ON THREE-POINT HADAMARD-TYPE FRACTIONAL BOUNDARY VALUE PROBLEMS

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\textbf{Abstract:} In this paper, we study the existence of solutions for a fractional boundary value problem involving Hadamard type fractional differential equations and three-point boundary conditions. Our analysis is based on Krasnoselskii-Zabreiko’s fixed point theorem. As an application, we discuss a Hadamard type boundary value problem with fractional integral boundary conditions. We emphasize that our results are new in the context of Hadamard fractional calculus and are well illustrated with the aid of examples.

\textbf{AMS Subject Classification:} 34A08, 34B15
\textbf{Key Words:} Hadamard fractional derivative, three-point boundary conditions, fixed point theorems

\section{1. Introduction}

The theory of fractional differential equations and inclusions has received much attention over the past years and has become an important field of investigation due to its extensive applications in numerous branches of physics, economics and engineer-
ing sciences [19, 23, 25, 26]. Fractional differential equations and inclusions provide appropriate models for describing real world problems, which cannot be described using classical integer order differential equations. Some recent contributions to the subject can be seen in [1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 16, 17, 24] and references cited therein.

It has been noticed that most of the work on the topic is based on Riemann-Liouville and Caputo type fractional differential equations. Another kind of fractional derivatives that appears side by side to Riemann-Liouville and Caputo derivatives in the literature is the fractional derivative due to Hadamard introduced in 1892 [18], which differs from the preceding ones in the sense that the kernel of the integral (in the definition of Hadamard derivative) contains logarithmic function of arbitrary exponent. Details and properties of Hadamard fractional derivative and integral can be found in [13, 14, 15, 19, 20, 21].

In this paper, we study the following boundary value problem of Hadamard type fractional differential equations:

\[
\begin{align*}
D^\alpha x(t) &= f(t, x(t)), \quad 1 < t < e, \quad 1 < \alpha \leq 2, \\
x(1) &= 0, \quad x(e) = \beta x(\eta), \quad 1 < \eta < e,
\end{align*}
\]  

(1.1)

where \(D^\alpha\) is the Hadamard fractional derivative of order \(\alpha\), \(f : [1, e] \times \mathbb{R} \to \mathbb{R}\) is a continuous function and \(\beta\) is a real constant.

We aim to establish an existence result for the problem (1.1) via Krasnoselskii-Zabreiko’s fixed point theorem. In Section 2, we present some basic concepts and solve a linear Hadamard type fractional differential boundary value problem. Section 3 contains the main result for the problem (1.1). Section 4 is devoted to the case of fractional integral boundary conditions, while in Section 4, some illustrative examples are discussed.

2. Preliminaries

This section is devoted to the background material needed to establish the main result of the paper.

**Definition 2.1.** [19] The Hadamard derivative of fractional order \(q\) for a function \(g : [1, \infty) \to \mathbb{R}\) is defined as

\[
D^q g(t) = \frac{1}{\Gamma(n - q)} \left( t \frac{d}{dt} \right)^n \int_1^t \left( \log \frac{s}{t} \right)^{n-q-1} \frac{g(s)}{s} ds,
\]

\[n - 1 < q < n, \quad n = [q] + 1,
\]

provided the integral exists, where \([q]\) denotes the integer part of the real number \(q\) and \(\log(\cdot) = \log_e(\cdot)\).
Definition 2.2. [19] The Hadamard fractional integral of order $q$ for a function $g$ is defined as

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_1^t \left(\log \frac{t}{s}\right)^{q-1} \frac{g(s)}{s} ds, \quad q > 0,$$

provided the integral exists.

Lemma 2.3. (Auxiliary Lemma) For $1 < \alpha \leq 2$ and $\zeta \in C([1,e], \mathbb{R})$, the unique solution of the problem

$$\begin{cases} 
D^\alpha x(t) = \zeta(t), & 1 < t < e \\
 x(1) = 0, & x(e) = \beta x(\eta),
\end{cases} \quad (2.1)$$

is given by

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{\zeta(s)}{s} ds$$

$$+ \frac{(\log t)^{\alpha-1}}{1 - \beta (\log \eta)^{\alpha-1}} \left[ \frac{\beta}{\Gamma(\alpha)} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{\alpha-1} \frac{\zeta(s)}{s} ds \right]$$

$$- \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{\zeta(s)}{s} ds, \quad t \in [1,e]. \quad (2.2)$$

Proof. As argued in [19], the solution of Hadamard differential equation in (2.1) can be written as

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{\zeta(s)}{s} ds + c_1 (\log t)^{\alpha-1} + c_2 (\log t)^{\alpha-2}. \quad (2.3)$$

Using the given boundary conditions, we find that $c_2 = 0$, and

$$c_1 = \frac{1}{1 - \beta (\log \eta)^{\alpha-1}} \left\{ \frac{\beta}{\Gamma(\alpha)} \int_1^\eta \left(\log \frac{\eta}{s}\right)^{\alpha-1} \frac{\zeta(s)}{s} ds 
\right.$$ 

$$- \frac{1}{\Gamma(\alpha)} \int_1^e \left(\log \frac{e}{s}\right)^{\alpha-1} \frac{\zeta(s)}{s} ds \right\}.$$

Substituting the values of $c_1$ and $c_2$ in (2.3), we obtain (2.2). This completes the proof.

Now we recall the Krasnol’sk’ii-Zabreiko’s fixed point theorem.
Theorem 2.4. [22] Let \((E, \| \cdot \|)\) be a Banach space, and \(T : E \to E\) be a completely continuous operator. Assume that \(A : E \to E\) is a bounded linear operator such that 1 is not an eigenvalue of \(A\) and
\[
\lim_{\|u\| \to \infty} \frac{\|Tu - Au\|}{\|u\|} = 0.
\]
Then \(T\) has a fixed point in \(E\).

In view of Lemma 2.3, the solution of the problem (1.1) can be written as
\[
x(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left( \log \frac{t}{s} \right)^{\alpha-1} \frac{f(s, x(s))}{s} ds \\
+ \frac{(\log t)^{\alpha-1}}{1 - \beta(\log \eta)^{\alpha-1}} \left[ \frac{\beta}{\Gamma(\alpha)} \int_1^\eta \left( \log \frac{\eta}{s} \right)^{\alpha-1} \frac{f(s, x(s))}{s} ds \right] \\
- \frac{1}{\Gamma(\alpha)} \int_1^e \left( \log \frac{e}{s} \right)^{\alpha-1} \frac{f(s, x(s))}{s} ds,
\]
(2.4)

3. Main Result

Consider the Banach space \(X = C([1, e], \mathbb{R})\) endowed with the norm \(\|x\| = \sup_{t \in [1, e]} |x(t)|\).

Theorem 3.1. Let \(f\) be a continuous function, satisfying \(f(a, 0) \neq 0\) for some \(a \in [1, e]\), and
\[
\lim_{|x| \to \infty} \frac{f(t, x)}{x} = \lambda(t), \quad \lambda_{\max} := \max_{t \in [1, e]} |\lambda(t)| < \frac{1}{\delta},
\]
with
\[
\delta = \frac{1}{\Gamma(\alpha + 1)} \left( 1 + \frac{1 + \beta(\log \eta)^{\alpha}}{|1 - \beta(\log \eta)^{\alpha-1}|} \right).
\]
Then the boundary value problem (1.1) has at least one nontrivial solution in \(X\).

Proof. Define an operator \(F : X \to X\) by
\[
Fx(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left( \log \frac{t}{s} \right)^{\alpha-1} \frac{f(s, x(s))}{s} ds \\
+ \frac{(\log t)^{\alpha-1}}{1 - \beta(\log \eta)^{\alpha-1}} \left[ \frac{\beta}{\Gamma(\alpha)} \int_1^\eta \left( \log \frac{\eta}{s} \right)^{\alpha-1} \frac{f(s, x(s))}{s} ds \right] \\
- \frac{1}{\Gamma(\alpha)} \int_1^e \left( \log \frac{e}{s} \right)^{\alpha-1} \frac{f(s, x(s))}{s} ds,
\]
(3.1)
It is clear that the mapping $F$ is well defined. In view of Krasnoselsk’ii-Zabreiko’s fixed point theorem, we look for fixed points for the operator $F$ in the Banach space $X$. We split the proof into three steps.

**Step 1.** $F$ is continuous.

Let us consider a sequence $\{x_n\}$ converging to $x$. For each $t \in [1,e]$, we have
\[
|F x_n(t) - F x(t)| \leq \frac{1}{\Gamma(\alpha)} \int_1^t \left( \log \frac{t}{s} \right)^{\alpha - 1} \frac{|f(s, x_n(s)) - f(s, x(s))|}{s} \, ds \\
+ \frac{1}{|1 - \beta(\log \eta)|^{\alpha - 1}} \left[ \frac{\beta}{\Gamma(\alpha)} \int_1^\eta \left( \log \frac{\eta}{s} \right)^{\alpha - 1} \frac{|f(s, x_n(s)) - f(s, x(s))|}{s} \, ds \\
+ \frac{1}{\Gamma(\alpha)} \int_1^e \left( \log \frac{e}{s} \right)^{\alpha - 1} \frac{|f(s, x_n(s)) - f(s, x(s))|}{s} \, ds \right].
\]

Thus
\[
\|F x_n - F x\| \leq \delta \|f(s, x_n(s)) - f(s, x(s))\|
\]
for some $\delta > 0$. Since the convergence of a sequence implies its boundedness, therefore, there exists a number $k > 0$ such that
\[
\|x_n\| \leq k, \quad \|x\| \leq k,
\]
and hence $f$ is uniformly continuous on the compact set $\{(t, x) : t \in [1,e], \|x\| \leq k\}$. Thus $\|F x_n - F x\| \leq \varepsilon$, $\forall n \geq n_0$. This shows that $F$ is continuous.

For any $R > 0$, we consider the closed set $C = \{x \in X; \|x\| \leq R\}$.

**Step 2.** We prove that $F(C)$ is relatively compact in $X$.

We set
\[
f_{\max} = \max_{t \in [1,e], \|x\| \leq R} |f(t, x)|.
\]
Then we have
\[
|F x(t)| \leq \frac{1}{\Gamma(\alpha)} \int_1^t \left( \log \frac{t}{s} \right)^{\alpha - 1} \frac{|f(s, x(s))|}{s} \, ds \\
+ \frac{1}{|1 - \beta(\log \eta)|^{\alpha - 1}} \left[ \frac{\beta}{\Gamma(\alpha)} \int_1^\eta \left( \log \frac{\eta}{s} \right)^{\alpha - 1} \frac{|f(s, x(s))|}{s} \, ds \\
+ \frac{1}{\Gamma(\alpha)} \int_1^e \left( \log \frac{e}{s} \right)^{\alpha - 1} \frac{|f(s, x(s))|}{s} \, ds \right].
\]
\[ \leq f_{\text{max}} \frac{1}{\Gamma(\alpha + 1)} \left\{ 1 + \frac{1 + \beta(\log \eta)^\alpha}{1 - \beta(\log \eta)^{\alpha - 1}} \right\}. \]

Thus \( \|Fx\| \leq f_{\text{max}} \delta \) and thus \( F(C) \) is uniformly bounded. For \( \tau_1, \tau_2 \in [1, e] \) with \( \tau_1 < \tau_2 \) we have

\[
\begin{align*}
|Fx(\tau_2) - Fx(\tau_1)| & \leq f_{\text{max}} \frac{1}{\Gamma(\alpha)} \int_1^{\tau_1} \left( \log \frac{\tau_1}{s} \right)^{\alpha - 1} \frac{1}{s} ds - \int_1^{\tau_2} \left( \log \frac{\tau_2}{s} \right)^{\alpha - 1} \frac{1}{s} ds \\
& \quad + f_{\text{max}} \frac{\beta(\log \eta)^\alpha}{1 - \beta(\log \eta)^{\alpha - 1}} \left[ \frac{\beta}{\Gamma(\alpha)} \int_1^{\eta} \left( \log \frac{\eta}{s} \right)^{\alpha - 1} \frac{1}{s} ds \right] \\
& \quad - \frac{1}{\Gamma(\alpha)} \int_1^{e} \left( \log \frac{e}{s} \right)^{\alpha - 1} \frac{1}{s} ds \\
& \leq f_{\text{max}} \frac{1}{\Gamma(\alpha)} \int_1^{\tau_1} \left[ \left( \log \frac{\tau_1}{s} \right)^{\alpha - 1} - \left( \log \frac{\tau_2}{s} \right)^{\alpha - 1} \right] \frac{1}{s} ds \\
& \quad + f_{\text{max}} \frac{\beta(\log \eta)^\alpha}{1 - \beta(\log \eta)^{\alpha - 1}} \left[ \frac{\beta}{\Gamma(\alpha)} \int_1^{\eta} \left( \log \frac{\eta}{s} \right)^{\alpha - 1} \frac{1}{s} ds \right] \\
& \quad - \frac{1}{\Gamma(\alpha)} \int_1^{e} \left( \log \frac{e}{s} \right)^{\alpha - 1} \frac{1}{s} ds \\
& \quad + f_{\text{max}} \frac{\beta(\log \eta)^\alpha}{1 - \beta(\log \eta)^{\alpha - 1}} \left[ \frac{\beta}{\Gamma(\alpha)} \int_1^{e} \left( \log \frac{\eta}{s} \right)^{\alpha - 1} \frac{1}{s} ds \right].
\end{align*}
\]

Then \( F(C) \) is equicontinuous as the right-hand side tends to 0 as \( \tau_1 \to \tau_2 \). By the Arzelá-Ascoli Theorem, the mapping \( F \) is completely continuous on \( X \). This completes the proof of Step 2.

Next consider the following boundary value problem

\[
\begin{align*}
D^\alpha x(t) &= \lambda(t)x(t), \quad 1 < t < e, \\
x(1) &= 0, \quad x(e) = \beta x(\eta).
\end{align*}
\]

(3.2)

Let us define an operator \( A : X \to X \) by

\[
A x(t) = \frac{1}{\Gamma(\alpha)} \int_1^{t} \frac{\alpha - 1}{s} \lambda(s)x(s) ds \\
+ \frac{(\log t)^{\alpha - 1}}{1 - \beta(\log \eta)^{\alpha - 1}} \left[ \frac{\beta}{\Gamma(\alpha)} \int_1^{\eta} \frac{\alpha - 1}{s} \lambda(s)x(s) ds \right].
\]
\[-\frac{1}{\Gamma(\alpha)} \int_1^e \left( \log \frac{e}{s} \right)^{\alpha-1} \frac{\lambda(s) x(s)}{s} ds \], \quad t \in [1, e]. \quad (3.3)\]

Obviously $A$ is a bounded linear operator. Furthermore, any fixed point of $A$ is a solution of the boundary value problem (3.2) and vice versa.

**Step 3.** We now assert that 1 is not an eigenvalue of $A$.

Suppose that the boundary value problem (3.2) has a nontrivial solution $x$. We have

\[
\|x\| = \|(A(x))\| = \sup_{t \in [1, e]} |Ax(t)| \\
\leq \lambda_{\text{max}} \sup_{t \in [1, e]} \left[ \frac{1}{\Gamma(\alpha)} \int_1^t \left( \log \frac{t}{s} \right)^{\alpha-1} \frac{|x(s)|}{s} ds \right. \\
\left. + \frac{1}{[1 - \beta(\log \eta)^{\alpha-1}]} \left[ \frac{\beta}{\Gamma(\alpha)} \int_1^{\eta} \left( \log \frac{\eta}{s} \right)^{\alpha-1} \frac{|x(s)|}{s} ds \right] \right. \\
\left. + \frac{1}{\Gamma(\alpha)} \int_1^e \left( \log \frac{e}{s} \right)^{\alpha-1} \frac{|x(s)|}{s} ds \right] \\
\leq \lambda_{\text{max}} \frac{1}{\Gamma(\alpha + 1)} \left( 1 + \frac{1 + \beta(\log \eta)^{\alpha}}{[1 - \beta(\log \eta)^{\alpha-1}]} \right) \|x\| \\
= \lambda_{\text{max}} \delta \|x\| \\
< \|x\|.
\]

This contradiction prove that the BVP (3.2) has no nontrivial solution. Thus, 1 is not an eigenvalue of $A$.

Finally, we prove that

\[
\lim_{\|x\| \to \infty} \frac{\|F(x) - Ax\|}{\|x\|} = 0.
\]

According to \( \lim_{|x| \to \infty} \frac{f(t, x)}{x} = \lambda(t) \), for any \( \varepsilon > 0 \), there exist some \( M > 0 \), such that

\[
|f(t, x) - \lambda(t) x| < \varepsilon |x| \quad \text{for} \quad |x| > M.
\]

Set

\[
M^* = \max_{t \in [1, e]} \{ \max_{|x| \leq M} f(t, x) \}
\]

and select \( R' > 0 \) such that

\[
M^* + \lambda_{\text{max}} M < \varepsilon R'.
\]
We denote 
\[ I_1 = \{ t \in [1,e] : |x(t)| \leq M \}, \quad I_2 = \{ t \in [1,e] : |x(t)| > M \}. \]

For any \( x \in X \) with \( \|x\| > R' \), \( t \in I_1 \), we have 
\[ |f(t,x) - \lambda(t)x| \leq |f(t,x)| + \lambda_{\text{max}}|x| \leq M^* + \lambda_{\text{max}}M \leq \varepsilon R' < \varepsilon \|x\|. \]

For any \( x \in X \) with \( \|x\| > R' \), \( t \in I_2 \), we have 
\[ |f(t,x) - \lambda(t)x| < \varepsilon \|x\|. \]

Then for any \( x \in X \) with \( \|x\| > R' \), we have 
\[ |f(t,x) - \lambda(t)x| < \varepsilon \|x\|. \]

Then we obtain 
\[ \|Fx - Ax\| = \sup_{t \in [1,e]} |(Fx - Ax)(t)| \leq \sup_{t \in [1,e]} \left[ \frac{1}{\Gamma(\alpha)} \int_1^t \left( \log \frac{s}{t} \right)^{\alpha-1} \frac{|f(s,x(s)) - \lambda(s)x(s)|}{s} ds + \frac{\log t - 1}{1 - \beta(\log \eta)^{\alpha-1}} \left[ \frac{\beta}{\Gamma(\alpha)} \int_1^\eta \left( \log \frac{\eta}{s} \right)^{\alpha-1} \frac{|f(s,x(s)) - \lambda(s)x(s)|}{s} ds \right] \right] \leq \varepsilon \lambda_{\text{max}} \left[ \frac{1}{\Gamma(\alpha + 1)} \left( 1 + \frac{1 + \beta(\log \eta)^{\alpha}}{1 - \beta(\log \eta)^{\alpha-1}} \right) \|x\| \right] = \varepsilon \lambda_{\text{max}} \delta \|x\|, \]

which, on taking the limit, yields 
\[ \lim_{\|x\| \to \infty} \frac{\|Fx - Ax\|}{\|x\|} = 0. \]

Consequently, Theorem 2.4 guarantees that the boundary value problem (1.1) has at least one nontrivial solution.

\[ \square \]
4. The Case of Fractional Integral Boundary Conditions

The method presented in previous section is applicable to a variety of boundary value problems. As an application, we consider the following Hadamard type boundary value problem with fractional integral boundary conditions given by

\[
\begin{align*}
D^\alpha x(t) &= g(t, x(t)), \quad 1 < t < e, \quad 1 < \alpha \leq 2, \\
x(1) &= 0, \quad x(e) = I^\beta x(\eta), \quad 1 < \eta < e,
\end{align*}
\]

(4.1)

where \(D^\alpha\) is the Hadamard fractional derivative of order \(\alpha\), \(I^\beta\) is the Hadamard fractional integral of order \(\beta\) and \(g : [1, e] \times \mathbb{R} \to \mathbb{R}\) is a continuous function.

**Lemma 4.1.** [8] For \(1 < \alpha \leq 2\) and \(\zeta \in C([1, e], \mathbb{R})\), the unique solution of the problem

\[
\begin{align*}
D^\alpha x(t) &= \zeta(t), \quad 1 < t < e \\
x(1) &= 0, \quad x(e) = I^\beta x(\eta),
\end{align*}
\]

(4.2)

is given by

\[
x(t) = I^\alpha \zeta(t) + \frac{(\log t)^{\alpha-1}}{\Omega} \left[ I^{\beta+\alpha} \zeta(\eta) - I^\alpha \zeta(e) \right],
\]

(4.3)

where

\[
\Omega = \frac{1}{1 - \frac{1}{\Gamma(\beta)} \int_1^\eta \left( \log \frac{\eta}{s} \right)^{\beta-1} \left( \log s \right)^{\alpha-1} \frac{s}{s} ds}.
\]

(4.4)

**Theorem 4.2.** Let \(g\) be a continuous function, satisfying \(g(a, 0) \neq 0\) for some \(a \in [1, e]\), and

\[
\lim_{|x| \to \infty} \frac{g(t, x)}{x} = \lambda(t), \quad \lambda_{\text{max}} := \max_{t \in [1, e]} |\lambda(t)| < \frac{1}{\delta_1},
\]

with

\[
\delta_1 = \frac{1}{\Gamma(\alpha + 1)} + \frac{1}{\Omega} \left( \frac{(\log \eta)^{\beta+\alpha}}{\Gamma(\beta + \alpha + 1)} + \frac{1}{\Gamma(\alpha + 1)} \right).
\]

Then the boundary value problem (4.1) has at last one nontrivial solution in \(X\).

**Proof.** In view of Lemma 4.1, lets us define an operator \(G : X \to X\) by

\[
Gx(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left( \log \frac{t}{s} \right)^{\alpha-1} \frac{g(s, x(s))}{s} ds \\
+ \frac{(\log t)^{\alpha-1}}{\Omega} \left[ \int_1^\eta \left( \log \frac{\eta}{s} \right)^{\beta+\alpha-1} \frac{g(s, x(s))}{s} ds \\
- \int_1^e \left( \log \frac{e}{s} \right)^{\alpha-1} \frac{g(s, x(s))}{s} ds \right], \quad t \in [1, e].
\]

(4.5)
We omit the further details as the remaining proof runs parallel to that of Theorem 3.1 with $\delta_1$ in place of $\delta$.

5. Examples

In this section we present some concrete examples to illustrate our results.

Example 5.1. Consider the boundary value problem

$$\begin{cases}
D^{3/2}x(t) = f(t, x(t)), & 1 < t < e, \\
x(1) = 0, & x(e) = \frac{3}{2}x(2).
\end{cases} \tag{5.1}$$

Here $\alpha = 3/2$, $\beta = 3/2$, $\eta = 2$,

$$\delta = \frac{1}{\Gamma(\alpha + 1)} \left\{ 1 + \frac{1 + \beta(\log \eta)^\alpha}{1 - \beta(\log \eta)^\alpha - 1} \right\} \approx 6.3938692.$$ 

If $f(t, x) = \frac{1}{20}(t^2 + 1)x(t), t \in [1, e]$, then $\lambda_{\text{max}} \delta \approx 0.4194527 < 1$ and hence by Theorem 3.1 the boundary value problem (5.1) has at least one solution.

Example 5.2. Consider the problem

$$\begin{cases}
D^{3/2}x(t) = f(t, x(t)), & 1 < t < e, \\
x(1) = 0, & x(e) = \Gamma^{3/2}x(2).
\end{cases} \tag{5.2}$$

Here $\alpha = 3/2$, $\beta = 3/2$, $\eta = 2$. With the given values, we have

$$\Omega = \frac{1}{\Gamma(\beta)} \int_{1}^{\eta} \left( \frac{\log \eta}{s} \right)^{\beta - 1} \left( \frac{\log s}{s} \right)^{\alpha - 1} ds = \frac{4}{4 - \sqrt{\pi} \log 2} \approx 1.27,$$

and

$$\delta_1 = \frac{1}{\Gamma(\alpha + 1)} + \frac{1}{\Gamma(\beta + \alpha + 1)} \left( \frac{(\log \eta)^{\beta + \alpha}}{\Gamma(\beta + \alpha + 1)} + \frac{1}{\Gamma(\alpha + 1)} \right) \approx 1.39.$$ 

By taking $f(t, x) = \frac{1}{20}(t^2 + 1)x(t), t \in [1, e]$, it is found that $\lambda_{\text{max}} \delta_1 \approx 0.576 < 1$ and hence by Theorem 4.2 there exists at least one solution for problem (5.2).
References


