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An Alternate Non-repetitive Fractional Colouring of Graphs*

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Abstract

Fractional colouring of a graph is an assignment of a set of $b \geq 1$ colour(s) to each of its vertices so that adjacent vertices receive disjoint sets of colours. This is a generalisation of the traditonal vertex colouring, and the case of $b = 1$ is the traditional one. A traditional vertex colouring of a graph G is non-repetitive if the sequence of colours assigned to the vertices in the first half way of any even path in G is not identical with the sequence of colours assigned to the vertices in the second half. The minimum number of colours required for such a colouring is termed as *thue chromatic number* of the graph. Non-repetetive fractional colouring can be defined in more than one ways. Zhong and Zhu defined one such and studied fractional thue chromatic number. Here we define another variant and discuss the corresponding fractional thue chromatic number.

Keywords: Graph colouring, non-repetitive colouring, fractional colouring, non-repetitive fractional colouring, fractional thue chromatic number, weak non-repetitive fractional colouring, weak fractional thue chromatic number.

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1 Introduction

Graph colouring is favourite to a good number and has applications in various areas. Chromatic number may be the most celebrated graph invariant. The

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concept of fractional colouring of graphs, a generalisation of vertex colouring, is in the literature before four decades[14]. It is developed as an integral component of fractionl graph theory as well as a useful tool in combinatorial analysis.

Also called b-tuple colouring[4, 10], b-fold colouring[12, 11], list colouring [2, 14] or set colouring[8], a fractional colouring is a map f that assigns a set of $b \geq 1$ colour(s) to each vertex in a graph G, from a set of $a \geq b$ colours, so that the set of colours assigned to adjacent vertices are disjoint.

Such a map f is called an *a:b*-colouring of the graph. A graph G that admits an *a:b*-colouring is referred to as *a:b*-colourable[12], *a/b*-colourable[10], or *a:b*-choosable[2, 5]. The set $f(u)$ of b colours assigned to a vertex u is called the colour class of that vertex. An *a:b*-colouring coincides with traditional vertex colouring when $b = 1$.

In 2002, Alon et.al.[1] introduced non-repetitive vertex colouring and studied the corresponding chromatic number - *the thue chromatic number*. A proper vertex colouring of a graph G is non-repetitive if the sequence of colours assigned to the vertices in the first half way of any even path in G is not identical with the sequence in the second half way. The minimum number of colours required for such a colouring of a graph is termed [1] as its *thue chromatic number*, and is denoted by $\pi(G)$. The naming after Thue is because, in 1906, he proved[13] the existance of an infinite non-repetitive sequence using only three symbols, which in graph theory terms reads - *any path has a non-repetitive vertex colouring using atmost three colours*. This is equivalent to the statement "the thue chromatic number of a path is at most 3". The seemingly simple result by Thue contributed to the theory of symbolic dynamics, a topic in mathematical theory of dynamical systems, due to J. Hadamard[7].

The least value of a for which a graph G admits an *a:b*-colouring is refered to as the b-fold chromatic number of G, and is denoted by $\chi_b(G)$. It is observed in [12] that $\chi_{b+b'}(G) \leq \chi_b(G) + \chi_{b'}(G)$ for any graph G. The *fractional chromatic number* $\chi_f(G)$ of a graph G is defined to be the infimum of all fractions $\frac{a}{b}$ where $a = \chi_f(G)$. Since its first appearance [14], the concept of fractionl colouring attracted the attention of many people including Berge(1978), Erdös(1979) and Fisher(1995).

The natural question of non-repetitive fractional colouring of graphs and the corresponding thue chromatic number left untouched till Zhong and Zhu[16] worked upon it in 2016. They used the symbol $\pi_f(G)$ to denote *fractional thue chromatic number* and studied it. According to them a b-fold colouring f of G is non-repetitive if, for every path-subgraph $v_1v_2 \ \ldots \ \ldots \ v_{2k}$ in G and $c_i \in f(v_i)$, $1 \leq i \leq 2k$ none of the b^k sequences $< c_1, \ c_2, \ \ldots \ \ldots, \ c_k >$ is identical with any of the b^k sequences $< c_{k+1}, \ c_{k+2}, \ \ldots \ \ldots, \ c_{2k} >$. This is posible if, and only if, the colour classes $f(v_i)$ and $f(v_{k+i})$ are **disjoint** for at least one pair of vertices $\{v_i, v_{k+i}\}$ in every path $v_1v_2, \ \ldots \ \ldots \ v_{2k}$. Because, otherwise there exist identical sequences $< c_1, \ c_2, \ \ldots \ \ldots, \ c_k >$ and $< c_{k+1}, \ c_{k+2}, \ \ldots \ \ldots, \ c_{2k} >$ with $c_i = c_{k+i} \in f(v_i) \cap f(v_{k+i})$. Lemma 1 in [16] is a special case of this observation.

In this article we introduce an alternate non-repetitive fractional colouring - *weak non-repetitve fractional colouring*, with a relaxed condition and discuss the invariant thereof. Througout this article, by C_n we mean a closed path (i.e. a cycle) of order n and by P_n, a path of order n which is not closed. Also assume that every graph is finite and simple, and follow [3] and [15] for graph related concepts and terminology not elaborated here.

2 Weak Fractional Thue Chromatic Number

A b-fold colouring f of a graph G of order n is a *weak non-repetitive fractional colouring* if the colour classes $f(u_i)$ and $f(u_{k+i})$ are **distinct** for some $i \in \{1,\ 2,\ 3 \dots\ \dots k\}$ in every even path $P = u_1u_2\ \dots\ \dots\ u_{2k}$ in G, $2 \leq k \leq \frac{n}{2}$. That is for every path P_{2k} in G, there must be an $i \in \{1,\ 2,\ 3 \dots\ \dots k\}$ such that $f(u_i)$ and $f(u_{k+i})$ differ in atleast one colour. We call the corresponding fractional chromatic number as the *weak fractional thue chromatic number* of G and denote it by $\pi_w(G)$.

Thus $\pi_w(G) = \inf\limits_b \left\{ \dfrac{a}{b} :\ G \text{ admits a weak non-repetitive } a{:}b\text{-colouring} \right\}$.

It is evident that any non-empty graph need at least $2b$ colours for an $a{:}b$-colouring. Thus, $\pi_w(G) \geq 2$ for any non-empty graph. On the other end, $\pi_w(K_n) = n$, since an $a{:}b$-colouring of the complete garaph K_n require $a = nb$ colours. Since any path in a subgraph H of G is a path in G also, an optimal b-fold colouring of H will use no more colours than that used in a b-fold colouring of G. Thus we have

Proposition 1: *Let G be any non-empty graph order n and H be a subgraph of G, the weak non-repetitive fractional chromatic number π_w satisfy*

(a) $2 \leq \pi_w(G) \leq n$ and (b) $\pi_w(H) \leq \pi_w(G)$. □

Now we estimate the weak fractional thue chromatic number for some common classes of graphs, beginning with bipartite graphs.

2.1 Complete Bipartite Graphs

When discussing bipartite graphs, we assume that $U = \{u_1,\ u_2,\ \dots\ \dots,\ u_m\}$ and $V = \{v_1,\ v_2,\ \dots\ \dots,\ v_n\}$ as the partite sets with $m \geq 2$ and $n \leq m$.

Theorem 2: *The weak fractional thue chromatic number of the complete bipartite graph $K_{m,n}$ is 2.*

Proof. Let $G = K_{m,n}$ and $b \geq n$ be an integer. $X = \{c_1,\ c_2,\ \dots,\ c_b\}$ and $Y = \{d_1,\ d_2,\ \dots\ \dots,\ d_{b+1}\}$ be two disjoint sets of colours. Define a b-fold colouring f of G by $f(u_i) = X$ for every $1 \leq i \leq m$ and $f(v_j) = Y - \{d_j\}$ for every $1 \leq j \leq n$. In this colouring the colour classes of adjacent vertices are disjoint since X and Y are disjoint and no edge of the form u_iu_j or v_iv_j is present in G. Since the colour classes $f(v_j)$ of vertices in V are all distinct and vertices of every path in G alternate between the partite sets U and V, the colouring f is weak non-repetitive. Further, this colouring uses $a = 2b + 1$ colours only. Since fractional chromatic number is defined as a limit as b approches infinity and we consider only finite graphs, the condition $b \geq n$, assumed at the beginning will not affect the value of $\inf_b \left(\frac{a}{b}\right)$. So $\pi_w(K_{m,n}) = 2$ □

Corollary 3: *The weak fractional thue chromatic number of any bipartite graph is* 2.

Proof. This follows from the theorem due to proposition 1(b), since any bipartite gaph is a sub-graph of some complete bipartite graph. □

Corollary 4: *Let $n > 1$ be an integer. If G is the path of order n, the even cycle of order $2n$ or a tree of order n. Then $\pi_w(G) = 2$.*

Proof. All these classes are bipartite, and hence follows from the above corollary. □

It is well known that the traditional chromatic number $\chi(P_n)$ of any non-trivial path P_n is 2. From the classical result of Thue [13], it follows that the thue chromatic number $\pi(P_n)$ is 3 for every $n > 3$ and 2 otherwise. Zhong and Zhu [16] observed that the fractional thue chromatic number $\pi_f(P_n)$ coincides with $\pi(P_n)$ for every $n > 1$. We now proved that the weak fractional thue chromatic number coincides with the traditional chromatic number for bipartite graphs. Thus we have the weak fractional thue chromatic number for any graph not having odd cycles as subgraphs. The cases of graphs containing odd cycles remains to be addressed. In this direction we worked with odd cycles and have a nice outcome.

2.2 Odd Cycles

Following the convention, we denote an odd cycle as C_{2k+1}, or as C_n if there is no confusion. To discuss colouring, the standard notation $v_0v_1 \ldots \ldots v_{2k-1}v_{2k}v_0$ also is used in the usual sense. We begin with a sharp bound for the total number of colours in a b-fold colouring for some special values of b.

Lemma 5: *For each pair $\{k, \lambda\}$ of non-zero positive integers and $b = k\lambda$, the least value of a for which C_{2k+1} admits a* a:b-*colouring is $(2k + 1)\lambda$.*

Proof. Consider an *a:b*-coluring of C_{2k+1}, $b = k\lambda$. Since colour classes of neighbouring vertices are disjoint, the maximum number of colour classes in which one single colour can appear is k. Hence, the maximum possbile appearances of all the a colours is ak. On the other hand, since b colours are used at every vertex, the actual total appearence of all of the used colours is $(2k+1)b$. Hence $(2k+1)b \leq ak$. That is $ak \geq (2k+1)k\lambda$. □

Lemma 6: *For each pair $\{k,\ \lambda\}$ of non-zero positive integers, the odd cycle C_{2k+1} possesses a weak non-repetitive a:b-colouring with $a = (2k+1)\lambda$ and $b = k\lambda$.*

Proof. Let $k \geq 1$ and $\lambda \geq 1$ be integers and let $n = 2k+1$ and $b = k\lambda$. Consider a set X of $n\lambda$ colours and a partition $\{X_0,\ X_1,\ X_2,\ \ldots,\ \ldots,\ X_{2k-1},\ X_{2k}\}$ of it into n subsets of λ colours each.

Let f be a colouring of $C_{2k+1} = v_0v_1 \ldots \ldots v_{2k-1}v_{2k}v_0$ in which colour classes are given by

$$f(v_s) = \begin{cases} \bigcup\limits_{j=0}^{\infty} X_{\frac{s}{2}+j} & \text{when } j \text{ is even} \\ \bigcup\limits_{j=0}^{\infty} X_{k+\frac{s+1}{2}+j} & \text{when } j \text{ is odd} \end{cases}$$

where the addition of indices is taken $mod(2k+1)$.

The colour classes are

$$\begin{array}{ll} f(v_0) = X_0 \cup X_1 \cup \ldots \cup X_{k-1} & f(v_1) = X_{k+1} \cup X_{k+2} \cup \ldots \cup X_{2k} \\ f(v_2) = X_1 \cup X_2 \cup \ldots \cup X_k & \quad \ddots \quad \ddots \quad \ddots \quad \ddots \\ \quad \ddots \quad \ddots \quad \ddots \quad \ddots & f(v_{2k-1}) = X_{2k} \cup X_0 \cup X_1 \cup \ldots \cup X_{k-2} \\ f(v_{2k}) = X_k \cup X_{k+1} \cup \ldots \cup X_{2k-1} & f(v_{2k+1}) = X_0 \cup X_1 \cup \ldots \cup X_{k-1}. \end{array}$$

From the choice of colour classes, it is clear that $f(v_s)$ and $f(v_{s+1})$ are disjoint for every s. Since $v_{2k+1} = v_0$, there is nothing wrong with the colour classes $f(v_{2k+1})$ and $f(v_0)$ being identical. Moreover, the colour classes $f(v_s)$ and $f(v_t)$ are distinct whenever $s \not\equiv t(mod(2k+1))$. Clearly f is a b-fold colouring and uses only $(2k+1)\lambda$ colours.

So, f is a weak non-repetitive $(n\lambda : k\lambda)$-colouring of C_{2k+1} for every $k \geq 1$. □

Lemma 7: *For each pair $\{k,\ \lambda\}$ of non-zero positive integers, the odd cycle C_{2k+1} admits a non-repetitve b-fold colouring using $a = (2k+1)\lambda + 3j$ colours where $\lambda = \lfloor b \div k \rfloor$ and $j = b - \lfloor b \div k \rfloor$.*

Proof. The case when b is a multiple of k is given by lemma 6.

When b is not a multiple of k, let $\lambda = \lfloor b \div k \rfloor$ and $j = \lambda - b$. Then $1 \leq j < k$ and $b = k\lambda + j$. For any such j, choose a set Y of $3j$ colours disjoint from the colour set X choosen in lemma 6 and partition it into three subsets, say $\{Y_0,\ Y_1,\ Y_2,\}$, of j colours each.

Let f be the $(k\lambda)$-colouring of C_{2k+1} described in the above lemma. Define another colouring f' of $C_{2k+1} = v_0v_1 \ldots \ \ldots \ v_{2k-1}v_{2k}v_0$ by

$$f'(v_s) = \begin{cases} f(v_s) \cup Y_0 & \text{when } s = 0 \\ f(v_s) \cup Y_1 & \text{when } s \text{ is odd} \\ f(v_s) \cup Y_2 & \text{when } s \text{ is non-zero and even} \end{cases}$$

From the defintion, is clear that $f'(v_s)$ and $f'(v_t)$ are disjoint (respectively distinct) whenever $f(v_s)$ and $f(v_t)$ are so. Thus f' is a weak non-repetitive b-fold colouring of C_{2k+1}, $b = k\lambda + j$. The total number of colours used is $a = (2k+1)\lambda + 3j$. The value of a is optimum since $\chi(C_{2k+1}) = 3$. □

Theorem 8: $\pi_w(C_{2k+1}) = 2 + \frac{1}{k}$.

Proof. Lemma 7 assuares an optimal weak non-repetitive a:b-colouring of C_{2k+1} with $b = k\lambda + j$, $a = (2k+1)\lambda + 3j$ for integers $k \geq 1$, $\lambda \geq 1$ and $0 \leq j \leq (k-1)$.

$$\text{Thus } \frac{a}{b} = \frac{(2k+1)\lambda + 3j}{k\lambda + j} = \frac{2k\lambda + 2j}{k\lambda + j} + \frac{\lambda + j}{k\lambda + j}.$$

Since we consider only finite graphs, k and, consequently, j are always finite.

$$\text{Hence } \pi_w(C_{2k+1}) = \lim_{b\to\infty}\left(\frac{a}{b}\right) = 2 + \lim_{\lambda\to\infty}\left(\frac{\lambda + j}{k\lambda + j}\right) = 2 + \frac{1}{k}.$$ □

Corollary 9: $\lim_{n\to\infty} \pi_w(C_n) = 2.$ □

2.3 Friendship Graphs

For an integer $n \geq 1$, the friendship graph F_n of order $2n+1$ and size $3n$ is the graph with n triangle sub-graphs having a single vertex in common. Since every non-trivial member in this class of graphs has at least two triangles as sub-graphs, they are neither bipartite, nor odd cycles. So this class is out of the classes we have discussed earlier.

Theorem 10: *The weak fractional thue chromatic number of any friendship graph is 3.*

Proof. Consider the friendship graph F_n. Let $\{u_0,\ u_1,\ u_2, \ldots,\ u_n, v_1,\ v_2, \ldots,\ v_n\}$ be the vertex set and $\{u_0u_i : 1 \leq i \leq n\} \cup \{u_0v_i : 1 \leq i \leq n\} \cup \{u_iv_i : 1 \leq i \leq n\}$ be the edge set. Due to the presence of tiangle sub-graphs, any b-fold colouring need at least three colour classes. So, $a \geq 3b$ and consequently $\pi_w(F_n) \geq 3$.

To prove the opposite inequality, cosider three pairwise disjoint sets of colours X, Y and Z of size b each. Define a b-fold colouring f of F_n by $f(u_0) = X$, $f(u_i) = Y$ for every $1 \leq i \leq n$ and $f(u_j) = Z$ for every $1 \leq j \leq n$. Clearly, adjacent vertices receive disjoint colour classes in this colouring. More over, any path P_4 in F_n contains the vertex u_0. Since a vertex can appear only once in P_n, the colouring f is non-repetitive also. Total number of colours used is $a = 3b$, and consequently, fractional thue chromatic cumber is 3. □

2.4 Graph Products

A bunch of graph products are difined and studed in literature. We here analyses two of the products for the relationship between the weak thue chromatic number of the product and the respective factor graphs.

2.4.1 Corona products

Let G and H be any two graphs and $V(G) = \{u_1, u_2, \dots u_n\}$. The corona product is obtained by taking one copy of G and n copies of H, and introducing adjacency between every vertex in the i^{th} copy of H to the vertex u_i in G, $1 \leq i \leq n$, in addition to the existing edges in G and in all copies of H. We use the symbol $G \odot H$ to denote the corona product of G and H. This operation is non-commutative. When H is non-empty, $G \odot H$ has triangle sub-graphs in it. So, in non-trivial situation, this class is different form the classes discussed earlier.

Theorem 11: $\pi_w(G \odot H) = \pi_w(G) + \pi_w(H)$.

Proof. Let G and H be two graphs with vertex sets $V(G) = \{u_1, u_2, \dots, u_n\}$ and $V(H) = \{v_1, v_2, \dots, v_m\}$. Let the minimum number of colours used for b-fold colourings of G and H be α and β respectively. Then $\pi_w G = \inf_b \left(\frac{\alpha}{b}\right)$ and $\pi_w H = \inf_b \left(\frac{\beta}{b}\right)$.

Assume that the set of α colours used in G is disjoint from the set of a_2 colours used in H. Consider the b-fold colouring of $G \odot H$ keeping the same colour classes at each vertex u_i as well as in each copy of v_j. Clearly this is a fractional colouring since there is no edge between vertices in different copies of H.

Let P_{2k} is any path in $G \odot H$. If P_{2k} is fully contained either in G or in a copy of H, then the colouring on P_{2k} is non-repetitive, since it is so already. Otherwise, note that for every vertex $v_1, v_2, \dots, v_m$ in any copy of H there is one and only one u_j adjacent each, and is common to all. So, if a path in $G \odot H$ is traced out from any u_j, it cannot trace to some other u_k. Therefore, all vertices $\{u_s\}$ in any path P in $G \odot H$ must induce a sub-path of P. This forbide the possibity of repetition on P_{2k}, when one end is in G and other end in some copy of H. Thus there exists a non-repetitive a:b-colouring of $G \odot H$ using $a = \alpha + \beta$ colours. However, no less colours will do; because, that will challenge the choice of α and β. This proves our claim. □

2.4.2 Lexicographic Products

The *lexicographic product* $G(H)$ of two graphs G and H is the graph with vertex set $V = V(G) \times V(H)$, the cartesian product of the vertex sets, and two vertices (u_i, v_s) and (u_j, v_t) are adjacent in $G(H)$ if either $u_i u_j$ is an edge in G or $u_i = u_j$ and $v_s v_t$ is an edge in H. This product is not symmetric.

Theorem 12: $\pi_w(G(H)) > \pi_w(G)\pi_w(H)$.

Proof. Consider two graphs G and H with vertex sets $V(G) = \{u_1,\ u_2,\ \ldots,\ u_m\}$ and $V(H) = \{v_1,\ v_2,\ \ldots,\ v_n\}$. Let $a(G) = \alpha$ and $a(H) = \beta$ be the minimum number of colours used in a b-fold colouring of the respective graph and let $\{c_1,\ c_2,\ \ldots,\ c_\alpha\}$ and $\{d_1,\ d_2,\ \ldots,\ \ldots,\ d_\beta\}$ respectively be the colour sets used.

For a fractional colouring of $G(H)$, consider the set $X = \{\partial_{(i,j)} :\ 1 \leq i \leq \alpha,\ 1 \leq j \leq \beta\}$ of colours. Define a map f of $G(H)$ as $\partial_{(i,j)} \in f((u_i, v_j))$ if, and only if c_i belongs to the colour class of u_i in the $\alpha{:}b$-colouring of G and d_j belongs to the colour class of v_j in the $\beta{:}b$-colouring of H considered.

Each of the projection maps of f are clearly the assumed b-fold colouring of the respecive factor. So, the reducion of number of colours used will challenge the choice of at least one of α and β.

Hence $\pi_w(G(H)) = \inf_b \left(\frac{\alpha\beta}{b}\right) > \inf_b \left(\frac{\alpha}{b}\right) \inf_b \left(\frac{\beta}{b}\right) = \pi_w(G)\pi_w(H).$

□

A consequence of the weak non-repetitive fractional colouring described in lemma 6 is - for every integers $k \geq 1$ and $\lambda \geq 1$ it is possible to construct a class of $2k+1$ pairwise distict collection of $k\lambda$ symbols each, out of a set of $(2k+1)\lambda$ symbols, in a recursive manner such that the collections can be arranged cyclically (as if on a dial) with disjoint consecutive entries. The resulting system is an optimal one also.

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