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On a production inventory system with customer impatience

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Abstract

$M/M/1$ production system under (s, S) policy with impatience of customers. Arrival process follows a Poisson process. Exponentially distributed Processing and production time. Each production is of 1 unit. We assume that when no item is available, new demands do not enter the queue; such customers are considered as lost. The production system is such that it take inconsequential time for the item produced to reach the retail shop. Customers in the waiting space are impatient when no item in the production site. The optimal values of maximum inventory level S and the production switching on level s have been studied for a cost function involving the steady state system characteristic measures.

AMS Subject Classifications: 60J10, 60K25, 90B05, 90B22.
Keywords: (s, S) production inventory system, impatient customers

1 Introduction

Queueing-inventory has its origin in Melikov et al. [5] and Sigman & Levy [7]. In Sigman and Levi [7] serving of item requires a positive arbitrary amount of time which leads to the formation of queue. A survey on inventory with positive service time in Krishnamoorthy et al. [2].

Krishnamoorthy and Viswanath [4] analyzing an $M/M/1$ production system under (s, S) policy. No customer joining the queue, when no item is available in the system, they obtain an absolute PFS. Recently Krishnamoorthy et al. [3] analyzed a supply chain model with a single server in which stocks are kept in both the production and the distribution center and when no item available in the distribution center, no customer joining the queue. They derive an absolute PFS. More details about production inventory system with positive service time is reffered from the paper by Krishnamoorthy and Viswanath [4].

Inventory problems are classified into two forms, first one is no customer leave the system under any circumstances. That means customers are assumed to be infinitely patient as if the inventory level becomes zero. Second one is the customers have no patience. That means customers leave the system due to impatience when the inventory level is zero. One may refer to Benjaafar et al. [1] for more details about an optimal control of a production system with impatience of customer.

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The remaining part of this article is organized as follows. In Section 2, the mathematical model is described and in Section 3, the steady state probabilities of the system is obtained. Some important system characteristics are computed in Section 4. Finally a cost function is constructed.

Notations:

$$N(t): \text{ number of customers in the system}$$
$$I(t): \text{ number of items present in the stock}$$
$$M(t) = \begin{cases} 0 & \text{production is in off mode} \\ 1 & \text{production is in on mode} \end{cases}$$

2 Mathematical formulation

Analyze an (s, S) production system with impatience of customers. Arrival follows a Poisson process with rate Λ. Exponentially distributed service and production time with parameter ν and α respectively. When the number of items reaches s, the production process is immediately switched on, which is kept in this mode until items in the production center becomes S and production is of 1 unit. When no item is available in the stock, no new arrival is permitted to join the queue; such demands are loss. The production system is such that it takes negligible time for the item produced to reach the retail shop. Customers in the waiting space are impatient and may leave the system, when no item is available, after an exponentially distributed time with the parameter β, $0 \leq \beta < \infty$ (see Figure 1).

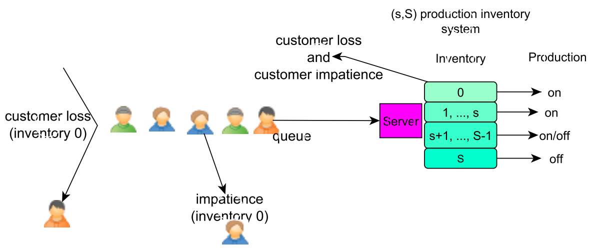

Figure 1: Pictorial representation of the model

The production process is on if $0 \leq I(t) \leq s$ and is off if $I(t) = S$. When $s+1 \leq I(t) \leq S-1$, that is on / off. Then $\Omega = \{(N(t), I(t), M(t)), t \geq 0\}$ is a continuous time Markov chain with state space $\{(m, i, 0), m \geq 0, s+1 \leq i \leq S\} \bigcup \{(m, i, 1), m \geq 0, 0 \leq i \leq S-1\}$. Thus the transition rates are:

1. Transition due to demands arrival

$$\begin{array}{lll} (m,i,0) \to (m+1,i,0): & \Lambda & \text{for } m \geq 0, s+1 \leq i \leq S \\ (m,i,1) \to (m+1,i,1): & \Lambda & \text{for } m \geq 0, 1 \leq i \leq S-1 \end{array}$$

2. Transition due to service completions

$$\begin{array}{lll} (m,i,0) \to (m-1,i-1,0): & \nu & \text{for } m \geq 1, s+2 \leq i \leq S \\ (m,i,1) \to (m-1,i-1,1): & \nu & \text{for } m \geq 1, 1 \leq i \leq S-1 \\ (m,s+1,0) \to (m-1,s,1): & \nu & \text{for } m \geq 1 \end{array}$$

3. Transition due to production process

$$\begin{array}{rll} (m,i,1)\to(m,i+1,1): & \alpha & \text{for } m\geq 0, 0\leq i\leq S-2 \\ (m,S-1,1)\to(m,S,0): & \alpha & \text{for } m\geq 0 \end{array}$$

4. Transition due to impatience of customers

$$(m,0,1)\to(m-1,0,1): \quad n\beta \quad \text{for } m\geq 1$$

Now describe the infinitesimal generator $\mathcal{W}$ is of the form

$$\mathcal{W}=\begin{pmatrix} Z_1^{(0)} & Z_0 & & & \\ Z_2^{(1)} & Z_1^{(1)} & Z_0 & & \\ & Z_2^{(2)} & Z_1^{(2)} & Z_0 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}. \tag{1}$$

Each matrix $Z_0, Z_1^{(i)}; i\geq 0, Z_2^{(i)}; i\geq 1$ are square matrices of order $2S-s$.

3 Steady state analysis

Let $\mathbf{x}$ be the steady state probability vector of $\mathcal{W}$. Then

$$\mathbf{x}\mathcal{W}=\mathbf{0} \qquad \mathbf{xe}=1. \tag{2}$$

Under Neuts-Rao truncation method, we derive the limiting probability vector $\mathbf{x}$ of the generator $\mathcal{W}$ by suitably identifying a positive integer, say, N such that defining $Z_1=Z_1^{(n)}, Z_2=Z_2^{(n)}$ for $n\geq N$ we use the following generator:

$$\mathcal{W}=\begin{pmatrix} Z_1^{(0)} & Z_0 & & & & & \\ Z_2^{(1)} & Z_1^{(1)} & Z_0 & & & & \\ & Z_2^{(2)} & Z_1^{(2)} & Z_0 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & Z_2^{(N-1)} & Z_1^{(N-1)} & Z_0 & \\ & & & & Z_2 & Z_1 & Z_0 \\ & & & & & \ddots & \ddots & \ddots \end{pmatrix}. \tag{3}$$

Stability condition: Let $\boldsymbol{\zeta}$ be the system state distribution of $Z=Z_0+Z_1+Z_2$. Then

$$\boldsymbol{\zeta}Z=\mathbf{0},\ \boldsymbol{\zeta}\mathbf{e}=1. \tag{4}$$

The components of $\boldsymbol{\zeta}$ are obtained as

$$\begin{aligned} \zeta(i,0) &= \zeta(S,0) && \text{for } s+1\leq i\leq S-1 \\ \zeta(i,1) &= \zeta(S,0)\sum_{j=1}^{S-i}\left(\frac{\nu}{\alpha}\right)^{j} && \text{for } s+1\leq i\leq S-1 \\ \zeta(s-i,1) &= \zeta(S,0)\sum_{j=1}^{S-s}\left(\frac{\nu}{\alpha}\right)^{j+i} && \text{for } 0\leq i\leq s. \end{aligned}$$

From $\boldsymbol{\zeta}\mathbf{e}=1$ we have

$$\zeta(S,0)=\left(\frac{\nu}{\alpha}-1\right)^2\left[\left(\frac{\nu}{\alpha}\right)^{S+2}-\left(\frac{\nu}{\alpha}\right)^{s+2}-(S-s)\left(\frac{\nu}{\alpha}-1\right)\right]^{-1}.$$

Theorem 1. *The production-inventory system is stable if and only if*

$$\Lambda(1-\zeta(0,1)) < \nu(1-\zeta(0,1)) + N\beta\zeta(0,1)$$

Proof. From Neuts [6] we have the given system is stable if and only if

$$\boldsymbol{\zeta} Z_0 \mathbf{e} < \boldsymbol{\zeta} Z_2 \mathbf{e}. \tag{5}$$

Now

$$\boldsymbol{\zeta} Z_0 \mathbf{e} = \Lambda\left(\sum_{i=1}^{S-1}\zeta(i,1) + \sum_{i=s+1}^{S}\zeta(i,0)\right) \tag{6}$$

$$\boldsymbol{\zeta} Z_2 \mathbf{e} = \nu\left(\sum_{i=1}^{S-1}\zeta(i,1) + \sum_{i=s+1}^{S}\zeta(i,0)\right) + N\beta\zeta(0,1) \tag{7}$$

and

$$\left(\sum_{i=0}^{S-1}\zeta(i,1) + \sum_{i=s+1}^{S}\zeta(i,0)\right) = 1. \tag{8}$$

Hence we get the required result. □

The vectors $\mathbf{x}_n, n \geq N$ are given by Nuets [6]

$$\mathbf{x}_{n+N-1} = \mathbf{x}_{N-1}M^n \text{ for } n \geq 1 \tag{9}$$

where M is the solution of

$$M^2 Z_2 + M Z_1 + Z_0 = O \tag{10}$$

and the vectors $\mathbf{x}_j, 0 \leq j \leq N-1$ are obtained from

$$\begin{aligned}
\mathbf{x}_0 {Z_1}^{(0)} + \mathbf{x}_1 {Z_2}^{(1)} &= \mathbf{0}, \\
\mathbf{x}_{j-1} Z_0 + \mathbf{x}_j {Z_1}^{(j)} + \mathbf{x}_{j+1} {Z_2}^{(j+1)} &= \mathbf{0}, \quad 1 \leq j \leq N-2 \\
\mathbf{x}_{N-2} Z_0 + \mathbf{x}_{N-1}\left(Z_1^{(N-1)} + M Z_2\right) &= \mathbf{0}
\end{aligned}$$

and

$$\sum_{i=1}^{N-2}\mathbf{x}_i \mathbf{e} + \mathbf{x}_{N-1}(I-M)^{-1}\mathbf{e} = 1 \tag{11}$$

where

$$\mathbf{x}_i = \mathbf{x}_0 \prod_{j=1}^{i} N_j, 1 \leq i \leq N-1 \tag{12}$$

with

$$N_j = \begin{cases} -Z_0\left({Z_1}^{(i)} + N_{j+1}{Z_2}^{(j+1)}\right)^{-1} & 1 \leq j \leq N-1, \\ -Z_0\left(Z_1^{(N-1)} + M Z_2\right)^{-1} & j = N-1 \end{cases} \tag{13}$$

and

$$\mathbf{x}_0\left[I + \sum_{i=1}^{N-2}\prod_{j=1}^{i} N_j + \prod_{j=1}^{N-1} N_j (1-M)^{-1}\right]\mathbf{e} = 1. \tag{14}$$

4

3.1 Some important system characteristics

- Expected number of customers in the system: $E_N = \sum_{m=1}^{\infty} m\mathbf{x}_m\mathbf{e}$.
- Expected number of items in the production-inventory system:
  $E_{inv} = \sum_{m=0}^{\infty}\left(\sum_{i=1}^{S-1} ix_m(i,1) + \sum_{i=s+1}^{S} ix_m(i,0)\right)$.
- Expected production rate: $E_{rp} = \alpha \sum_{m=0}^{\infty}\sum_{i=0}^{S-1} x_m(i,1)$.
- Expected loss rate: $E_{loss} = \Lambda \sum_{m=0}^{\infty} x_m(0,1)$.
- Expected rate of impatience: $E_{imp} = \beta \sum_{m=1}^{\infty} mx_m(0,1)$.
- Expected rate at which production is switched on: $E_{on} = \nu \sum_{m=1}^{\infty} x_m(s+1,1)$.
- Expected rate at which production is switched off: $E_{off} = \alpha \sum_{m=0}^{\infty} x_m(S-1,1)$.

4 Numerical illustration

In this section we provide some numerical examples.

Tables 1, 2, 3, 4 show the system performance with variation in values of underlying parameters.

Λ	E_N	E_{INV}	E_L	E_{IMP}
5	0.7518	0.6840	1.2513	0.0429
6	1.0246	0.5197	1.9803	0.0649
7	1.3727	0.4227	2.8481	0.0931
8	1.8201	0.3596	3.8381	0.1293
9	2.3958	0.3164	4.9275	0.1759

Table 1: Effect of Λ : Fix $S = 30, s = 12, \nu = 11, \alpha = 2, \beta = 0.1$

ν	E_N	E_{INV}	E_L	E_{IMP}
6	1.2752	0.9454	1.2820	0.5661
7	0.9601	0.8797	1.2307	0.4632
8	0.7675	0.8384	1.1812	0.3918
9	0.6382	0.8103	1.1366	0.3394
10	0.5458	0.7900	1.0971	0.2994

Table 2: Effect of ν : Fix $S = 20, s = 8, \Lambda = 5, \alpha = 2, \beta = 1.5$

α	E_N	E_{INV}	E_L	E_{IMP}
2	1.3727	0.4227	2.8481	0.0931
3	1.5061	0.7777	2.3492	0.0813
4	1.5943	1.3664	1.7913	0.0642
5	1.6586	2.5392	1.2067	0.0443
6	1.7086	5.8876	0.6105	0.0228

Table 3: Effect of α: Fix $S = 40, s = 23, \Lambda = 7, \nu = 11, \beta = 0.1$

β	E_N	E_{INV}	E_L	E_{IMP}
0.5	1.2739	0.7978	1.6614	0.2840
1	1.0627	0.8487	1.3953	0.3954
1.5	0.9601	0.8797	1.2307	0.4632
2	0.8975	0.9009	1.1164	0.5099
2.5	0.8548	0.9166	1.0318	0.5443

Table 4: Effect of β : Fix $S = 20, s = 8, \nu = 7, \Lambda = 5, \alpha = 2$

5 Analysis of cost function

Based on the above system characteristics for evaluate the optimality of s and S, we construct a cost function

$$CF = D_{inv}E_{inv} + D_{rp}E_{rp} + D_{loss}E_{loss} + HE_{on} + D_{imp}E_{imp}$$

where D_{inv} is the holding cost / unit time / item, D_{rp} is the cost of production / unit time / item, D_{loss} is the cost incurred due to loss of customers, H is the fixed cost for starting the production and D_{imp} is the cost incurred due to impatience of customers.

We assign the following values to: $H = \$2000, D_{inv} = \$50, D_{rp} = \$200, D_{loss} = \$400, D_{imp} = \$250$. The numbers in the bold form indicate optimal value of S and

S	CF $s = 1$	CF $s = 2$	CF $s = 3$
8	674.5587	676.9700	680.3378
9	668.2563	669.6951	671.6098
10	665.4503	666.3513	667.5182
11	664.4981	665.0789	665.8206
12	**664.4905**	**664.8713**	**665.3543**
13	664.9234	665.1753	665.4943
14	665.5211	665.6887	665.9008
15	666.1391	666.2509	666.3925
16	666.7071	666.7817	666.8763

Table 5: Effect of S on cost: Fix $\Lambda = 3, \nu = 5, \alpha = 2, \beta = 1.5$

corresponding minimum costs (see Table 5).

Also we obtain the cost function is convex in nature given in Figure 2.

Conclusion:

In this article we studied an $M/M/1$ model with production system under (s, S)-policy. A cost function in S was constructed to numerically investigate the optimal value. At the production center, measures such as the expected number of up and down crossings

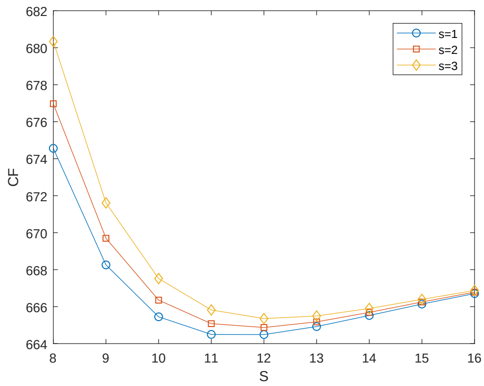

Figure 2: S verses CF

of s, while production is on is of interest. In a future paper we proposed to extend the present work to generally distributed service and production time.

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