^{x_{n-1}}...\int_{a_1}^{x_1} h_1(x_1,t_1,x_2,t_2,...,x_n,t_n)u(t_1,t_2,...,t_n)dt_1dt_2...dt_n\right\} +$$
$$L_n^{-1}\left\{\int_{a_n}^{b_n}\int_{a_{n-1}}^{b_{n-1}}...\int_{a_1}^{b_1} h_2(x_1,t_1,x_2,t_2,...,x_nt_n,)u(t_1,t_2,...,t_n)dt_1dt_2...dt_n\right\}.$$

Now, the function $f(x_1, x_2, ..., x_n)$ can be divided into two parts, namely $f_{01,02,...,0n}(x_1, x_2, ..., x_n)$ and $f_{11,12,...,1n}(x_1, x_2, ..., x_n)$ under this assumption, we set:
$f(x_1, x_2, ..., x_n) = f_{01,02,...,0n}(x_1, x_2, ..., x_n) + f_{11,12,...,1n}(x_1, x_2, ..., x_n)$.
According to the above assumption, and by relationship, we construct the following multi-dimensional variational iteration formula:-

$$\begin{aligned} u_{01,02,...,0n}(x_1, x_2, ..., x_n) &= f_{01,02,...,0n}(x_1, x_2, ..., x_n) \\ u_{11,12,...,1n}(x_1, x_2, ..., x_n) &= f(x_1, x_2, ..., x_n) - L_n^{-1} u_{n1+11,n2+12,...,nn+1n}(x_1, x_2, ..., x_n) \end{aligned}$$

to illustrate this method, consider the following examples.

1-(i) Let different derivative $\sum_{i=1}^{2} v_i = \frac{3}{4}$ and $0 < \sum_{i=1}^{2} v_i \leq 1$, then:
Consider the boundary value problem which consists of the 1st fractional order two-dimensional integro-differential equation for the problem with constant coefficient:

$$\begin{aligned} \frac{\partial^{\frac{3}{4}} u(x,y)}{\partial x^{\frac{1}{2}} \partial y^{\frac{1}{4}}} &= \frac{4}{3}\left[\frac{x^{\frac{3}{2}} y^{\frac{3}{4}}}{\Gamma(\frac{1}{4})}\right] - 1 + e^{x^2+y^2} - x^2 y^2 e^{x^2+y^2} \\ &+ \int_0^y \int_0^x t_1^2 t_2^2 e^{x t_1 + y t_2} u(t_1, t_2) dt_1 dt_2, \end{aligned}$$

with appropriate non-local boundary condition:

$$u(0,0) + 2u(1,1) + 3\int_0^1 \int_0^1 u(t_1, t_2) dt_1 dt_2 = 3.$$

Then, by taking or using the (MVIM) of the fractional two-dimensional integro-differential equation and by using the property, one can get:

$$J_1^{\frac{1}{2}} x^{\frac{3}{2}} = \frac{\Gamma(\frac{3}{2}+1)}{\Gamma(\frac{1}{2}+\frac{3}{2}+1)} x^{\frac{1}{2}+\frac{3}{2}} = \frac{3}{4}x^2$$

$$J_2^{\frac{1}{4}} y^{\frac{3}{4}} = \frac{\Gamma(\frac{3}{4}+1)}{\Gamma(\frac{1}{4}+\frac{3}{4}+1)} y^{\frac{1}{4}+\frac{3}{4}} = \frac{3}{4}y.$$

Assuming $f_{0,0}(x,y) = x^2 y$ and $f_{1,1}(x,y) = -J_1^{\frac{1}{2}} J_2^{\frac{1}{4}} 1 - J_1^{\frac{1}{2}} J_2^{\frac{1}{4}} e^{x^2+y^2} - J_1^{\frac{1}{2}} J_2^{\frac{1}{4}} x^2 y^2 e^{x^2+y^2}$, and choosing $u_{0,0}(x,y) = f_{0,0}(x,y) = x^2 y$, we have:-

$$\begin{aligned} u_{0,0}(x,y) &= x^2 y \\ u_{1,1}(x,y) &= x^2 y - J_1^{\frac{1}{2}} J_2^{\frac{1}{4}} 1 - J_1^{\frac{1}{2}} J_2^{\frac{1}{4}} e^{x^2+y^2} - J_1^{\frac{1}{2}} J_2^{\frac{1}{4}} x^2 y^2 e^{x^2+y^2} \\ &- L_{xy}^{-1}\left[\int_0^y \int_0^x x^2 y^2 e^{x t_1 + y t_2} t_1^2 t_2 dt_1 dt_2\right] \\ u_{n_i+1}(x,y) &= x^2 y, \quad n_i \geq 1, \quad i = 1, 2. \end{aligned}$$

In view of the above equation, it follows that:
$u(x,y) = x^2 y$,
is the exact solution of the non-local boundary conditions.

(ii) Now if we consider the initial and boundary value problem which consists of the 1st order linear two-dimensional Volterra integro-differential equation:

$$u_x(x,y) = -xy + \int_0^y \int_0^x u(s,t) ds dt$$

together with the initial and boundary conditions $u(0,y)=1$. Then,

$$J_1 x = \frac{\Gamma(2)}{\Gamma(3)}x^2 = \frac{1}{2!}x^2$$

Assuming $f_{0,0}(x,y)=x^2$ and choosing $u_{0,0}(x,y)=f_{0,0}(x,y)=x^2$, we have:

$$\begin{aligned} u_{0,0}(x,y) &= x^2 \\ u_{1,1}(x,y) &= x^2 - L_{xy}^{-1}\left[\int_0^y\int_0^x u(s,t)dsdt\right] \\ u_{n_i+1}(x,y) &= x^2, \quad n_i \geq 1, \quad i=1,2. \end{aligned}$$

In view of the above equation, it follows that:

$$u(x,y)=x^2.$$

We note that in particular if $\sum_{i=1}^{2} v_i = 1$, then the derivative is the classical ordinary derivatives.

2- When different derivative $\sum_{i=1}^{3} v_i = \frac{5}{3}$ and $1 < \sum_{i=1}^{3} v_i \leq 2$, then:

Consider the 2nd fractional order two-dimensional integro-differential equation:

$$\begin{aligned} \frac{\partial^{\frac{5}{3}}}{\partial x^{\frac{2}{3}}\partial y^{\frac{1}{4}}\partial z^{\frac{3}{4}}} &= -x^2yz^3e^{x+y+z}u(x,y,z) + \frac{x^{\frac{7}{3}}y^{\frac{7}{4}}z^{\frac{1}{4}}}{\Gamma(\frac{13}{3})\Gamma(\frac{9}{4})\Gamma(\frac{5}{2})} \\ &+ \int_0^z\int_0^y\int_0^x x^2y^2t_1t_2t_3e^{x+y+z}u(t_1,t_2,t_3)dt_1dt_2dt_3, \end{aligned}$$

with appropriate non-local boundary condition:

$$\frac{\partial u(x,0)}{\partial y} + u(x,0) + \int_0^1\int_0^1 u(t_1,t_2)dt_1dt_2 = 0.$$

Then, by taking or using the MVIM of the fractional three-dimensional integro-differential equation and by using property, we get:

$$J_1^{\frac{2}{3}}x^{\frac{7}{3}} = \frac{\Gamma(\frac{7}{3}+1)}{\Gamma(\frac{2}{3}+\frac{7}{3}+1)}x^{\frac{2}{3}+\frac{7}{3}} = \frac{7!}{3!3!}x^3$$

$$J_2^{\frac{1}{4}}y^{\frac{7}{4}} = \frac{\Gamma(\frac{7}{4}+1)}{\Gamma(\frac{1}{4}+\frac{7}{4}+1)}y^{\frac{1}{4}+\frac{7}{4}} = \frac{7!}{2!4!}y^2$$

$$J_3^{\frac{3}{4}}z^{\frac{1}{4}} = \frac{\Gamma(\frac{1}{4}+1)}{\Gamma(\frac{3}{4}+\frac{1}{4}+1)}z^{\frac{1}{4}+\frac{7}{4}} = \left(\frac{1}{4}\right)!z^2.$$

Then, by letting $u_0(x,y,z)=x^3y^2z^2$ and $u_1(x,y,z)=0$. Now, we select: $u_0(x,y,z)=x^3y^2z^2$ to find:

$$\begin{aligned} u_0(x,y,z) &= x^3y^2z^2 \\ u_1(x,y,z) &= x^3y^2z^2 - J_1^{\frac{2}{3}}J_2^{\frac{1}{4}}J_3^{\frac{3}{4}}x^2yz^3e^{x+y+z}x^3y^2z + J_1^{\frac{2}{3}}J_2^{\frac{1}{4}}J_3^{\frac{3}{4}}\frac{x^{\frac{7}{3}}y^{\frac{7}{4}}z^{\frac{1}{4}}}{\Gamma(\frac{13}{3})\Gamma(\frac{9}{4})\Gamma(\frac{5}{2})} \\ &+ L_{xyz}^{-1}J_1^{\frac{2}{3}}J_2^{\frac{1}{4}}J_3^{\frac{3}{4}}\int_0^z\int_0^y\int_0^x x^2y^2t_1t_2t_3e^{x+y+z}t_1^3t_2^2t_3^2dt_1dt_2dt_3 \\ u_{n_i+1}(x,y,z) &= x^3y^2z^2, \quad n_i \geq 1, \quad i=1,2,3. \end{aligned}$$

In view of the above equation, it follows that:-

$$u(x,y,z) = x^3y^2z^2,$$

is the exact solution of the non-local boundary conditions.

3- When different derivative $\sum_{i=1}^{2} v_i = \frac{9}{4}$ and $2 < \sum_{i=1}^{2} v_i \leq 3$, then:

Consider the 3rd fractional order three-dimensional integro-differential equation:

$$\frac{\partial^{\frac{9}{4}}}{\partial y^{\frac{9}{4}}} = \sin(x)\cos(y)u(x,y) + \frac{1}{\Gamma(\frac{3}{4})}x^{\frac{9}{4}} - \frac{1}{4}y^{\frac{7}{4}} + \int_0^y \int_0^x \sin t_1 \cos t_2 u(t_1,t_2)dt_1dt_2.$$

With appropriate non-local boundary condition:

$$u(0,y) + u(x,0) + u_x(0,y) + \int_0^1 \int_0^1 u(t_1,t_2)dt_1dt_2 = 0.$$

Then, by taking or using the MVIM of the fractional two-dimensional integro-differential equation and by using property, we get:

$$J_2^{\frac{9}{4}} y^{\frac{7}{4}} = \frac{\Gamma(\frac{7}{4}+1)}{\Gamma(\frac{9}{4}+\frac{7}{4}+1)} y^{\frac{9}{4}+\frac{7}{4}} = \frac{7!}{4!4!}y^4.$$

Assuming $u_0(x,y) = y^4$ and $u_1(x,y) = 0$ and we select $u_0(x,y) = y^4$, we have:

$$\begin{aligned}
u_0(x,y) &= y^4 \\
u_1(x,y) &= y^4 - J_2^{\frac{9}{4}} y^4 \sin x \cos y + J_2^{\frac{9}{4}} \frac{1}{\Gamma(\frac{3}{4})} x^{\frac{9}{4}} - J_2^{\frac{9}{4}} \frac{1}{4} y^{\frac{7}{4}} + L_y^{-1} J_2^{\frac{9}{4}} \int_0^y \int_0^x t_2^4 \sin t_1 \cos t_2 dt_1 dt_2 \\
u_{n_i+1}(x,y) &= y^4, \quad n_i \geq 1, \quad i = 1,2.
\end{aligned}$$

In view of the above equation, it follows that:-

$$u(x,y) = y^4,$$

is the exact solution of the non-local boundary conditions.

4- When $\sum_{i=1}^{2} v_i = \frac{1}{2}$, $\sum_{i=1}^{2} v_i = \frac{2}{3}$ and $0 < \sum_{i=1}^{2} v_i \leq 1$, then: Consider the 1st fractional order two-dimensional integro-differential equation:

$$\frac{\partial^{\frac{1}{2}} u(x,y)}{\partial y^{\frac{1}{2}}} + \frac{\partial^{\frac{2}{3}} u(x,y)}{\partial x^{\frac{2}{3}}} = \frac{1}{3!}x^3 + \frac{1}{\Gamma(\frac{1}{3})}x^{\frac{1}{3}} + \frac{4}{3}y^{\frac{3}{2}} + \int_0^y \int_0^x e^{t_1 t_2} u(t_1,t_2)dt_1dt_2.$$

With appropriate non-local boundary condition:

$$u(0,y) + 2u(x,0) + \int_0^1 \int_0^1 u(t_1,t_2)dt_1dt_2 = 1.$$

Then, by taking or using the MVIM of the fractional two-dimensional integro-differential equation and by using property, we get:

$$J_1^{\frac{2}{3}} x^3 = \frac{\Gamma(3+1)}{\Gamma(\frac{2}{3}+3+1)} x^{\frac{2}{3}+3} = \frac{\Gamma(4)}{\Gamma(\frac{14}{3})} x^{\frac{11}{3}} = \left(\frac{3}{11}\right)! x^{\frac{11}{3}}$$

$$J_1^{\frac{2}{3}} x^{\frac{1}{3}} = \frac{\Gamma(\frac{1}{3}+1)}{\Gamma(\frac{2}{3}+\frac{1}{3}+1)} x^{\frac{2}{3}+\frac{1}{3}} = \left(\frac{1}{3}\right)! x$$

$$J_2^{\frac{1}{2}} y^{\frac{3}{2}} = \frac{\Gamma(\frac{3}{2}+1)}{\Gamma(\frac{1}{2}+\frac{3}{2}+1)} x^{\frac{1}{2}+\frac{3}{2}} = \frac{3!}{4} y^2.$$

Assuming $f_{0,0}(x,y) = x + y^2$ and $f_{1,1}(x,y) = -J_1^{\frac{2}{3}} \left\{ (\frac{3}{11})! x^{\frac{11}{3}} - x + y^2 \right\}$ and choosing $u_{0,0}(x,y) = f_{0,0}(x,y) = x + y^2$, we can get:

$$\begin{aligned} u_{0,0}(x,y) &= x + y^2 \\ u_{1,1}(x,y) &= -J_1^{\frac{2}{3}} \left\{ \left(\frac{3}{11} \right)! x^{\frac{11}{3}} - x + y^2 \right\} + L_{xy}^{-1} J_1^{\frac{2}{3}} J_2^{\frac{1}{2}} \int_0^y \int_0^x e^{t_1 t_2} (t_1 + t_2^2) dt_1 dt_2 \\ u_{n_i+1}(x,y) &= x + y^2, \quad n_i \geq 1, \quad i = 1, 2. \end{aligned}$$

In view of the above equation, it follows that:-

$$u(x,y) = x + y^2$$

is the exact solution of the non-local boundary conditions.

5- (i) When different derivatives $\sum_{i=1}^{3} v_i = \frac{7}{2}$ and $3 < \sum_{i=1}^{3} v_i \leq 4$ and $\sum_{i=1}^{3} v_i = \frac{5}{2}$ and $2 < \sum_{i=1}^{3} v_i \leq 3$, then:

Consider the 2nd fractional order two-dimensional integro-differential equation:

$$\begin{aligned} \frac{\partial^{\frac{7}{2}} u(x,y,z)}{\partial x^{\frac{4}{3}} \partial y^{\frac{3}{2}} \partial z^{\frac{2}{3}}} + \frac{\partial^{\frac{5}{2}} u(x,y)}{\partial x^{\frac{4}{3}} \partial y^{\frac{3}{2}} \partial z^{\frac{2}{3}}} &= x^{\frac{5}{4}} y^{\frac{1}{2}} z^{\frac{1}{3}} + z^{\frac{2}{5}} e^{y^4 z^5} y^{\frac{1}{4}} u(x,y,z) \\ &+ \int_0^z \int_0^y \int_0^x xy^3 z u(t_1,t_2,t_3) dt_1 dt_2 dt_3, \end{aligned}$$

with appropriate non-local boundary condition:

$$\frac{\partial u(0,y,z)}{\partial x} + \frac{\partial^2 u(x,0,z)}{\partial y^2} + \frac{\partial u(x,y,0)}{\partial z} \int_0^1 \int_0^1 \int_0^1 u(t_1,t_2,t_3) dt_1 dt_2 dt_3 = 0.$$

Then, by taking or using the MVIM of the fractional three-dimensional integro-differential equation and by using property, we get:

$$J_{1,1}^{\frac{4}{3}} x^{\frac{5}{4}} = \frac{\Gamma(\frac{5}{4}+1)}{\Gamma(\frac{4}{3}+\frac{5}{4}+1)} x^{\frac{4}{3}+\frac{5}{4}} = \frac{\Gamma(\frac{9}{4})}{\Gamma(\frac{43}{12})} x^{\frac{31}{12}},$$

$$J_{1,2}^{\frac{3}{4}} x^{\frac{5}{4}} = \frac{\Gamma(\frac{5}{4}+1)}{\Gamma(\frac{3}{4}+\frac{5}{4}+1)} x^{\frac{3}{4}+\frac{5}{4}} = \frac{\Gamma(\frac{9}{4})}{\Gamma(3)} x^2,$$

$$J_{2,1}^{\frac{3}{2}} y^{\frac{1}{2}} = \frac{\Gamma(\frac{1}{2}+1)}{\Gamma(\frac{3}{2}+\frac{1}{2}+1)} y^{\frac{3}{2}+\frac{1}{2}} = \frac{\Gamma(\frac{3}{2})}{\Gamma(3)} y^2,$$

$$J_{2,1}^{\frac{3}{2}} y^{\frac{1}{4}} = \frac{\Gamma(\frac{1}{4}+1)}{\Gamma(\frac{3}{2}+\frac{1}{4}+1)} y^{\frac{3}{2}+\frac{1}{4}} = \frac{\Gamma(\frac{5}{4})}{\Gamma(\frac{11}{4})} y^{\frac{7}{4}},$$

$$J_{2,2}^{\frac{7}{4}} y^{\frac{1}{2}} = \frac{\Gamma(\frac{1}{2}+1)}{\Gamma(\frac{7}{4}+\frac{1}{2}+1)} y^{\frac{7}{4}+\frac{1}{2}} = \frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{13}{4})} y^{\frac{9}{4}},$$

$$J_3^{\frac{2}{3}} z^{\frac{1}{3}} = \frac{\Gamma(\frac{1}{3}+1)}{\Gamma(\frac{2}{3}+\frac{1}{3}+1)} z^{\frac{2}{3}+\frac{1}{3}} = \frac{\Gamma(\frac{4}{3})}{\Gamma(2)} z,$$

$$J_3^{\frac{2}{3}} z^{\frac{2}{5}} = \frac{\Gamma(\frac{2}{5}+1)}{\Gamma(\frac{2}{3}+\frac{2}{5}+1)} z^{\frac{2}{3}+\frac{2}{5}} = \frac{\Gamma(\frac{7}{5})}{\Gamma(\frac{31}{15})} z^{\frac{16}{15}}.$$

Then, by letting $u_{0,0,0}(x,y,z) = x^2y^2z$ and $u_{1,1,1}(x,y,z) = 0$. Now, we select, $u_0(x,y,z) = x^2y^2z$ to find:

$$\begin{aligned}
u_{0,0,0}(x,y,z) &= x^2y^2z \\
u_{1,1,1}(x,y,z) &= x^2y^2z + J_1^{\frac{7}{2}}\left[x^{\frac{5}{4}}y^{\frac{1}{2}}z^{\frac{1}{3}} + z^{\frac{2}{5}}e^{y^4z^5}y^{\frac{1}{4}}x^2y^2z\right] \\
&+ J_1^{\frac{7}{2}}\left[\int_0^z\int_0^y\int_0^x xy^3zt_1^2t_2^2t_3dt_1dt_2dt_3,\right] + J_2^{\frac{5}{2}}\left[x^{\frac{5}{4}}y^{\frac{1}{2}}z^{\frac{1}{3}} + z^{\frac{2}{5}}e^{y^4z^5}y^{\frac{1}{4}}x^2y^2z\right] \\
&+ J_2^{\frac{5}{2}}\left[\int_0^z\int_0^y\int_0^x xy^3zu(t_1,t_2,t_3)dt_1dt_2dt_3,\right] \\
&+ L_{xyz}^{-1}J_1^{\frac{2}{3}}J_2^{\frac{1}{4}}J_3^{\frac{3}{4}}\int_0^z\int_0^y\int_0^x x^2y^2t_1t_2t_3e^{x+y+z}t_1^2t_2^2t_3dt_1dt_2dt_3, \\
&\quad u_{n_i+1}(x,y,z) = x^2y^2z.
\end{aligned}$$

In view of the above equation, it follows that:-

$$u(x,y,z) = x^2y^2z, \quad n_i \geq 1, \quad i = 1,2,3.$$

is the exact solution of the non-local boundary conditions.

(ii) Consider the initial and boundary value problem which consists of the 4th order linear three-dimensional Volterra integro-differential equation:

$$u_{yyyy}(x,y,z) = x^2y^6z^2 + \int_0^z\int_0^y\int_0^x u(t_1,t_2,t_3)dt_1dt_2dt_3$$

together with the initial and boundary conditions
$u(x,0,z) = u_y(x,0,z) = u_{yy}(x,0,z) = u_{yyy}(x,0,z) = 0$. Then,

$$J_2^4 y^6 = \frac{\Gamma(7)}{\Gamma(11)} y^{10} = \frac{6!}{10!} y^{10}$$

Assuming $f_{0,0,0}(x,y,z) = y^{10}$ and choosing $u_(0,0,0) = f(x,y,z) = y^{10}$, we have:

$$\begin{aligned}
u_{0,0,0}(x,y,z) &= y^{10} \\
u_{1,1,1}(x,y,z) &= y^{10} - L_{xy}^{-1}\left[\int_0^z\int_0^y\int_0^x u(t_1,t_2,t_3)dt_1dt_2dt_3\right] \\
u_{n_i+1}(x,y,z) &= y^{10}, \quad n_i \geq 1, \quad i = 1,2.
\end{aligned}$$

In view of the above equation, it follows that:-

$$u(x,y) = y^{10}.$$

We note that in particular if $\sum_{i=1}^{3} v_i = 4$, then the derivative is the classical ordinary derivatives.

6- Let different derivative $\sum_{i=1}^{2} v_i = \frac{14}{3}$ and $4 < \sum_{i=1}^{2} v_i \leq 5$, then: Consider the 5th fractional order two-dimensional integro-differential equation:

$$\frac{\partial^{\frac{14}{3}} u(x,y)}{\partial x^{\frac{10}{3}} \partial y^{\frac{4}{3}}} = \frac{\Gamma(7)\Gamma(4)}{\Gamma(\frac{11}{3})\Gamma(\frac{8}{3})} x^{\frac{8}{3}} y^{\frac{5}{3}} + \int_0^y \int_0^x \frac{e^{xt_1}}{(x-y)^2 e^{yt_2}} u(t_1,t_2) dt_1 dt_2,$$

with appropriate non-local boundary condition:

$$u(0,0) + 2\frac{\partial u(x,0)}{\partial y} + \frac{\partial^3 u(x,0)}{\partial y^3} + \frac{\partial^2 u(0,y)}{\partial x^2} + \int_0^1 \int_0^1 u(t_1,t_2) dt_1 dt_2 = 0.$$

Then, by taking or using the MVIM of the fractional two-dimensional integro-differential equation and by using property, one can get:

$$J_1^{\frac{10}{3}} x^{\frac{8}{3}} = \frac{\Gamma(\frac{8}{3}+1)}{\Gamma(\frac{10}{3}+\frac{8}{3}+1)} x^{\frac{10}{3}+\frac{8}{3}} = \frac{\frac{8}{3}!}{6!} x^{\frac{18}{3}} = \frac{8!}{3!6!} x^6,$$

$$J_2^{\frac{4}{3}} y^{\frac{5}{3}} = \frac{\Gamma(\frac{5}{3}+1)}{\Gamma(\frac{4}{3}+\frac{5}{3}+1)} y^{\frac{4}{3}+\frac{5}{3}} = \frac{\Gamma(\frac{8}{3})}{\Gamma(\frac{12}{3})} y^{\frac{9}{3}} = \frac{5!}{3!3!} y^3.$$

Assuming $f_{0,0}(x,y) = x^6 y^3$ and choosing $u_{0,0}(x,y) = f_{0,0}(x,y) = x^2 y$, we have:

$$\begin{aligned} u_{0,0}(x,y) &= x^6 y^3 \\ u_{1,1}(x,y) &= x^6 y^3 - L_{xy}^{-1} J_1^{\frac{10}{3}} J_2^{\frac{4}{3}} \left[\int_0^y \int_0^x \frac{e^{xt_1}}{(x-y)^2 e^{yt_2}} t_1^6 t_2^3 dt_1 dt_2 \right] \\ u_{n_i+1}(x,y) &= x^6 y^3, n_i \geq 1, i = 1,2. \end{aligned}$$

In view of the above equation, it follows that:-

$$u(x,y) = x^6 y^3$$

is the exact solution of the non-local boundary conditions.

3 Conclusion

In this work, we propose a reliable multi-dimensional modification for He's variational iteration method for solving multi-dimensional fractional order integro-differential equations with the given conditions. It is illustrated that the method is a very powerful, highly accurate and efficient to find approximate solutions as well as numerical solutions.

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