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Achromatic Number of Some Graphs

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Abstract

Achromatic colouring is vertex colouring of a graph in such a way that every pair of distinct colours appears on the the endpoints of atleast one edge. The maximum possible number of colours in the complete colouring of the graph is the achromatic number $\Psi(G)$. We discuss the achromatic number of ($T_{m,n}$), S_n and D[T_n].

AMS Subject Classification: 05C15

Key Words and Phrases: Achromatic colouring, Achromatic number, Middle graph, Central graph, Tadpole graph, Spider graph, Double Triangular Snake graph.

1 Introduction

The Graphs we consider here be simple, undirected and finite with vertex set V(G) and edge set E(G). Let S be the set of colours and V be the set of vertices of G, a map f : V $\rightarrow$ S is said to have a proper vertex colouring, if no two adjacent vertices in G share a common colour [4]. The minimum number of colours needed to colour a graph G is called its chromatic number and is denoted by $\chi(G)$.

A complete colouring of a simple graph G is a proper vertex colouring such that each pair of colours appears together on at least one edge [10]. An achromatic colouring of a graph G is a complete colouring and the maximum number of colours used in it is the achromatic number of G, which is denoted by $\Psi(G)$ [3].

Harary, Hedetniemi and Prins introduced the concept of achromatic number [1]. Shousaku Hara and Atsuhiro Nakamoto obtained a lower bound for achromatic number of outer planar graph [5]. The achromatic number of various graph classes and its Central and Middle graphs have been studied in [7], [12], [8], [10] and [11].

Definition 1. (Central graph): The graph obtained by subdividing each edge of the given graph G exactly once and joining all the non-adjacent vertices of it is the Central graph C(G) of G .

Definition 2. (Middle graph): A graph M(G) with vertex set $V(G) \cup E(G)$ and edge set $\{xy : xy \in E(G)\} \cup \{xy : x \in V(G), y \in E(G)$ and x is an end vertex of y$\}$ is the middle graph of G.

Definition 3. (Achromatic Colouring): Consider a graph G with vertices $\{v_1, v_2, \ldots v_n\}$ and edges $\{e_1, e_2, \ldots, e_n\}$. Let e_i be the edge joining the vertices $v_i v_{i+1}$ and $\{C_1, C_2, \ldots, C_n\}$ be the given set of colours. According to achromatic colouring, for every pair of distinct colours, there is at least one edge in the graph whose endpoints are assigned those colours [13]. We assign colours C_i and C_{i+1} to the end vertices v_i and v_{i+1} of the edge e_i. The other vertices are coloured in such a way that each pair of distinct colours have been used at the end vertices of at least one edge and the achromatic number is the maximum number of such colours used.

Here in this paper, we determine the achromatic number of Central and Middle graph of Tadpole graph, Double Triangular Snake graph and Spider graph.

2 Tadpole graph

The Tadpole graph ($T_{m,n}$) consists of a cycle C_m and a path P_n, which is joined by an edge.

Theorem 4. *For the Central graph of Tadpole graph, the achromatic number is given by*

$$\Psi(C((T_{m,n}))) = \begin{cases} n+m & m \in N, n \leq 2 \\ n+m-1 & m \in N, n > 2 \end{cases}$$

$\forall\, m \geq 3$, $n \in N$,

Proof. Consider a Tadpole graph with (m+n) vertices and (m+n) edges. Let $V(C_m) = \{x_1, x_2, \ldots x_m\}$ and $V(P_n) = \{y_1, y_2, \ldots y_n\}$, and the connecting edge is x_1y_1. We subdivide the edges of given graph G exactly once and join all the non-adjacent vertices of G to get the Central graph C(G). Let the vertices x'_i subdivides the edges $x_1x_2, x_2x_3, \ldots x_mx_1$ and let y'_j subdivides the edges $x_1y_1, y_1y_2, \ldots y_{n-1}y_n$, where $1 \leq i \leq m$, $1 \leq j \leq n$.
Consider the colours $\{C_1, C_2, \ldots C_{m+n}\}$.

Case 1: n $\leq$ 2
Assign the colours C_1, C_2,...C_n to the vertices y_j, $1 \leq j \leq n$ and the colours C_{n+i} to the vertices x_i, $1 \leq i \leq m$. Now assign the colours C_1, C_2,...C_{m+n} to the vertices x'_i and y'_j from definition 3 to get all pairs of distinct colours. The maximum possible colours used is n+m.
Therefore $\Psi(C((T_{m,n}))) =$ n+m.
Case 2: n > 2
Assign the colours C_1, C_2,...C_n to y_j, $1 \leq j \leq n$, and the colours C_{n+i} to the vertices x_2, x_3,...x_m respectively, where $1 \leq i \leq m-1$. Colour the vertex x_1 with C_1 and the remaining vertices x'_i and y'_j with C_1, C_2,...C_{n+m-1} using the definition 3. The maximum possible colours used is n+m-1.
Therefore $\Psi(C((T_{m,n}))) =$ n+m-1. □

Illustration

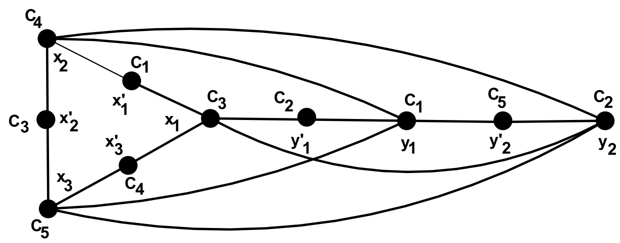

Figure 1: $\Psi(C((T_{3,2}))) = 5$

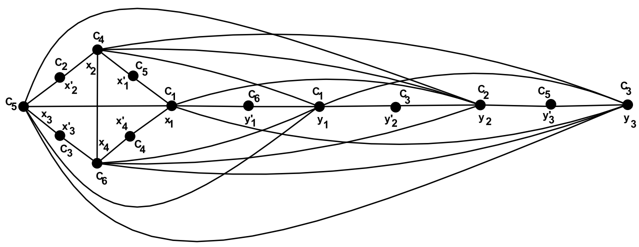

Figure 2: $\Psi(C((T_{4,3}))) = 6$

Theorem 5. *The Achromatic number of middle graph of $(T_{4,n})$, n = 1, 2,..., is* $\Psi(M((T_{4,n}))) = \lfloor \frac{4+n}{3} \rfloor + 4$.

Proof. Consider the (4,n) Tadpole graph. Let $V(C_4)= \{x_1, x_2, x_3, x_4\}$ and $V(P_n)= \{y_1, y_2,...y_n\}$, the edge y_1x_1 connects the cycle C_4 with the path P_n. Let $M((T_{4,n}))$ be the middle graph with 2(4+n) vertices. Introduce the vertices x'_i, where $1 \leq i \leq 4$ and y'_j, where $1 \leq j \leq n$ in the cycle C_4 and path P_n respectively

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of M($T_{4,n}$). Assign the colours to the vertices as follows :
Consider {C_1, C_2,...C_{4+n}}. Colour y'_1 with C_1, assign different colours to the adjacent vertices of y'_1. Colour the remaining vertices using the definition 3 to get all possible pair of colours. The maximum colours used is $\lfloor \frac{4+n}{3} \rfloor + 4$.
Therefore $\Psi(M(T_{4,n})) = \lfloor \frac{4+n}{3} \rfloor + 4$. □

Illustration

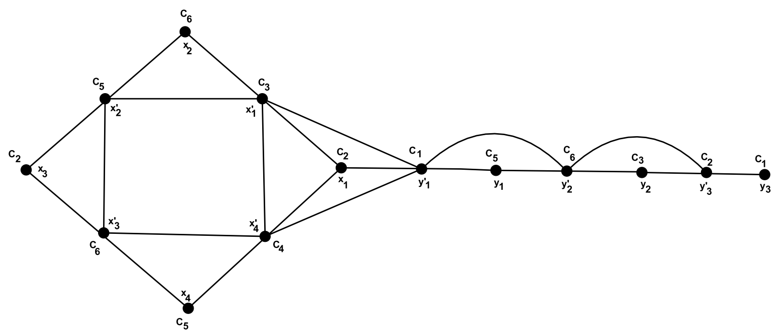

Figure 3: $\Psi(M((T_{4,3}))) = 6$

Proposition 6. For any Tadpole graph, $3 \leq \Psi((T_{m,n})) \leq \lfloor \frac{m+n}{3} \rfloor + 2$ □

3 Spider graph

A Spider graph S_n is formed by subdividing n edges of a star graph $K_{1,n}$.

Theorem 7. *The achromatic number of Central graph of a spider graph is* $\Psi(C(S_n))$ *= 2n+1* $\forall$ *n = 2, 3,...*

Proof. Consider the spider graph S_n. Let v_0, v_1,...v_{2n} be the vertex set of S_n, where v_0 be the root vertex.
Let $C(S_n)$ be the central graph of Spider graph S_n. Let u_1, u_2...u_{2n} be the newly introduced vertices in $C(S_n)$.
Consider the colour class {C_0, C_1,...C_{2n}}. Assign the colours C_1, C_2,...C_{2n} to v_1, v_2,...v_{2n} and assign C_0 to v_0, u_2, u_4,...u_{2n}. Colour the remaining u_{2n-1} vertices making use of definition 3 to get all possible pair of colours.
Therefore $\Psi(C(S_n))$ = 2n+1, $\forall$n= 2, 3,... □

Illustration

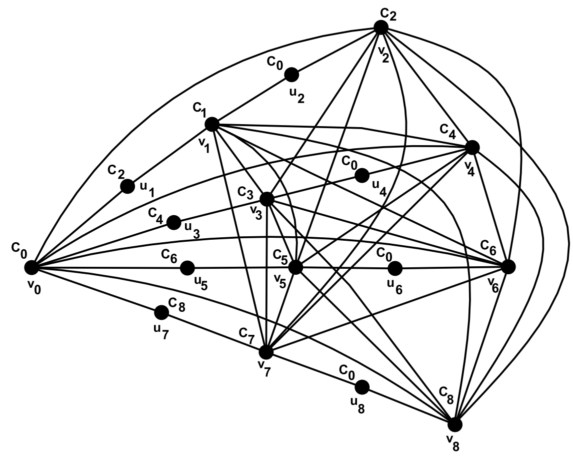

Figure 4: $\Psi(C(S_4)) = 9$

Theorem 8. *For any Spider graph S_n, the achromatic number $\Psi(M(S_n)) =$ n+3 $\forall$ n $\geq$ 2.*

Proof. Consider a Spider graph S_n having (2n+1) vertices. Let v_0, v_1,...v_{2n} be the vertex set of S_n where v_0 be the root vertex.
Consider $M(S_n)$ with (4n+1) vertices. Let u_1, u_2,...u_{2n} be the newly introduced vertices in $M(S_n)$. Now colour the vertices using the colours $\{C_0, C_1,...C_{2n}\}$. Assign C_0 to v_0, since v_0 is adjacent to u_j, ($1 \leq j \leq$ 2n-1), assign C_1, C_3,... C_{2n-1} to the vertices u_j. Colour the vertex u_2 with C_2 , and u_i , ($4 \leq i \leq 2n$) with C_4. Also, using definition 3, colour the vertices v_j, ($3 \leq j \leq$ 2n-1), v_1, v_i, ($2 \leq i \leq 2n$), with C_2, C_4, C_0 respectively.
Hence $\Psi(M(S_n))$ = n+3. □

Illustration

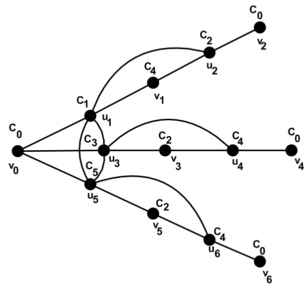

Figure 5: $\Psi(M(S_3)) = 6$

Proposition 9. For any Spider graph S_n, $3 \leq \Psi(S_n) \leq \lfloor \frac{m+n}{2} \rfloor + 2$ □

4 Double Triangular Snake graph

A double triangular Snake graph consists of two triangular snakes joined by a common path. This can be formed from a path y_1, y_2,...y_{n+1} by joining y_i and y_{i+1} to a new vertex z_i and to a new vertex x_i for $1 \leq i \leq n$, denoted by $D[T_n]$.

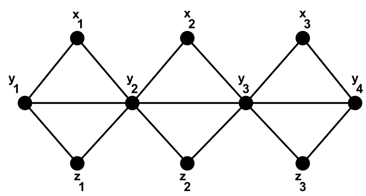

Figure 6: $D[T_3]$

Theorem 10. *The achromatic number of central graph of a Double Triangular Snake graph $D[T_n]$ is $\Psi(C(D[T_n])) = 3n+1$, $\forall n$.*

Proof. Consider a double triangular snake graph $D[T_n]$ having (3n+1) vertices. Let $\{x_1, x_2...x_n, y_1, y_2, ...y_{n+1}, z_1, z_2,...z_n\}$ be the vertices of $D[T_n]$. Now by definition of central graph, each edge y_iy_{i+1}, x_jy_i and y_iz_j, i =1, 2,...n+1 and j =1, 2,...n of $D[T_n]$ are subdivided by the vertices v_i, u_j and w_j, i =1, 2,...n and j =1, 2,...n+1 respectively to get $C(D[T_n])$. Now colour the vertices as follows :
Consider the colour class $\{C_1, C_2,...C_{8n+1}\}$. Assign the vertex x_1 having the maximum degree with colour C_1. Since (3n) number of vertices share edges with x_1 assign

different colours to these (3n) vertices. Now the remaining vertices are coloured with the same colours using the definition 3 to get all possible pair of colours. The maximum number of colours used is (3n+1).
Therefore $\Psi(\mathrm{C}(\mathrm{D}[\mathrm{T}_n])) = 3\mathrm{n}+1$. □

Illustration

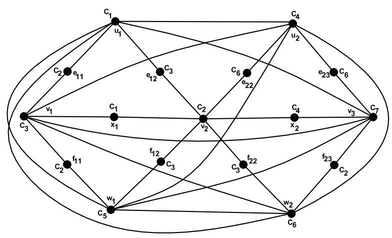

Figure 7: $\Psi(C(D[T_2]) = 7$

Proposition 11. For any Double Triangular Snake Graph, $\Psi(\mathrm{M}(\mathrm{D}[\mathrm{T}_n])) = 2\mathrm{n}+3, \forall \mathrm{n}$ □

5 Conclusion

We tried to find out the achromatic number of some graph families with special emphasis to their central and middle graph.

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