HYERS-ULAM-RASSIAS STABILITY OF A QUADRATIC-ADDITIVE TYPE FUNCTIONAL EQUATION IN FUZZY SPACES

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Abstract: In this paper, we investigate a fuzzy version of stability for the functional equation

\[ f((a-1)x+y)+(a-1)f(x-y)+f(ax)-((a+1)f(x)\]
\[-(a-1)^2f(-x)-f(y)-(a-1)f(-y)=0\]

in the sense of A. K. Mirmostafaee and M. S. Moslehian.

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1. Introduction

In 1940, S. M. Ulam[23] posed a classical question in the theory of functional equations: “when is it true that a mapping, which approximately satisfies a functional equation, must be somehow close to an exact solution of the equation?”, which is called a stability problem of the functional equation. In the next year, D. H. Hyers [6] gave a partial solution of Ulam’s problem for the case of approximate additive mappings. Subsequently, his result was generalized by T. Aoki [1] for additive mappings, and by Th. M. Rassias [22] for linear mappings, to considering the stability problem with unbounded Cauchy differences. During the last decades, the stability
problems of functional equations have been extensively investigated by a number of mathematicians, see [4], [5], [12], [14], [16], [17].

In 1984, A. K. Katsaras [13] defined a fuzzy norm on a linear space to construct a fuzzy structure on the space. Since then, several types of fuzzy norm have been introduced in different points of view. In particular, T. Bag and S.K. Samanta [2], following Cheng and Mordeson [3], gave an idea of a fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [15]. In 2008, A. K. Mirmostafaee and M. S. Moslehian obtained stability results for the Cauchy functional equation [21] and the quadratic functional equation [20] in a fuzzy sense. Now, for a fixed integer $a > 1$, we consider the functional equation:

$$f((a - 1)x + y) + (a - 1)f(x - y) + f(ax) - (a^2 + a - 1)f(x) - (a - 1)^2f(-x) - f(y) - (a - 1)f(-y) = 0 \quad (1.1)$$

which is called the quadratic-additive type functional equation. A solution of (1.1) is called a quadratic-additive mapping.

In this paper, we get a general stability result of the functional equation (1.1) in the fuzzy normed linear space in the manner of A. K. Mirmostafaee and M. S. Moslehian [20](see [7]-[11], [18],[19]).

### 2. Stability of the Functional Equation (1.1)

We use the definition of a fuzzy normed space given in [2] to exhibit a reasonable fuzzy version of stability for the Cauchy additive and quadratic type functional equation in the fuzzy normed linear space.

**Definition 2.1.** ([2]) Let $X$ be a real linear space. A function $N : X \times \mathbb{R} \to [0, 1]$ is said to be a fuzzy norm on $X$ if the following conditions are true for all $x, y \in X$ and all $s, t \in \mathbb{R}$:

1. $(N_1) \; N(x, t) = 0$ for $t \leq 0$;
2. $(N_2) \; x = 0$ if and only if $N(x, t) = 1$ for all $t > 0$;
3. $(N_3) \; N(cx, t) = N(x, t/|c|)$ if $c \neq 0$;
4. $(N_4) \; N(x + y, s + t) \geq \min\{N(x, s), N(y, t)\}$;
5. $(N_5) \; N(x, \cdot)$ is a nondecreasing function on $\mathbb{R}$ and $\lim_{t \to \infty} N(x, t) = 1$.

The pair $(X, N)$ is called a fuzzy normed linear space. Let $(X, N)$ be a fuzzy normed linear space. Let $\{x_n\}$ be a sequence in $X$. Then $\{x_n\}$ is said to be convergent if there exists $x \in X$ such that $\lim_{n \to \infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, $x$ is called the limit of the sequence $\{x_n\}$ and we denote it by $N \lim_{n \to \infty} x_n = x$. 
A sequence \( \{x_n\} \) in \( X \) is called Cauchy if for each \( \varepsilon > 0 \) and each \( t > 0 \) there exists \( n_0 \) such that for all \( n \geq n_0 \) and all \( p > 0 \) we have \( N(x_{n+p} - x_n, t) > 1 - \varepsilon \). It is known that every convergent sequence in a fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space.

Let \( (X, N) \) be a fuzzy normed space and \( (Y, N') \) a fuzzy Banach space. For a given mapping \( f : X \to Y \), we use the abbreviations

\[
Af(x, y) := f(x + y) - f(x) - f(y),
Qf(x, y) := f(x + y) + f(x - y) - 2f(x) - 2f(y),
D_af(x, y) := f((a - 1)x + y) + (a - 1)f(x - y) + f(ax) - (a^2 + a - 1)f(x) - (a - 1)^2 f(-x) - f(y) - (a - 1)f(-y)
\]

for all \( x, y \in X \), where \( a \) is a fixed natural number greater than 1.

**Lemma 2.2.** If \( f : X \to Y \) is a mapping such that \( D_af(x, y) = 0 \) for all \( x, y \in X\backslash\{0\} \) with \( f(0) = 0 \), then \( D_af(x, y) = 0 \) for all \( x, y \in X \).

**Proof.** By the hypothesis, we get

\[
D_af(x, 0) = \frac{a - 1}{a(a - 2)} D_af(x, (1 - a)x) - \frac{1}{a(a - 2)} D_af(-x, (a - 1)x)
- \frac{2a - 3}{2(a - 2)} D_af(x, x) + \frac{1}{2(a - 2)} D_af(-x, -x) = 0
\]

for all \( x \in X \backslash\{0\} \) if \( a > 2 \) and

\[
D_2f(x, 0) = \frac{1}{2} D_2f(x, x) = 0
\]

for all \( x \in X \backslash\{0\} \) if \( a = 2 \). It is easy to show that \( D_af(0, y) = 0 \) for all \( y \in X \) as we desired. \(\square\)

Observe that the equations \( Af = 0 \), \( Qf = 0 \), and \( D_2f = 0 \) represent the Cauchy functional equation, the quadratic functional equation, and the quadratic-additive type functional equation, respectively. A solution of \( Af = 0 \) is call an additive mapping and a solution of \( Qf = 0 \) is call a quadratic mapping. The following lemma shows that if \( f \) is an even or an odd mapping satisfying the equality \( D_2f(x, y) = 0 \) for all \( x, y \in X \), then \( f \) is a quadratic mapping or an additive mapping, respectively.
Lemma 2.3. A mapping $f : X \to Y$ satisfies the inequality $D_af(x, y) = 0$ for all $x, y \in X$ if and only if there exist a quadratic mapping $g : X \to Y$ and an additive mapping $h : X \to Y$ such that

$$f(x) = g(x) + h(x)$$

for all $x \in X$.

Proof. (Necessity) We decompose $f$ into the even part and the odd part by putting

$$g(x) = \frac{f(x) + f(-x)}{2}, \quad h(x) = \frac{f(x) - f(-x)}{2}$$

for all $x \in X$. Notice that $f(0) = -\frac{D_a f(0,0)}{(2a+1)(a-1)} = 0$. From the equalities

$$Qg(x, y) = \frac{D_ag(y, ax) - D_ag(x + y, x)}{a(a-1)} - \frac{D_ag(-x, y - x)}{a} + \frac{(2a-1)D_ag(x, x)}{2a(a-1)}$$

$$= 0,$$

$$Ah(x, y) = \frac{D_ah(y, (a-1)x) - (a-2)D_ah(x, 0) - D_ah(x + y, 0)}{a(a-1)} - \frac{D_ah(-x, y)}{a} + \frac{(a-1)D_ah(x, x) + D_ah(x + y, x + y) - D_ah(y, y)}{2a(a-1)}$$

$$= 0$$

for all $x, y \in X$, we conclude that $g$ is a quadratic mapping and $h$ is an additive mapping.

(Sufficiency) If $g$ is a quadratic mapping, then

$$D_2g(x, y) = Qg(x, y) + Qg(x, x) = 0$$

for all $x, y \in X$. Notice that $g(nx) = n^2g(x)$ and $g(x) = g(-x)$ for all $x \in X$ and all $n \in N$. Assume that $D_kg(x, y) = 0$ for all $x, y \in X$ and an integer $k \geq 2$. Then

$$D_{k+1}g(x, y) = D_kg(x, x + y) + kQg(x, y) + g((k+1)x) - g(kx) - (4k + 1)g(x) = 0$$

for all $x, y \in X$. By induction, we have $D_ah(x, y) = 0$ for all $x \in X$ and all integers $a \geq 2$. If $h$ is an additive mapping, then $h(nx) = nh(x)$ and $h(x) = h(-x)$ for all $x \in X$ and all $n \in N$. So $h$ satisfies the equality

$$D_ah(x, y) = Ah((a-1)x, y) + (a-1)Ah(x, -y) + h(ax) + h(a-1)x - (2a-1)h(x) = 0$$
Proof for all \( x, y \in X \) and all integers \( a \geq 1 \).

Now we get the general stability result of the functional equation (1.1) in a fuzzy space. For given \( q > 0 \), the mapping \( f \) is called a fuzzy \( q \)-almost quadratic-additive mapping if

\[
N'(D_0 f(x,y), t + s) \geq \min \{N(x,s^q), N(y,t^q)\}
\]

for all \( x, y \in X \setminus \{0\} \) and all \( s, t \in (0, \infty) \).

**Theorem 2.4.** Let \( q \) be a positive real number with \( q \neq \frac{1}{2}, 1 \). And let \( f \) be a fuzzy \( q \)-almost quadratic-additive mapping from a fuzzy normed space \((X, N)\) into a fuzzy Banach space \((Y, N')\) with \( f(0) = 0 \). Then there is a unique quadratic-additive mapping \( F : X \to Y \) such that

\[
N'(F(x) - f(x), t) \geq \begin{cases} 
\sup_{t < 1} \left\{ N(x, (a - a^p)t^q) \right\} & \text{if } q > 1, \\
\sup_{t < 1} \left\{ N\left(x, \frac{(a^2 - a^p)q(a^p - a)^q}{(a^2 - a)^q}\right) \right\} & \text{if } 1/2 < q < 1, \\
\sup_{t < 1} \left\{ N(x, (a^p - a^2)t^q) \right\} & \text{if } 0 < q < 1/2
\end{cases}
\]

for each \( x \in X \) and \( t > 0 \), where \( p = 1/q \).

**Proof.** We will prove the theorem in three cases, \( q > 1, \frac{1}{2} < q < 1, \) and \( 0 < q < \frac{1}{2} \).

**Case 1.** Let \( q > 1 \) and let \( J_0 f : X \to Y \) be a mapping defined by

\[
J_n f(x) = \frac{1}{2} \left( a^{-2n} (f(a^n x) + f(-a^n x)) + a^{-n} (f(a^n x) - f(-a^n x)) \right)
\]

for all \( x \in X \). Notice that \( J_0 f(x) = f(x) \) and

\[
J_j f(x) - J_{j+1} f(x) = \frac{a^{j+1} - 1}{4 \cdot a^{2j+2}} D_0 f(-a^j x, -a^j x) - \frac{a^{j+1} + 1}{4 \cdot a^{2j+2}} D_0 f(a^j x, a^j x)
\]

for all \( x \in X \) and \( j \geq 0 \). Together with (N3), (N4) and (2.1), this equation implies that if \( n + m > m \geq 0 \) then

\[
N'(J_m f(x) - J_{n+m} f(x), \sum_{j=m}^{n+m-1} \frac{1}{a} \left( \frac{a^p}{a} \right)^j t^p)
\]

\[
\geq N' \left( \sum_{j=m}^{n+m-1} (J_j f(x) - J_{j+1} f(x)), \sum_{j=m}^{n+m-1} \frac{1}{a} \left( \frac{a^p}{a} \right)^j t^p \right)
\]

\[
\geq \min \bigcup_{j=m}^{n+m-1} \left\{ N' \left( J_j f(x) - J_{j+1} f(x), \frac{1}{a} \left( \frac{a^p}{a} \right)^j t^p \right) \right\}
\]

\[
\geq \min \bigcup_{j=m}^{n+m-1} \left\{ \min \left\{ N' \left( -\frac{(a^{j+1} + 1)}{4 \cdot a^{2j+2}} D_0 f(a^j x, a^j x), \frac{(a^{j+1} + 1)}{2 \cdot a^{2j+2}} t^p \right) \right\} \right\}
\]
\[ N' \left( \left( \frac{a^{j+1}-1}{4 \cdot a^{2j+2}} D_a f\left( -a^j x, -a^j x \right), \frac{(a^{j+1}-1)a^{j \mu + \nu}}{2 \cdot a^{2j+2}} \right) \right) \right) \]

\[ \geq \min \bigcup_{j=m}^{n+m-1} \{ N(a^j x, a^j t) \} = N(x, t) \]  

(2.4)

for all \( x \in X \) and \( t > 0 \). Let \( \varepsilon > 0 \) be given. Since \( \lim_{t \to \infty} N(x, t) = 1 \), there is \( t_0 > 0 \) such that

\[ N(x, t_0) \geq 1 - \varepsilon. \]

We observe that for some \( \hat{t} > t_0 \), the series \( \sum_{j=0}^{\infty} \frac{1}{a} \left( \frac{a^j}{a} \right)^j \hat{t}^p \) converges for \( p = \frac{1}{q} < 1 \).

It guarantees that, for an arbitrary given \( c > 0 \), there exists \( n_0 \geq 0 \) such that

\[ \sum_{j=m}^{n+m-1} \frac{1}{a} \left( \frac{a^j}{a} \right)^j \hat{t}^p < c \]

for each \( m \geq n_0 \) and \( n > 0 \). By (N5) and (2.4), we have

\[ N'(J_m f(x) - J_{n+m} f(x), c) \]

\[ \geq N' \left( J_m f(x) - J_{n+m} f(x), \sum_{j=m}^{n+m-1} \frac{1}{a} \left( \frac{a^j}{a} \right)^j \hat{t}^p \right) \]

\[ \geq N(x, \hat{t}) \geq N(x, t_0) \geq 1 - \varepsilon. \]

for all \( x \in X \). Hence \( \{ J_n f(x) \} \) is a Cauchy sequence in the fuzzy Banach space \((Y, N')\), and so we can define a mapping \( F : X \to Y \) by

\[ F(x) := N' - \lim_{n \to \infty} J_n f(x). \]

Moreover, if we put \( m = 0 \) in (2.4), we have

\[ N'(f(x) - J_n f(x), t) \geq N \left( x, \frac{t^q}{\left( \sum_{j=0}^{n-1} \frac{1}{a} \left( \frac{a^j}{a} \right)^j \right)^q} \right) \]  

(2.5)

for all \( x \in X \).

Next we will show that \( F \) is the desired quadratic additive mapping. Using (N4), we have

\[ N'(D_a f(x, y), t) \]

\[ \geq \min \left\{ N' \left( (F - J_n f)((a - 1)x + y), \frac{t}{14} \right), N' \left( (a - 1)(F - J_n f)(x - y), \frac{t}{14} \right), \right. \]

\[ \left. N' \left( (F - J_n f)(ax), \frac{t}{14} \right), N' \left( (a^2 + a - 1)(J_n f - F)(x), \frac{t}{14} \right) \right\}. \]
By (2.5), we have

\[
N'(a-1)^2(J_nf-F)(-x), \frac{t}{14}\bigg),
N'(J_nf-F)(y), \frac{t}{14}\bigg),
N'(a-1)(J_nf-F)(-y), \frac{t}{14}\bigg),
N'(DJ_nf(x,y), \frac{t}{2}\bigg)\}
\]

(2.6)

for all \(x, y \in X\) and \(n \in \mathbb{N}\). The first seven terms on the right hand side of (2.6) tend to 1 as \(n \to \infty\) by the definition of \(F\) and (N2), and the last term holds

\[
N'(D_aJ_nf(x,y), \frac{t}{2}) \geq \min \left\{ N'(\frac{D_a(f(\pm a^n x, \pm a^n y)}{2 \cdot a^{2n}}, \frac{t}{8}), N'(\frac{D_a(-a^n x, -a^n y)}{2 \cdot a^{2n}}, \frac{t}{8}) \right\}
\]

for all \(x, y \in X\). By (N3) and (2.1), we obtain

\[
N'(\frac{D_a(f(\pm a^n x, \pm a^n y))}{2 \cdot a^{2n}}, \frac{t}{8}) = N'(\frac{D_a(f(\pm a^n x, \pm a^n y)}{2 \cdot a^{2n}}, \frac{t}{8})
\]

\[
\geq \min \left\{ N\left(\frac{a^n x}{8}, \frac{a^{2n} t}{q}\right), N\left(\frac{a^n y}{8}, \frac{a^{2n} t}{q}\right) \right\}
\]

\[
\geq \min \left\{ N\left(\frac{a^{(q-1)n} x}{2q}, \frac{t}{q}\right), N\left(\frac{a^{(q-1)n} y}{2q}, \frac{t}{q}\right) \right\}
\]

and

\[
N'(\frac{D_a(f(\pm a^n x, \pm a^n y))}{2 \cdot a^n}, \frac{t}{8}) \geq \min \left\{ N\left(\frac{a^{(q-1)n} x}{2q}, \frac{t}{q}\right), N\left(\frac{a^{(q-1)n} y}{2q}, \frac{t}{q}\right) \right\}
\]

for all \(x, y \in X\) and \(n \in \mathbb{N}\). Since \(q > 1\), together with (N5), we can deduce that the last term of (2.6) also tends to 1 as \(n \to \infty\). It follows from (2.6) that

\[
N'(D_a f(x,y), t) = 1
\]

for each \(x, y \in X\) and \(t > 0\). By (N2) and Lemma 2.2, this means that \(D_a f(x,y) = 0\) for all \(x, y \in X\). Next we approximate the difference between \(f\) and \(F\) in a fuzzy sense. For an arbitrary fixed \(x \in X\) and \(t > 0\), choose \(0 < \varepsilon < 1\) and \(0 < t' < t\). Since \(F\) is the limit of \(\{J_n f(x)\}\), there is \(l \in \mathbb{N}\) such that

\[
N'(F(x) - J_n f(x), t - t') \geq 1 - \varepsilon.
\]

By (2.5), we have

\[
N'(F(x) - f(x), t) \geq \min \left\{ N'(F(x) - J_n f(x), t - t'), N'(J_n f(x) - f(x), t') \right\}
\]

\[
\geq \min \left\{ 1 - \varepsilon, N\left(\frac{t}{\sum_{j=0}^{n-1} \left(\frac{a^q}{a}\right)^j}\right) \right\}
\]
\[ \geq \min \left\{ 1 - \varepsilon, N(x, (a - a^n)q^t q) \right\}. \]

Because \(0 < \varepsilon < 1\) is arbitrary, we get the inequality (2.2) in this case. Finally, to prove the uniqueness of \(F\), let \(F' : X \rightarrow Y\) be another quadratic-additive mapping satisfying (2.2). Then by (2.3), we get

\[
\begin{align*}
F(x) - J_n F(x) &= \sum_{j=0}^{n-1} (J_j F(x) - J_{j+1} F(x)) = 0 \\
F'(x) - J_n F'(x) &= \sum_{j=0}^{n-1} (J_j F'(x) - J_{j+1} F'(x)) = 0
\end{align*}
\]

(2.7)

for all \(x \in X\) and all \(n \in \mathbb{N}\). Together with (N4) and (2.2), this implies that

\[
N'(F(x) - F'(x), t)
= N'(J_n F(x) - J_n F'(x), t)
\geq \min \left\{ N' \left( J_n F(x) - J_n f(x), \frac{t}{2} \right), N' \left( J_n f(x) - J_n F'(x), \frac{t}{2} \right) \right\}
\geq \min \left\{ N' \left( \frac{(F - f)(n^t x)}{2 \cdot n^2}, \frac{t}{8} \right), N' \left( \frac{(f - F')(a^n x)}{2 \cdot a^{2n}}, \frac{t}{8} \right), N' \left( \frac{(F - f)(a^n x)}{2 \cdot a^n}, \frac{t}{8} \right), N' \left( \frac{(f - F')(a^n x)}{2 \cdot a^n}, \frac{t}{8} \right) \right\}
\geq \sup_{t < t} N \left( x, a^{(q-1)n} 2^{-2q} (a - a^n)q^t q \right)
\]

for all \(x \in X\) and all \(n \in \mathbb{N}\). Observe that, for \(q = \frac{1}{p} > 1\), the last term of the above inequality tends to 1 as \(n \to \infty\) by (N5). This implies that \(N'(F(x) - F'(x), t) = 1\) and so we get

\[ F(x) = F'(x) \]

for all \(x \in X\) by (N2).

**Case 2.** Let \(\frac{1}{2} < q < 1\) and let \(J_n f : X \rightarrow Y\) be a mapping defined by

\[ J_n f(x) = \frac{1}{2} \left( a^{-2n} f(a^n x) + f(-a^n x) \right) + a^n \left( f \left( \frac{x}{a^n} \right) - f \left( -\frac{x}{a^n} \right) \right) \]

for all \(x \in X\). Then we have \(J_0 f(x) = f(x)\) and

\[
\begin{align*}
J_j f(x) - J_{j+1} f(x) &= -\frac{1}{4 \cdot a^{2j+2}} D_a f(a^j x, a^j x) - \frac{1}{4 \cdot a^{2j+2}} D_a f(-a^j x, -a^j x) \\
&\quad + \frac{a^j}{4} D_a f \left( \frac{x}{a^{j+1}}, \frac{x}{a^{j+1}} \right) - \frac{a^j}{4} D_a f \left( -\frac{x}{a^{j+1}}, -\frac{x}{a^{j+1}} \right)
\end{align*}
\]

for all \(x, b \in X\). Then observe that, for \(a, b \in X\), the left hand side of (2.7) satisfies
for all \( x \in X \) and \( j \geq 0 \). If \( n + m > m \geq 0 \), then we have

\[
N' \left( J_m f(x) - J_{n+m} f(x), \sum_{j=m}^{n+m-1} \left( \frac{1}{a^2} \left( \frac{a^p}{a^2} \right)^j + \frac{1}{a^p} \left( \frac{a}{a^p} \right)^j \right) t^p \right)
\]

\[
\geq \min_{j=m}^{n+m-1} \left\{ N' \left( -\frac{D_a f(a^j x, a^j x)}{4 \cdot a^{2j+2}}, \frac{a^{jp} t^p}{2 \cdot a^{2j+2}} \right) \right\}
\]

\[
= N(x, t)
\]

for all \( x \in X \) and \( t > 0 \). In the similar argument following (2.4) of the previous case, we can define the limit \( F(x) := N' - \lim_{n \to \infty} J_n f(x) \) of the Cauchy sequence \( \{J_n f(x)\} \) in the Banach fuzzy space \( Y \). Moreover, putting \( m = 0 \) in the above inequality, we have

\[
N'(f(x) - J_n f(x), t) \geq N \left( x, \frac{t^q}{\sum_{j=0}^{n-1} \left( \frac{1}{a^q} \left( \frac{a^q}{a^2} \right)^j + \frac{1}{a^q} \left( \frac{a}{a^p} \right)^j \right)^q} \right)
\]

(2.8)

for each \( x \in X \) and \( t > 0 \). To prove that \( F \) is a quadratic additive mapping, we have enough to show that the last term of (2.6) in Case 1 tends to 1 as \( n \to \infty \). By (N3) and (2.1), we get

\[
N' \left( D_a J_n f(x, y), \frac{t^q}{2} \right)
\]

\[
\geq \min \left\{ N' \left( \frac{D_a f(a^nx, a^ny)}{2 \cdot a^{2n}}, \frac{t^q}{8} \right), N' \left( \frac{a^n D_a f(x, y)}{2 \cdot a^n}, \frac{t^q}{8} \right), N' \left( \frac{a^n}{2} D_a f \left( \frac{x}{a^n}, \frac{y}{a^n} \right), \frac{t^q}{8} \right) \right\}
\]

\[
\geq \min \left\{ N(x, a^{2q-1}n^2-3qy^2), N(y, a^{2q-1}n^2-3qy^2), N(x, a^{(1-q)n^2-3qy^2}), N(y, a^{(1-q)n^2-3qy^2}) \right\}
\]
for each \( x, y \in X \) and \( t > 0 \). Observe that all the terms on the right hand side of the above inequality tend to 1 as \( n \to \infty \), since \( \frac{1}{2} < q < 1 \). Hence, together with the similar argument after (2.6), we can say that \( D_n f(x, y) = 0 \) for all \( x, y \in X \). Recall, in Case 1, the inequality (2.2) follows from (2.5). By the same reasoning, we get (2.7) from (2.8) in this case. Now to prove the uniqueness of \( F \), let \( F' \) be another quadratic additive mapping satisfying (2.2). Then, together with (N4), (2.2), and (2.7), we have

\[
N'(F(x) - F'(x), t) = N'(J_n F(x) - J_n F'(x), t)
\]

\[
\geq \min \left\{ N' \left( J_n F(x) - J_n f(x), \frac{t}{2} \right), N' \left( J_n f(x) - J_n F'(x), \frac{t}{2} \right) \right\}
\]

\[
\geq \min \left\{ N' \left( \frac{(F-f)(a^nx)}{2 \cdot a^{2n}}, \frac{t}{8} \right), \frac{(f-F')(a^nx)}{2 \cdot a^{2n}}, \frac{t}{8} \right),
N' \left( \frac{(F-f)(-a^nx)}{2 \cdot a^{2n}}, \frac{t}{8} \right), \frac{(f-F')(-a^nx)}{2 \cdot a^{2n}}, \frac{t}{8} \right),
N' \left( \frac{a^n}{2} \left( \frac{F-f}{\frac{x}{a^n}} \right), \frac{t}{8} \right), \frac{a^n}{2} \left( \frac{f-F'}{\frac{x}{a^n}} \right), \frac{t}{8} \right),
N' \left( \frac{a^n}{2} \left( \frac{-x}{a^n} \right), \frac{t}{8} \right), \frac{a^n}{2} \left( \frac{a^p-a}{a^q} \right)^q t^q) \right) \right\}
\]

\[
\geq \min \left\{ \sup_{t' < t} N \left( x, a^{(2q-1)n}2^{-2q}(a^2-a)^{-q}(a^2-a^p)^q(a^p-a)^q t' \right), \sup_{t' < t} N \left( x, a^{(1-q)n}2^{-2q}(a^2-a)^{-q}(a^2-a^p)^q(a^p-a)^q t' \right) \right\}
\]

for all \( x \in X \) and \( n \in \mathbb{N} \). Since \( \lim_{n \to \infty} a^{(2q-1)n} = \lim_{n \to \infty} a^{(1-q)n} = \infty \) in this case, both terms on the right hand side of the above inequality tend to 1 as \( n \to \infty \) by (N5). This implies that \( N'(F(x) - F'(x), t) = 1 \) and so \( F(x) = F'(x) \) for all \( x \in X \) by (N2). **Case 3.** Finally, we take \( 0 < q < \frac{1}{2} \) and define \( J_n f : X \to Y \) by

\[
J_n f(x) = \frac{1}{2} \left( a^n f(\frac{x}{a^n}) - f(\frac{-x}{a^n}) \right) + a^n \left( f(\frac{x}{a^n}) - f(\frac{-x}{a^n}) \right)
\]

for all \( x \in X \). Then we have \( J_0 f(x) = f(x) \) and

\[
J_j f(x) - J_{j+1} f(x) = \frac{a^{2j} + a^j}{4} D_a f \left( x, a^{j+1} \right) + \frac{a^{2j} - a^j}{4} D_a f \left( \frac{-x}{a^{j+1}}, \frac{-x}{a^{j+1}} \right)
\]

which implies that if \( n + m > m \geq 0 \) then

\[
N' \left( J_n f(x) - J_{n+m} f(x), \sum_{j=m}^{n+m-1} \left( \frac{a^{2j}}{a^p} \right) \frac{t}{a^p} \right)
\]
Therefore, we can say that (2.2). Then by (2.7), we get

\[ F(x) \]

for all \( x \in X \) and \( t > 0 \). Similar to the previous cases, it leads us to define the mapping \( F : X \to Y \) by \( F(x) := N' - \lim_{n \to \infty} J_n f(x) \). Putting \( m = 0 \) in the above inequality, we have

\[
N'(f(x) - J_n f(x), t) \geq N \left( x, \frac{t^q}{\frac{1}{\alpha^p} \sum_{j=0}^{n-1} \left( \frac{a^j}{\alpha^p} \right)^j} \right) \tag{2.9}
\]

for all \( x \in X \) and \( t > 0 \). Notice that

\[
N' \left( D_a J_n f(x, y), \frac{t}{2} \right) \geq \min \left\{ N' \left( \frac{a^{2n}}{2} D_a f \left( \frac{x}{a^n}, \frac{y}{a^n} \right), \frac{t}{8} \right), N' \left( \frac{a^{2n}}{2} D_a f \left( \frac{-x}{a^n}, \frac{-y}{a^n} \right), \frac{t}{8} \right) \right\}
\]

for each \( x, y \in X \) and \( t > 0 \). Since \( 0 < q < \frac{1}{2} \), other terms on the right hand side tend to 1 as \( n \to \infty \), which implies that the last term of (2.6) tends to 1 as \( n \to \infty \). Therefore, we can say that \( D_a f \equiv 0 \). Moreover, using the similar argument after (2.6) in Case 1, we get the inequality (2.2) from (2.9) in this case. To prove the uniqueness of \( F \), let \( F' : X \to Y \) be another quadratic additive function satisfying (2.2). Then by (2.7), we get

\[
N'(F(x) - F'(x), t) \geq \min \left\{ N' \left( J_n F(x) - J_n f(x), \frac{t}{2} \right), N' \left( J_n f(x) - J_n F'(x), \frac{t}{2} \right) \right\}
\]
for all $x \in X$ and all $n \in \mathbb{N}$. Observe that, for $0 < q < \frac{1}{2}$, the last term tends to 1 as $n \to \infty$ by (N5). This implies that $N'(F(x) - F'(x), t) = 1$ and $F(x) = F'(x)$ for all $x \in X$ by (N2).

**Remark 2.5.** Consider a mapping $f : X \to Y$ satisfying (2.1) for all $x, y \in X \setminus \{0\}$ and a real number $q < 0$. Take any $t > 0$. If we choose a real number $s$ with $0 < 2s < t$, then we have

$$N'(D_af(x, y), t) \geq N'(D_af(x, y), 2s) \geq \min\{N(x, s^q), N(y, s^q)\}$$

for all $x, y \in X \setminus \{0\}$. Since $q < 0$, we have $\lim_{s \to 0^+} s^q = \infty$. This implies that

$$\lim_{s \to 0^+} N(x, s^q) = \lim_{s \to 0^+} N(y, s^q) = 1$$

and so

$$N'(D_af(x, y), t) = 1$$

for all $x, y \in X \setminus \{0\}$ and $t > 0$. By (N2), it allows us to get $D_af(x, y) = 0$ for all $x, y \in X$. In other words, $f$ is itself a quadratic additive mapping if $f$ is a fuzzy $q$-almost quadratic-additive mapping for the case $q < 0$.

**Corollary 2.6.** Let $f$ be an even mapping satisfying all of the conditions of Theorem 2.4 with $f(0) = 0$. Then there is a unique quadratic mapping $F : X \to Y$ such that

$$N'(F(x) - f(x), t) \geq \sup_{t' < t} N\left(x, \left(|a^2 - a^p|t'\right)^q\right)$$

(2.10)

for all $x \in X$ and $t > 0$, where $p = 1/q$.

**Proof.** Let $J_n f$ be defined as in Theorem 2.4. Since $f$ is an even mapping, we obtain

$$J_n f(x) = \begin{cases} \frac{f(a^nx) + f(-a^nx)}{a^2q^{2n}} & \text{if } q > \frac{1}{2}, \\ \frac{1}{2} (a^{2n}(f(x/a^n) + f(-x/a^n))) & \text{if } 0 < q < \frac{1}{2} \end{cases}$$
for all $x \in X$. Notice that $J_0 f(x) = f(x)$ and

$$J_j f(x) - J_{j+1} f(x) = \begin{cases} \frac{-D a f(a^j x, a^{j+1} x)}{2 a^{j+1}} & \text{if } q > \frac{1}{2}, \\ \frac{a^{j+2}}{2} D a f \left( \frac{x}{a^{j+1}}, \frac{x}{a^{j+1}} \right) & \text{if } 0 < q < \frac{1}{2} \end{cases}$$

for all $x \in X$ and $j \in \mathbb{N} \cup \{0\}$. From these, using the similar method in Theorem 2.4, we obtain the quadratic-additive mapping $F$ satisfying (2.10). Notice that $F$ is also even, $F(0) = 0$, and $D a F(x, y) = 0$ for all $x, y \in X$. This means that $F$ is a quadratic mapping. □

**Corollary 2.7.** Let $f$ be an odd mapping satisfying all of the conditions of Theorem 2.4. Then there is a unique additive mapping $F : X \rightarrow Y$ such that

$$N'(F(x) - f(x), t) \geq \sup_{t' < t} N \left( x, (|a - a^p| t')^q \right)$$

for all $x \in X$ and $t > 0$, where $p = 1/q$.

**Proof.** Let $J_n f$ be defined as in Theorem 2.4. Since $f$ is an odd mapping, we obtain

$$J_n f(x) = \begin{cases} \frac{f(a^n x) - f(-a^n x)}{2 a^n} & \text{if } q > 1, \\ \frac{p}{2} \left( f \left( \frac{x}{a^n} \right) - f \left( -\frac{x}{a^n} \right) \right) & \text{if } 0 < q < 1 \end{cases}$$

for all $x \in X$. Notice that $J_0 f(x) = f(x)$ and

$$J_j f(x) - J_{j+1} f(x) = \begin{cases} \frac{-1}{2 a^{j+1}} D a f(a^j x, a^{j+1} x) & \text{if } q > 1, \\ \frac{a^{j+2}}{2} D a f \left( \frac{x}{a^{j+1}}, \frac{x}{a^{j+1}} \right) & \text{if } 0 < q < 1 \end{cases}$$

for all $x \in X$ and $j \in \mathbb{N} \cup \{0\}$. From these, using the similar method in Theorem 2.4, we obtain the quadratic-additive mapping $F$ satisfying (2.11). Notice that $F$ is also odd and $D a F(x, y) = 0$ for all $x, y \in X$. This means that $F$ is an additive mapping. □

We can use Theorem 2.4 to get a classical result in the framework of normed spaces. Let $(X, \| \cdot \|)$ be a normed linear space. Then we can define a fuzzy norm $N_X$ on $X$ by following

$$N_X(x, t) = \begin{cases} 0, & t \leq \| x \| \\ 1, & t > \| x \| \end{cases}$$

where $x \in X$ and $t \in \mathbb{R}$, see [20]. Suppose that $f : X \rightarrow Y$ is a mapping into a Banach space $(Y, \| \cdot \|)$ such that

$$\| D a f(x, y) \| \leq \| x \|^p + \| y \|^p$$
for all \( x, y \in X \), where \( p > 0 \) and \( p \neq 1, 2 \). Let \( N_Y \) be a fuzzy norm on \( Y \). Then we get

\[
N_Y(D_a f(x, y), s + t) = \begin{cases} 
0, & s + t \leq |||D_a f(x, y)||| \\
1, & s + t > |||D_a f(x, y)||| 
\end{cases}
\]

for all \( x, y \in X \) and \( s, t \in \mathbb{R} \). Consider the case \( N_Y(D_a f(x, y), s + t) = 0 \). This implies that

\[
|||x|||^p + |||y|||^p \geq |||D_a f(x, y)||| \geq s + t
\]

and so either \( |||x|||^p \geq s \) or \( |||y|||^p \geq t \) in this case. Hence, for \( q = \frac{1}{p} \), we have

\[
\min\{N_X(x, s^q), N_X(y, t^q)\} = 0
\]

for all \( x, y \in X \) and \( s, t > 0 \). Therefore, in every case, the inequality

\[
N_Y(D_a f(x, y), s + t) \geq \min\{N_X(x, s^q), N_X(y, t^q)\}
\]

holds. It means that \( f \) is a fuzzy \( q \)-almost quadratic additive mapping, and by Theorem 2.2, we get the following stability result.

**Corollary 2.8.** Let \((X, \| \cdot \|)\) be a normed linear space and let \((Y, ||| \cdot |||)\) be a Banach space. If

\[
|||D_a f(x, y)||| \leq |||x|||^p + |||y|||^p
\]

for all \( x, y \in X \), where \( p > 0 \) and \( p \neq 1, 2 \), then there is a unique quadratic-additive mapping \( F : X \rightarrow Y \) such that

\[
|||F(x) - f(x)||| \leq \begin{cases} 
\frac{|||x|||^p}{a^p - a} & \text{if } 0 < p < 1, \\
\frac{|||x|||^p}{a^p - a} & \text{if } 1 < p < 2, \\
\frac{|||x|||^p}{a^p - a^2} & \text{if } p > 2
\end{cases}
\]

for all \( x \in X \).

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**References**


