NEARLY COUNTABLY COMPACT SPACES

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Abstract: New class of spaces is introduced in this paper as a generalization of countably compact spaces called nearly countably compact spaces. Some characterizations and results about this new class of spaces are also presented. We give a definition of more generalized kind of spaces and we call it nearly regular countably compact spaces. Also, we study the effect of some mappings on the nearly countably compact spaces.

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1. Introduction

Different generalizations are made on spaces involving those covers with regular open sets as nearly compact spaces which introduced in 1969 by Singal and Asha [9] and nearly Lindeöf spaces defined as a generalization of nearly compact spaces by Balasubramanian in 1982 [1]. In this paper, we give an analogous definition of those spaces have been mentioned above. This new kind of space is a generalization of countably compact spaces called nearly countably compact spaces. Also, we present some characterizations of this new class of spaces in terms of semireg-
ularization topologies; in which we prove that a topological space \((X, \tau)\) is nearly countably compact if and only if \((X, \tau^*)\) is countably compact. Moreover, nearly regular countably compact spaces are defined as a generalization of nearly countably compact spaces. In such kind of spaces, we used the notion of regular cover presented in [2]. In the other hand, the effect of some kind of mappings is studied, such as \(R\)-maps and some others, and they preserve the nearly countably compact spaces and the image of nearly countably compact spaces is again nearly countably compact spaces under \(\alpha\)-open and almost continuous functions.

In this paper, a space \(X\) stands for a topological space \((X, \tau)\) with no separation axioms assumed unless mentioned. The interior and closure of a subset \(A\) in a space \(X\) are denoted by \(\text{Int}(A)\) and \(\text{Cl}(A)\) respectively. Recall that a subset \(A\) of a space \(X\) is called regular open if \(A = \text{Int}(\text{Cl}(A))\) and regular closed if \(A = \text{Cl}(\text{Int}(A))\). It is clear that every regular open set is an open set. The semiregularization topology of a space \(X\) is the topology whose base is the set of all regular open sets in the space \(X\) and denoted by \(\tau^*\). If \(\tau = \tau^*\) then \(X\) is said to be semiregular.

2. Nearly Countably Compact Spaces

In this section, we define nearly countably compact spaces and we provide an example of a nearly countably compact topological space but not countably compact. Also, we study some characterizations in terms of semiregularization topologies and some other results.

Definition 2.1. A topological space \(X\) is said to be nearly countably compact if for every countable open cover \(\{U_n : n \in \mathbb{N}\}\) of \(X\), there is a finite subfamily \(\{U_{n_i}\}_{i=1}^m\) where \(m \in \mathbb{N}\) such that

\[
X = \bigcup_{i=1}^m \text{Int}(\text{Cl}(U_{n_i}))
\]

Lemma 2.1. A topological space \(X\) is nearly countably compact if and only if every countable cover \(\{U_n : n \in \mathbb{N}\}\) of regular open sets of \(X\) has a finite subcover \(\{U_{n_i}\}_{i=1}^m\) of \(X\).

Proof. Let \(X\) be a nearly countably compact space and \(X = \bigcup_{n \in \mathbb{N}} U_n\), where \(U_n = \text{Int}(\text{Cl}(U_n))\) for all \(n \in \mathbb{N}\). Since \(X\) is nearly countably compact space, there is a finite subcover of \(X\), so \(X = \bigcup_{i=1}^m \text{Int}(\text{Cl}(U_{n_i})) = \bigcup_{i=1}^m U_{n_i}\).

Conversely, let \(X = \bigcup_{n \in \mathbb{N}} U_n\), where \(U_n\) is an open in \(X\) for all \(n \in \mathbb{N}\). We know that \(U_n \subseteq \text{Int}(\text{Cl}(U_n))\) for all \(n \in \mathbb{N}\), so \(\{\text{Int}(\text{Cl}(U_n)) : n \in \mathbb{N}\}\) is a countable open cover of \(X\). Hence, it has a countable subfamily \(\{\text{Int}(\text{Cl}(U_{n_i}))\}_{i=1}^m\) such that \(X = \bigcup_{i=1}^m \text{Int}(\text{Cl}(U_{n_i}))\). Thus, \(X\) is nearly countably compact.

\[
X = \bigcup_{i=1}^m \text{Int}(\text{Cl}(U_{n_i}))
\]

\[
\text{Int}(\text{Cl}(U_{n_i}))
\]
Since every regularly open set is open, we have the following corollary:

**Corollary 2.1.** Any countably compact space is nearly countably compact space.

The converse needs not be true in general as shown in the next example:

**Example 1.** Consider the cocompact topology on real numbers \((\mathbb{R}, \tau_{coc})\). This space is not countably compact. On the other hand, let \(\{U_n : n \in \mathbb{N}\}\) be a countable open cover of \((\mathbb{R}, \tau_{coc})\) and \(U_k \in \{U_n : n \in \mathbb{N}\}\) for any \(k \in \mathbb{N}\). Then \(\text{Cl}(U_k) = \mathbb{R}\), and \(\text{Int}(\text{Cl}(U_k)) = \mathbb{R}\), that means \(\text{Int}(\text{Cl}(U_k))\) covers \(\mathbb{R}\) so \(\{\text{Int}(\text{Cl}(U_k))\}\) is a finite subfamily of \(\{U_n : n \in \mathbb{N}\}\) that covers \(\mathbb{R}\). Therefore, \((\mathbb{R}, \tau_{coc})\) is nearly countably compact space.

The following lemma is easily can be proved.

**Lemma 2.2.** The regularly open sets of a topological space \((X, \iota)\) are the same as the regularly open sets of its semiregularization \((X, \iota^*)\).

**Theorem 2.1.** A topological space \((X, \tau)\) is nearly countably compact if and only if \((X, \tau^*)\) is nearly countably compact.

**Proof.** Assume that \((X, \tau)\) is nearly countably compact. Let \(U\) be a countable cover of regularly open sets of \((X, \iota^*)\). By using 2.2, we see that \(U\) is countable cover by regularly open sets in \((X, \iota)\) which has a finite subcover by the nearly countably compactness of \((X, \iota)\). Therefore, \((X, \iota^*)\) is nearly countably compact.

Conversely, suppose that \((X, \tau^*)\) is nearly countably compact space. Let \(U\) be a countable cover of regularly open sets of \((X, \iota)\). From 2.2, the collection \(U\) is countable cover by regularly open sets in \((X, \tau^*)\), so \(U\) has a finite subcover by the nearly countably compactness of \((X, \iota^*)\). Therefore, \((X, \iota)\) is nearly countably compact.

The following proposition gives a characterization of nearly countably compact spaces.

**Theorem 2.2.** A topological space \((X, \tau)\) is nearly countably compact if and only if every countable collection of regularly closed subsets of \(X\) satisfying the finite intersection property has a nonempty intersection.

**Proof.** Let \((X, \tau)\) be countably compact space and \(\{F_n\}_{n=1}^{\infty}\) be a countable collection of regularly closed subsets of \(X\) satisfying the finite intersection property.

Assume that \(\bigcap_{n=1}^{\infty} F_n = \emptyset\), so the collection of regularly open sets \(\{X \setminus F_n\}_{n=1}^{\infty}\) covers \(X\). Since \(X\) is nearly countably compact, we have \(X = \bigcup_{i=1}^{m} X \setminus F_{n_i}\) for some
\( m \in \mathbb{N}, \) so \( \bigcap_{i=1}^{m} F_{n_i} = \emptyset \) which contradicts our assumption of the finite intersection property. Therefore, \( \bigcap_{n=1}^{\infty} F_n \neq \emptyset. \)

Conversely, assume that every countable collection of regularly closed subsets of \( X \) satisfying the finite intersection property has nonempty intersection. Let \( X = \bigcup_{n=1}^{\infty} U_n \) where \( U_n = \text{Int}(\text{Cl}(U_n)) \) for all \( n = 1, 2, \ldots \). From the assumption, there is \( m \in \mathbb{N} \) such that \( \bigcap_{i=1}^{m} X \setminus U_{n_i} = \emptyset \) and so \( X = \bigcup_{i=1}^{m} U_{n_i}. \)

By lemma 2.1, we get \( X \) is a nearly countably compact space.

Now, we give a definition of nearly regular countably compact spaces as an analogous definition of nearly regular-Lindelöf spaces [3].

**Theorem 2.3.** Let \( X \) be a nearly countably compact space and \( A \) be a regular closed subset of \( X \). Then \( A \) is nearly countably compact relative to \( X \).

**Proof.** Let \( \{U_n : n \in \mathbb{N}\} \) be a regularly open cover of \( A \). Then

\[
\{U_n : n \in \mathbb{N}\} \bigcup \{X \setminus A\}
\]

is a regularly open cover of \( X \), so it has a finite subcover, say \( U_{n_1} \cup \ldots \cup U_{n_k} \cup (X \setminus A) \). Thus \( A = \bigcup_{i=1}^{k} U_{n_i} \). Thus, \( A \) is nearly countably compact space.

**Definition 2.2.** [2] An open cover \( \{U_{\alpha} : \alpha \in \Delta\} \) of a topological space \( X \) is called regular cover if, for every \( \alpha \in \Delta \), there exists a nonempty regular closed subset \( C_{\alpha} \) in \( X \) such that \( C_{\alpha} \subseteq U_{\alpha} \) and \( X = \bigcup_{\alpha \in \Delta} \text{Int}(\text{Cl}(C_{\alpha})). \)

**Definition 2.3.** A topological space \((X, \tau)\) is called nearly regular countably compact if every countable regular cover \( \{U_n\}_{n=1}^{\infty} \) of \( X \) has a finite subfamily \( \{U_{n_1}, U_{n_2}, \ldots, U_{n_m}\} \) such that \( X = \bigcup_{i=1}^{m} \text{Int}(\text{Cl}(U_{n_i})). \)

**Theorem 2.4.** A space \( X \) is nearly regular countably compact if and only if for every countable family \( \{C_n : n \in \mathbb{N}\} \) of closed subsets of \( X \) such that, for each \( n \in \mathbb{N} \), there exists an open set \( A_n \) such that \( C_n \subseteq A_n \) with \( \bigcap_{n \in \mathbb{N}} \text{Cl}(A_n) = \emptyset \), there exists a finite subfamily \( \{C_{n_1}, C_{n_2}, \ldots, C_{n_m}\} \) such that \( \bigcap_{k=1}^{m} C_{n_k} = \emptyset. \)

**Proof.** Let \( \{C_n : n \in \mathbb{N}\} \) be a countable family of regularly closed subsets of \( X \) such that, for each \( n \in \mathbb{N} \), there exists an open set \( A_n \) such that \( C_n \subseteq A_n \) and \( \bigcap_{n \in \mathbb{N}} \text{Cl}(A_n) = \emptyset \). Then \( X = \bigcup_{n \in \mathbb{N}} X \setminus (\text{Cl}(A_n)). \) Thus, we have that \( C_n \subseteq A_n \subseteq \)
Int(Cl(A_n)) ⊆ Cl(A_n) for every n ∈ N. So X/(Cl(A_n)) ⊆ X/(Int(Cl(A_n))) ⊆ X\A_n ⊆ X/C_n, for every n ∈ N. Thus X = \bigcup_{n ∈ N} X\C_n so \{X\C_n : n ∈ N\} is a countable cover of open sets in X such that, for every n ∈ N, we have X/(Int(Cl(A_n))) ∈ X/C_n and X = \bigcup_{n ∈ N} X\C_n = \bigcup_{n ∈ N} Int(X\C_n). Now, for each n ∈ N, X/(Int(Cl(A_n))) is regular closed set in X. Therefore; \{X\C_n : n ∈ N\} is a countable regular cover of X, so there exists a countable subfamily \{X\C_{n_1}, X\C_{n_2}, ..., X\C_{n_m}\} such that X = \bigcup_{k=1}^{m} Cl(Int(X\C_{n_k})) = \bigcup_{k=1}^{m} Cl(X\C_{n_k}) = \bigcup_{k=1}^{m} X\C_{n_k} = X\bigcap_{k=1}^{m} Int(C_{n_k}) and so \bigcap_{k=1}^{m} Int(C_{n_k}) = ∅.

Conversely, let \{U_n : n ∈ N\} be a countable regular cover of X, then for each n ∈ N, there exists a regularly closed subset C_n of X such that Int(C_n) ⊆ C_n ⊆ U_n and X = \bigcup_{n ∈ N} Int(C_n). See that the family \{X\U_n : n ∈ N\} is a countable family of closed sets such that for each n ∈ N, there is an open set X\C_n ⊇ X\U_n and

\[ \bigcap_{n ∈ N} Cl(X\C_n) = \bigcap_{n ∈ N} X\C_{n_k} = X\bigcap_{n ∈ N} Int(C_{n_k}) = ∅. \]

By the assumption, we have \bigcap_{k=1}^{m} (X\U_{n_k}) = X\bigcap_{k=1}^{m} U_{n_k} = ∅. Thus, X = \bigcup_{k=1}^{m} U_{n_k}.

This proves that X is nearly regular countably compact.

3. Mappings on Nearly Countably Compact Spaces

Definition 3.1. A function f : (X, τ) → (Y, σ) is called

1. R-map [4] if f⁻¹(V) is regularly open in X for every regularly open set V in Y.

2. Almost continuous [10] if f⁻¹(V) is open in X for every regularly open set V in Y.

3. Almost α-open [7] f(U) ⊆ Int(Cl(Int(f(U)))) for every regularly open set U in X.

4. Weakly open [8] f(U) ⊆ Int(f(Cl(U))) for every open set U in X.

Theorem 3.1. Let f : (X, τ) → (Y, σ) be an R-map from X onto Y. If X is nearly countably compact then so is Y.
Proof. Let \( \{ V_n : n \in \mathbb{N} \} \) be a countable regularly open cover of \( Y \), then
\[
\{ f^{-1}(V_n) : n \in \mathbb{N} \}
\]

is a countable regularly open cover of \( X \). Since \( X \) is nearly countably compact, there exists a finite subfamily \( \{ f^{-1}(V_{n_1}), ..., f^{-1}(V_{n_k}) : i = 1, 2, ..., k \} \) such that \( X = \bigcup_{i=1}^{k} f^{-1}(V_{n_i}) \). Hence, \( Y = f(X) = f(\bigcup_{i=1}^{k} f^{-1}(V_{n_i})) = \bigcup_{i=1}^{k} f(f^{-1}(V_{n_i})) = \bigcup_{i=1}^{k} V_{n_i} \).

Therefore, \( Y \) is nearly countably compact.

**Theorem 3.2.** [5] Every almost continuous and almost \( \alpha \)-open function is an \( R \)-map.

**Corollary 3.1.** Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be an almost continuous and \( \alpha \)-open function from \( X \) onto \( Y \). If \( X \) is nearly countably compact then so is \( Y \).

**Theorem 3.3.** Let \( f : (X, \tau) \rightarrow (Y, \sigma) \) be an almost continuous and weakly open function from \( X \) onto \( Y \). If \( X \) is nearly countably compact then so is \( Y \).

Proof. Let \( \{ V_n : n \in \mathbb{N} \} \) be a countable regularly open cover of \( Y \). So,
\[
\{ f^{-1}(V_n) : n \in \mathbb{N} \}
\]

is an open cover of \( X \). Since \( f^{-1}(V_n) \subseteq \text{Int}(\text{Cl}(f^{-1}(V_n))) \) for all \( n \in \mathbb{N} \), we have that \( \{ \text{Int}(\text{Cl}(f^{-1}(V_n))) : n \in \mathbb{N} \} \) is a countable regularly open cover of \( X \) and so it has a countable subfamily \( \{ \text{Int}(\text{Cl}(f^{-1}(V_{n_1}))), ..., \text{Int}(\text{Cl}(f^{-1}(V_{n_k}))) : i = 1, ...k \} \) such that \( X = \bigcup_{i=1}^{k} \text{Int}(\text{Cl}(f^{-1}(V_{n_i}))) \). Since \( f \) is weakly open and \( \text{Int}(\text{Cl}(f^{-1}(V_{n_i}))) \) is an open set for all \( i = 1, ...k \), we have
\[
f(\text{Int}(\text{Cl}(f^{-1}(V_{n_i})))) \subseteq \text{Int}(f(\text{Cl}(\text{Int}(f^{-1}(V_{n_i}))))).
\]

Thus,
\[
Y = f(X) = f(\bigcup_{i=1}^{k} \text{Int}(\text{Cl}(f^{-1}(V_{n_i})))) = \bigcup_{i=1}^{k} f(\text{Int}(\text{Cl}(f^{-1}(V_{n_i})))) \subseteq \bigcup_{i=1}^{k} \text{Int}(f(\text{Cl}(\text{Int}(f^{-1}(V_{n_i}))))).
\]

\[
\subseteq \bigcup_{i=1}^{k} \text{Int}(f(\text{Cl}(f^{-1}(V_{n_i})))) \subseteq \bigcup_{i=1}^{k} \text{Int}(f(\text{Cl}(f^{-1}(V_{n_i})))) = \bigcup_{i=1}^{k} V_{n_i}. \]
So \( \{V_n\}_{i=1}^k \) is a finite subcover of \( Y \). Therefore, \( Y \) is nearly countably compact.

References


