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On a queueing-inventory system with impatience of customers

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Abstract

Analyzes an $M/M/1$ model with a storage system under (s, Q) and (s, S) policy having impatient customers in the buffer. Arrival process is Poisson. Exponentially distributed service and replenishment time. Assume (a) if $i > s$, then new customer joins; (b) if $i \leq s$ and $n < i$, then new arrival joins; (c) if $i \leq s$ and $i \leq n$, then the new arrivals do not join; (d) if $i \leq s$, customers in the buffer are impatient where i, n, s denote the number of items in the inventory, number of customers in the system and re-order level, respectively. Some numerical examples are provided.

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Keywords: Inventory, Impatience.

1 Introduction

In many real life situations positive service time is involved before an item is delivered to the customer. The inventory models with processing time was first introduced in Sigman and Levi [7]. They assumed the case of serving of an item requiring a random amount of time, thus leading to the formation of queue.

$M/M/1$ model with a storage system where the exponentially distributed replenishment time was first discussed by Schwarz et al. [5]. They obtain PFS under the assumption no customer enters the system when no item is available. The queueing networks with attached inventory was considered by Schwarz et al. [6]. The order for replenishment at each service station is made when the inventory level reaches s. They discussed about the product form solution of the steady state distributions.

A queueing-inventory system under the (s, Q) policy with lost sales in which arrival occur according to a Poisson process and exponentially distributed service time was discussed by Saffari et al. [4]. Krishnamoorthy et al. [2] is a survey paper which includes more papers related to positive service time. In this paper we have a deduction which appeared in Dhanya Shajin [1].

The remaining part of this article is organized as follows. In section 2, the model description is given. The system state probability vector is derived in Section 3. Also several system characteristic are indicated. In Section 4, numerical illustrations of the models are presented. Cost functions under (s, Q)-policy, (s, S)-policy and corresponding optimization problems are investigated in Section 5.

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2 Model description

$M/M/1$ model with a storage system is consider. Arrival process follows Poisson with rate Λ. Exponentially distributed service and lead time with parameters ν and η respectively. Distribution of lead time is independent of that of service time. Also we assume the following assumptions: (i) If $i > s$, then new demands join. (ii) If $n < i \leq s$, then new arrival enters. (iii) If $i \leq s$ and $i \leq n$, then the new arrival does not enter. (iv) If $i \leq s$, customers are impatient, that is, customers leave the buffer depending on the number of items available, after an exponentially distributed amount of time with the parameter α. We analyze two distinct replenishment policies: (a) (s, Q) policy (b) (s, S) policy.

Let $N(t)$ denote the number of customers in the system and $I(t)$ denote the number of items in the inventory at time t. Then $\Omega = \{(N(t), I(t)); t \geq 0\}$ is a Markov Chain with state space $\{(m, i), m \geq 0, 0 \leq i \leq S\}$ and corresponding matrix $\tilde{\mathcal{Q}}$ is

$$\tilde{\mathcal{Q}} = \begin{pmatrix} Z_{00} & Z_{01} & & & \\ Z_{10} & Z_{11} & Z_{12} & & \\ & Z_{21} & Z_{22} & Z_{23} & \\ & & \ddots & \ddots & \ddots \end{pmatrix}. \tag{1}$$

which is an infinitesimal generator and all the sub-matrices are square matrices of order $S + 1$.

2.1 Analysis of (s, Q)-policy

The transition rates are:

- Transition due to arrival:

$$\begin{array}{lll} (m, i) \to (m+1, i): & \Lambda & \text{for } 0 \leq m \leq s-1, m+1 \leq i \leq S \\ (m, i) \to (m+1, i): & \Lambda & \text{for } m \geq s, s+1 \leq i \leq S \end{array}$$

- Transition due to service:

$$(m, i) \to (m-1, i-1): \quad \nu \quad \text{for } m \geq 1, 1 \leq i \leq S$$

- Transition due to impatience:

$$\begin{array}{lll} (m, i) \to (m-1, i): & (m-i)\alpha & \text{for } 1 \leq m \leq s, 0 \leq i \leq m-1 \\ (m, i) \to (m-1, i): & (m-i)\alpha & \text{for } m > s, 0 \leq i \leq s \end{array}$$

- Transition due to replenishment:

$$(m, i) \to (m, i+Q): \quad \eta \quad \text{for } m \geq 0, 0 \leq i \leq s$$

To find the steady state distribution of $\tilde{\mathcal{Q}}$ we use Neuts Rao Truncation method: Choose $Z_{m\ m-1} = Z_2, Z_{m\ m+1} = Z_0$ and $Z_{m\ m} = Z_1$ for $m \geq N$.

$$\tilde{\mathcal{Q}} = \begin{pmatrix} Z_{00} & Z_{01} & & & & & \\ Z_{10} & Z_{11} & Z_{12} & & & & \\ & Z_{21} & Z_{22} & Z_{23} & & & \\ & & & \ddots & \ddots & & \\ & & & Z_{N-1\ N-2} & Z_{N-1\ N-1} & Z_{N-1\ N} & \\ & & & & Z_2 & Z_1 & Z_0 \\ & & & & & \ddots & \ddots & \ddots \end{pmatrix}$$

2

Define $Z = Z_0 + Z_1 + Z_2$. Let $\boldsymbol{\pi}$ be the system state probability distribution of the generator Z. Then $\boldsymbol{\pi} Z = 0$ and $\boldsymbol{\pi}\mathbf{e} = 1$.

The set of equations are:

$$\begin{aligned} -\eta\pi_0 + \nu\pi_1 &= 0, \\ -(\eta+\nu)\pi_i + \nu\pi_{i+1} &= 0,\ 1 \le i \le s \\ -\nu\pi_i + \nu\pi_{i+1} &= 0,\ s+1 \le i \le Q-1 \\ \eta\pi_i - \nu\pi_{Q+i} + \nu\pi_{Q+i+1} &= 0,\ 0 \le i \le s-1 \\ \eta\pi_s - \nu\pi_S &= 0. \end{aligned} \tag{2}$$

By solving

$$\pi_i = \begin{cases} \frac{\eta}{\nu}(\frac{\eta+\nu}{\nu})^{i-1}\pi_0,\ 1 \le i \le s \\ \frac{\eta}{\nu}(\frac{\eta+\nu}{\nu})^{s}\pi_0,\ s+1 \le i \le Q \end{cases}$$

$$\pi_{Q+i} = \frac{\eta}{\nu}\left(\frac{\eta+\nu}{\nu}\right)^{i-1}\left[\left(\frac{\eta+\nu}{\nu}\right)^{s-(i-1)} - 1\right]\pi_0,\ 1 \le i \le s.$$

Using $\boldsymbol{\pi}\mathbf{e} = 1$, we get

$$\pi_0 = \left[1 + (S-s)\frac{\eta}{\nu}\left(\frac{\eta+\nu}{\nu}\right)^s\right]^{-1}.$$

From Neuts [3] on the positive recurrence of Z we have $\boldsymbol{\pi} Z_0\mathbf{e} < \boldsymbol{\pi} Z_2\mathbf{e}$. Computation yields the stability condition as

$$\frac{\Lambda}{\nu} < \frac{1 - \left(1 + (S-s)\frac{\eta}{\nu}\left(\frac{\eta+\nu}{\nu}\right)^s\right)^{-1}}{1 - \left(\frac{\eta+\nu}{\nu}\right)^s\left(1 + (S-s)\frac{\eta}{\nu}\left(\frac{\eta+\nu}{\nu}\right)^s\right)^{-1}}. \tag{3}$$

2.2 Analysis: (s, S) policy

- Transition due to replenishment:

$$(m, i) \to (m, S):\quad \eta \quad \text{for } m \ge 0, 0 \le i \le s$$

Remaining transition rates are same as that given in Section 2.1.

Now define $\tilde{Z} = Z_0 + Z_1 + Z_2$. Let $\boldsymbol{\phi} = (\phi_0, \phi_1, ..., \phi_S)$ be the steady state probability distribution of $\tilde{Z}$. Then $\boldsymbol{\phi}$ satisfies the equation $\boldsymbol{\phi}\tilde{Z} = 0$ and $\boldsymbol{\phi}\mathbf{e} = 1$.

From $\boldsymbol{\phi}\tilde{Z} = 0$ we have

$$\begin{aligned} -\eta\phi_0 + \nu\phi_1 &= 0, \\ -(\eta+\nu)\phi_i + \nu\phi_{i+1} &= 0,\ 1 \le i \le s \\ -\nu\phi_i + \nu\phi_{i+1} &= 0,\ s+1 \le i \le S-1 \\ \eta\left(\phi_0 + \phi_1 + ... + \phi_s\right) - \nu\phi_S &= 0. \end{aligned} \tag{4}$$

Solve these equations

$$\phi_i = \begin{cases} \frac{\eta}{\nu}(\frac{\eta+\nu}{\nu})^{i-1}\phi_0,\ 1 \le i \le s \\ \frac{\eta}{\nu}(\frac{\eta+\nu}{\nu})^{s}\phi_0,\ s+1 \le i \le S. \end{cases}$$

Now $\boldsymbol{\phi}\mathbf{e} = 1$ implies

$$\phi_0 = \frac{1}{\left(\frac{\eta+\nu}{\nu}\right)^s\left(1 + \frac{\eta}{\nu}(S-s)\right)}.$$

The stability condition is (see Section 2.1)

$$\frac{\Lambda}{\nu} < \frac{1-\left(\frac{\eta+\nu}{\nu}\right)^{-s}\left[1+\frac{\eta}{\nu}(S-s)\right]^{-1}}{1-\left[1+\frac{\eta}{\nu}(S-s)\right]^{-1}}. \tag{5}$$

3 Steady state distribution

Let $\mathbf{x} = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, ...)$ be the system state distribution of $\tilde{\mathcal{Q}}$. Then $\mathbf{x}\tilde{\mathcal{Q}} = \mathbf{0}$ and $\mathbf{xe} = 1$.

The vectors $\mathbf{x}_n, n \geq N$ are given by Nuets [3]

$$x_{n+N-1} = x_{N-1}M^n \text{ for } n \geq 1 \tag{6}$$

where M is the minimal solution of

$$Z_0 + MZ_1 + M^2Z_2 = O \tag{7}$$

and the vectors $\mathbf{x}_n, 0 \leq n \leq N-1$ are obtained by solving the following set of equations

$$\begin{aligned} \mathbf{x}_0 Z_{00} + \mathbf{x}_1 Z_{10} &= \mathbf{0}, \\ \mathbf{x}_{j-1} Z_{j-1\ j} + \mathbf{x}_j Z_{j\ j} + \mathbf{x}_{j+1} Z_{j+1\ j} &= \mathbf{0}, \quad 1 \leq j \leq N-2 \\ \mathbf{x}_{N-2} Z_{N-2\ N-1} + \mathbf{x}_{N-1}\left(Z_{N-1\ N-1} + MZ_2\right) &= \mathbf{0} \end{aligned}$$

and the normalizing condition

$$\sum_{i=1}^{N-2} \mathbf{x}_i \mathbf{e} + \mathbf{x}_{N-1}(I-M)^{-1}\mathbf{e} = 1 \tag{8}$$

where

$$\mathbf{x}_i = \mathbf{x}_0 \prod_{j=1}^{i} V_j, 1 \leq i \leq N-1 \tag{9}$$

with

$$V_j = \begin{cases} -Z_{j-1\ j}\left(Z_{j\ j} + V_{j+1}Z_{j+1\ j}\right) & 1 \leq j \leq N-1, \\ -Z_{N-2\ N-1}\left(Z_{N-1\ N-1} + MZ_2\right)^{-1} & j = N-1. \end{cases} \tag{10}$$

From the normalizing condition we have

$$\mathbf{x}_0 \left[I + \sum_{i=1}^{N-2} \prod_{j=1}^{i} V_j + \prod_{j=1}^{N-1} V_j (I-M)^{-1} \right] \mathbf{e} = 1. \tag{11}$$

3.1 System characteristics

1. Mean number of customers in the system: $E_C = \sum_{i=1}^{\infty} \sum_{j=0}^{S} i x_i(j)$

2. Expected inventory level in the system: $E_I = \sum_{i=0}^{\infty} \sum_{j=1}^{S} j x_i(j)$

3. Expected re-order rate: $E_K = \nu \sum_{i=1}^{\infty} x_i(s+1)$

4. Expected loss rate of customers: $E_L = \Lambda \left[\sum_{i=0}^{s} \sum_{j=0}^{i} x_i(j) + \sum_{i=s+1}^{\infty} \sum_{j=0}^{s} x_i(j) \right]$

5. Expected rate of impatience: $E_{imp} = \alpha \left[\sum_{i=0}^{s} \sum_{j=0}^{i} x_i(j) + \sum_{i=s+1}^{\infty} \sum_{j=0}^{s} x_i(j) \right]$

6. Expected Rate of Replenishment: $E_R = \eta \sum_{i=0}^{\infty} \sum_{j=0}^{s} x_i(j)$

4 Numerical illustrations

In this section we provide some numerical examples.

Tables 1, 2, 3 and 4 indicate the system performance with variation in values of underlying parameters.

Λ	(s, Q)-policy					(s, S)-policy				
	E_C	E_I	E_L	E_{IMP}	E_R	E_C	E_I	E_L	E_{IMP}	E_R
1	0.2491	6.9983	0.0010	0.0000	0.1663	0.2492	7.1942	0.0010	0.0000	0.1537
1.5	0.4212	6.7456	0.0063	0.0000	0.2479	0.4219	7.0365	0.0057	0.0000	0.2214
2	0.6343	6.4970	0.0209	0.0003	0.3261	0.6378	6.8817	0.0184	0.0003	0.2829
2.5	0.8953	6.2603	0.0498	0.0011	0.3989	0.9070	6.7338	0.0435	0.0011	0.3380
3	1.2128	6.0428	0.0987	0.0036	0.4645	1.2433	6.5960	0.0864	0.0036	0.3866

Table 1: Effect of Λ: Fix $(S, Q, s, \eta, \nu, \alpha, N) = (10, 6, 4, 2, 5, 0.2, 150)$

ν	(s, Q)-policy					(s, S)-policy				
	E_C	E_I	E_L	E_{IMP}	E_R	E_C	E_I	E_L	E_{IMP}	E_R
4	1.9470	6.0841	0.1101	0.0096	0.4522	2.0234	6.6186	0.0997	0.0098	0.3784
5	1.2128	6.0428	0.0987	0.0036	0.4645	1.2433	6.5960	0.0864	0.0036	0.3866
6	0.8715	6.0239	0.0949	0.0016	0.4707	0.8879	6.5836	0.0814	0.0016	0.3907
7	0.6769	6.0142	0.0936	0.0008	0.4742	0.6873	6.5756	0.0793	0.0008	0.3930
8	0.5521	6.0090	0.0932	0.0004	0.4764	0.5594	6.5701	0.0784	0.0004	0.3945

Table 2: Effect of ν: Fix $(S, Q, s, \eta, \Lambda, \alpha, N) = (10, 6, 4, 2, 3, 0.2, 150)$

η	(s, Q)-policy					(s, S)-policy				
	E_C	E_I	E_L	E_{IMP}	E_R	E_C	E_I	E_L	E_{IMP}	E_R
1	0.9916	4.8394	0.3958	0.0074	0.4061	1.0704	5.7981	0.3269	0.0082	0.3167
2	1.2128	6.0428	0.0987	0.0036	0.4645	1.2433	6.5960	0.0864	0.0036	0.3866
3	1.2992	6.5128	0.0350	0.0018	0.4807	1.3154	6.8924	0.0317	0.0018	0.4184
4	1.3450	6.7562	0.0152	0.0010	0.4872	1.3553	7.0444	0.0141	0.0009	0.4364
5	1.3735	6.9038	0.0076	0.0006	0.4906	1.3807	7.1362	0.0071	0.0005	0.4480

Table 3: Effect of η: Fix $(S, Q, s, \nu, \Lambda, \alpha, N) = (10, 6, 4, 5, 3, 0.2, 150)$

5 Optimization problem

In this section we provide the optimal (s, Q) and (s, S) pairs and corresponding minimal costs. For computing the minimal costs of the given models we introduce the cost

α	(s,Q)-policy					(s,S)-policy				
	E_C	E_I	E_L	E_{IMP}	E_R	E_C	E_I	E_L	E_{IMP}	E_R
0.1	1.2169	6.0422	0.1005	0.0021	0.4647	1.2477	6.5952	0.0880	0.0021	0.3868
0.2	1.2128	6.0428	0.0987	0.0036	0.4645	1.2433	6.5960	0.0864	0.0036	0.3866
0.3	1.2093	6.0434	0.0972	0.0046	0.4643	1.2394	6.5967	0.0849	0.0047	0.3864
0.4	1.2061	6.0439	0.0958	0.0054	0.4641	1.2360	6.5974	0.0837	0.0054	0.3862
0.5	1.2033	6.0443	0.0946	0.0060	0.4639	1.2329	6.5981	0.0825	0.0060	0.3861

Table 4: Effect of α: Fix $(S,Q,s,\nu,\Lambda,\eta,N) = (10,6,4,5,3,2,150)$

functions:

$$CF(s,Q) = [D_0 + QD_1]E_R + D_2E_I + D_3E_C + D_4E_L + D_5E_{IMP} \qquad \text{for } (s,Q)-\text{policy}$$

$$CF(s,S) = \left[D_0 + \sum_{i=0}^{s}(S-i)D_1\right]E_R + D_2E_I + D_3E_C + D_4E_L + D_5E_{IMP} \qquad \text{for } (s,S)-\text{policy}$$

where
D_0 : Fixed Cost for placing an order
D_1 : Procurement cost / unit
D_2 : Holding cost of inventory / unit / unit time
D_3 : Holding cost of customers / unit / unit time
D_4 : Cost due to the loss of customers / unit / unit time
D_5 : Cost due to impatience of customers / unit / unit time

We assign the following values to the parameters: $D_0 = \$100, D_1 = \$10, D_2 = \$1, D_3 = \$0.5, D_4 = \$20, D_5 = \$15, \Lambda = 3, \nu = 5, \alpha = 1.5, \eta = 2, N = 150$.

	(s,Q)-policy				
optimal pair	(1,24)	(2,24)	(3,24)	(4,24)	(5,24)
cost in \$	56.5627	56.5303	57.0436	57.9104	58.9254
	(s,S)-policy				
optimal pair	(1,22)	(2,22)	(3,24)	(4,28)	(5,33)
cost in \$	84.3479	113.6468	145.5002	179.2941	214.187

Table 5: Optimal $(s,S)\&(s,Q)$ pairs and corresponding minimum cost

From the Table 5 we observe $(s,Q) = (2,24)$ and $(s,S) = (1,22)$ as the optimal pairs and the corresponding costs are \$56.5303 and \$84.3479 respectively. Also we obtain the cost function is convex in nature (see Figure 1).

Conclusion

In this article we studied $M/M/1$ model with a storage system. Impatience of customers and loss of customers were considered. For investigating the optimal values we constructed cost functions. In a future paper we proposed to extend the present work to generally distributed service and replenishment time.

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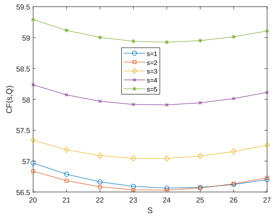
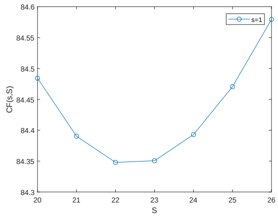
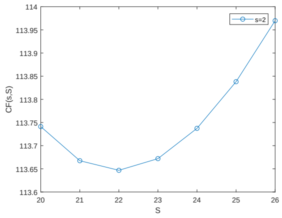


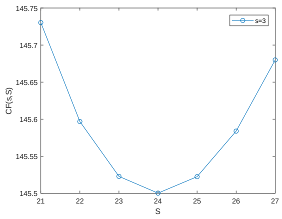
(a) S verses $CF(s,Q)$



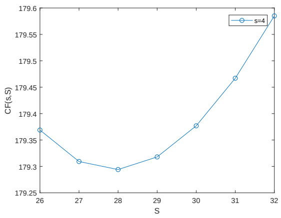
(b) S verses $CF(1,S)$



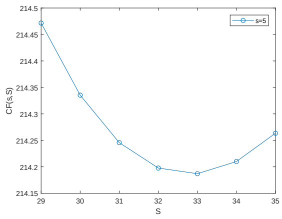
(c) S verses $CF(2,S)$



(d) S verses $CF(3,S)$



(e) S verses $CF(4,S)$



(f) S verses $CF(5,S)$

Figure 1: Cost function

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